Number Conservation Property of Elementary Cellular Automata under Asynchronous Update

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This paper studies the number conservation property of elementary cellular automata (ECAs) under asynchronous update. In asynchronous update, any number of cells can be updated in each time step. It is, however, shown in the beginning of the study that no elementary cellular automaton (ECA), in general, can be claimed as number conserving under asynchronous update. Our goal was to search for some ECAs that conserve the number of 1s and 0s of any initial configuration for at least one update pattern. As a result of this search, we get a set of 64 ECAs. Each of these ECAs can sometimes show the number conservation property under asynchronous update. The probability of showing the number conservation property of these 64 ECAs may be small, but it is nonzero. However, in the update of these *number-conserving* ECAs, only passive transitions may be observed. We proceed by searching some ECAs (from the set of 64 ECAs) that show the number conservation property even with active transitions against some initial configurations. Here we identify 49 ECAs that obey this criterion. We finally extract nine ECAs from these 49 ECAs that can show the number conservation property with only active transitions against each of the possible initial configurations except for two homogeneous configurations: one with cell state 0 and the other with cell state 1. We conclude our study after presenting an asynchronous update scheme for these number-conserving ECAs.

1. Introduction

Since their introduction in the 1980s, elementary cellular automata (ECAs) have received wide attention from researchers due to their simplicity and capability of modeling physical systems. ECAs are the simplest cellular automata (CAs) where systems are binary and one-

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dimensional, and where the behavior of each cell depends only on itself and its two nearest neighbors. The ECAs that are extensively used in real-world modeling are primarily synchronous, where all the cells of an automaton update their states simultaneously. However, it is observed that in many real systems, components are not always synchronous. When asynchronism is incorporated in an automaton model, a drastic change in the behavior of the model is observed [1–6]. The cells of an *asynchronous* cellular automaton (CA) are independent, and any number of cells may be updated in a step of the automaton's evolution. In recent times, the CA models under asynchronous update have gained a huge popularity [5, 7–9].

On the other hand, the number conservation property of CAs has attracted researchers for a long time, due to their similarity with the physical law of conservation [10–15]. The number conservation property of binary CAs refers to the invariance of the number of 1s and 0s during the evolution of the automata. Such CAs essentially model the particle systems that are governed by local interaction rules. ECAs 184 (traffic rule), 226, 170, and 240 show the number conservation property under the periodic boundary condition. Whether any ECAs follow the number conservation property under asynchronous update, however, has not yet been investigated. In fact, this is an untouched issue in the domain of one-dimensional CAs. In this scenario, we take up this research to study the number conservation property of ECAs under asynchronous update.

We start our study by asking a question, can we get some ECAs that follow the number conservation property under asynchronous update? We utilize the concept of *update pattern* [7] in this study. An update pattern $\langle u_t \rangle_{t \in \mathbb{N}}$ of an ECA under asynchronous update is a sequence of sets u_t that notes the cells that are updated at discrete time t. An update pattern along with an initial configuration can show the evolution of an ECA. While we explore the ECAs under asynchronous update, it is revealed that no ECA (except ECA 204, the identity rule), in general, can be claimed as number conserving (see Section 3). To be number-conserving, an ECA has to always conserve the number of 1s and 0s of any configuration. Since it cannot be seen in advance which cells are going to be updated at time t, the prediction of future configurations is impossible. This is an informal logic why ECA, in general, cannot be claimed as number conserving under asynchronous update.

At this point we modulate our first question. We now ask, can we get an ECA that conserves the number of 1s of any initial configuration during its evolution for at least one update pattern? Interestingly, we get an affirmative answer to this question. In Section 3, a set of ECAs is identified (see Table 3) that can show the number conserva-

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tion property for some update patterns. This implies there is a possibility (maybe very low) that these ECAs can show the number conservation property under asynchronous update.

However, there are some ECAs in Table 3 that show the number conservation property with only passive transitions. Obviously, this is a trivial case. In Section 4, we proceed with our study in search of ECAs (from Table 3) that can be number conserving for an update pattern with active transitions. As a result of this search, we identify 49 ECAs (see Table 4). Section 4 finally identifies nine ECAs that are capable of showing the number conservation property against each of the possible initial configurations (except two homogeneous configurations: one with cell state 0 and the other with cell state 1) with active transitions only. We conclude our study with an asynchronous update scheme (see Section 5) for these number-conserving ECAs.

Before proceeding further, we next present some basics of CAs.

2. Cellular Automata

CAs, introduced by von Neumann [16], are mathematical modeling tools that have been studied extensively as models of physical systems. The CAs evolve in discrete space and time. In their simplest form, as proposed by Wolfram [17], a CA consists of an array of cells, each of which stores a binary state at time t. The next state of a cell is affected by its present state and the present states of its two nearest neighbors. These CAs are referred to as elementary cellular automata. In this work, we concentrate only on ECAs.

We consider that ECAs use the periodic boundary condition, where the ECA cells are arranged in a *ring*. In an ECA, the next state of a cell i is determined as:

$$S_i^{t+1} = f\left(S_{i-1}^t, S_i^t, S_{i+1}^t\right), \tag{1}$$

where f is the next state function, and S_{i-1}^t, S_i^t, and S_{i+1}^t are the present states of the left neighbor, self, and right neighbor of the i^{th} cell at time t. The function $f : \{0, 1\}^3 \mapsto \{0, 1\}$ can be expressed as a lookup table (see Table 1). The decimal equivalent of the next eight states is called a *rule* [17]. There are 2^8 (256) rules, hence 256 ECAs, in two-state, three-neighborhood dependency. Three such rules, 184, 222, and 228, are shown in Table 1.

The first row of Table 1 notes the combinations of the present states (PS) of three neighbors. Borrowing vocabulary from switching theory, we refer to each combination as a rule min term (RMT), because it can be viewed as the min term of a three-variable

$(S_{i-1}^t, S_i^t, S_{i+1}^t)$ switching function. Column 101 of Table 1 is the RMT 5. The next states (NS) corresponding to this RMT are 0 for ECA 222 and 1 for ECAs 184 and 228.

PS:	111	110	101	100	011	010	001	000	Rule
(RMT)	(7)	(6)	(5)	(4)	(3)	(2)	(1)	(0)	
(i) NS:	1	0	1	1	1	0	0	0	184
(ii) NS:	1	1	0	1	1	1	1	0	222
(iii) NS:	1	1	1	0	0	1	0	0	228

Table 1. Lookup table for ECA rules 184, 222, and 228.

The collection of states of all cells $(S_1^t, S_2^t, \ldots, S_n^t)$ at time t is called a *configuration* or a (global) state of the CA at that time. There are 2^n states in an n-cell ECA. Out of these states, some homogeneous states can be found. We write **0** and **1** to indicate homogeneous ECA states with cell state 0 and 1, respectively. Similarly, by **01** we indicate a homogeneous state of even size in which cell states 0 and 1 alternate. Another three homogeneous states can be found: **001** (state size is a multiple of 3), **011** (state size is a multiple of 3), and **0011** (state size is a multiple of 4).

However, an ECA state can also be viewed as a sequence of RMTs. For example, the state 1110 in the periodic boundary condition can be viewed as ⟨3765⟩, where 3, 7, 6, and 5 are corresponding RMTs on which the transitions of the first, second, third, and fourth cells can be made. The i^{th} RMT in the RMT sequence of an ECA state is formed with the states of cell $i-1$, cell i, and cell $i+1$. In the sequence of RMTs, however, two consecutive RMTs are related. If 5 (101) is the i^{th} RMT in some sequence, then the $(i+1)^{\text{th}}$ RMT is either 2 (010) or 3 (011). Similarly, if 0 (000) or 4 (100) is the i^{th} RMT, then 0 (000) or 1 (001) is the $(i+1)^{\text{th}}$ RMT. The relations of two consecutive RMTs in a sequence of RMTs are noted in Table 2.

Classically, all the cells of an ECA are updated simultaneously (a *synchronous* ECA). In this work, however, we introduce asynchronous update of ECA cells. That is, any number of cells may be updated in a step of the ECA's evolution. In a synchronous ECA, a global clock is assumed, which forces the cells to be updated simultaneously. Whereas in the asynchronous case, the clocks of the cells are independent, so the cells are updated independently. Wolfram refers to a class of CAs as *sequential* CAs, where an arbitrary cell is updated in each time step [6]. These CAs are also referred to as *fully asynchronous* CAs [8]. However, we consider here the more general case where one or more, even all the ECA's cells, may be updated in a time step.

i^{th} RMT	$(i+1)^{\text{th}}$ RMT
0	0, 1
1	2, 3
2	4, 5
3	6, 7
4	0, 1
5	2, 3
6	4, 5
7	6, 7

Table 2. Relationship between i^{th} and $(i+1)^{\text{th}}$ RMTs in an RMT sequence.

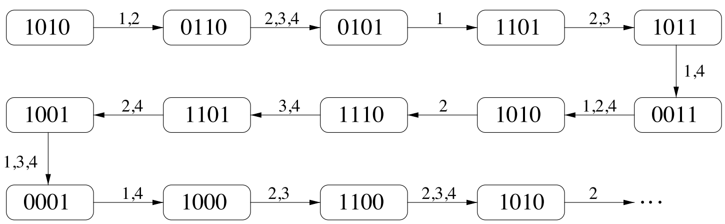

Figure 1. Partial state transition diagram of ECA 184. The cells updated during state transition are noted over the arrows.

During their evolution with time, ECAs generate a sequence of states. However, the next state of an ECA depends not only on the rule, but also on the cells that are updated at that time. We denote the set of cells updated at time t as u_t. Therefore, one can get an *update pattern* $U_Q = \langle u_1, u_2, \ldots, u_t, \ldots \rangle$, which is associated with an initial state Q. If an update pattern with an initial state Q is given, the state transitions of the ECA can be identified. A partial state transition diagram of four-cell ECA 184 with the periodic boundary condition is shown in Figure 1. (To reduce the space requirement of the figures, we consider in most of the examples of this paper that the number of cells is four.) The states are noted in rounded rectangles, whereas the cells updated during state transitions are noted over arrows. The update pattern for this transition

$$U_{1010} = \langle \{1, 2\}, \{2, 3, 4\}, \{1\}, \{2, 3\}, \{1, 4\}, \{1, 2, 4\}, \{2\}, \{3, 4\}, \{2, 4\}, \{1, 3, 4\}, \{1, 4\}, \{2, 3\}, \{2, 3, 4\}, \ldots \rangle$$

is associated with ECA state 1010. It is, therefore, obvious that the state transition of ECAs under asynchronous update depends on both the ECA rule and the update pattern.

We now proceed with the number conservation issue of ECAs under asynchronous update.

3. The Number Conservation Property

The number conservation property of an ECA refers to the conservation of 1s and 0s of any initial state during the ECA's evolution. ECA 184 of Figure 1 does not conserve the number of 1s of the initial state (i.e., 1010). A state transition diagram of an ECA verifies whether the ECA follows the number conservation property or not. However, the state transition diagram is changed if the update pattern changes. We start our study on the number conservation issue after asking the following question.

Question 1. Can we get some ECAs that conserve the number of 1s and 0s of *any* initial state under asynchronous update?

Note the emphasis on the word "any." If an ECA under asynchronous update shows the number conservation property against some specific initial states, we do not consider the ECA as number conserving. To qualify as number-conserving, the ECA has to follow the property for all possible initial states. However, we get a disappointing answer for Question 1, which is stated in Theorem 1. Before going into that theorem, we define the following.

Definition 1. RMT r of an ECA is active if a cell of the ECA flips its state (1 to 0 or 0 to 1) on r. The cell is referred to as an active cell, and the transition is referred to as an active transition. Otherwise, the RMT r (and corresponding cell and transition) is passive.

For example, RMT 2 of ECA 184 (see Table 1) is active. Here, a change occurs (from 1 to 0) in a cell when the cell acts on RMT 2. The cell is active and the transition that occurs in the cell is active. On the other hand, RMT 7 of ECA 184 is passive. In asynchronous update, if a cell is selected for update and the cell flips its state from 0 to 1 or 1 to 0, we call the transition active. If a cell is selected for update and no change occurs in the cell's state, then we call the transition passive.

Theorem 1. No ECA, except ECA 204, under asynchronous update can be claimed as a number-conserving ECA.

Proof. Transition of states of an ECA under asynchronous update depends on the ECA rule, the initial state, and the update pattern

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$\langle u_1, u_2, \ldots \rangle$ (in the case of ECA 204, the update pattern does not matter because all the RMTs are passive). Since any cell can be updated at any point in time, so the elements of each u_t (i.e., the cells to be updated at time t, $t \in \mathbb{N}$) cannot be known. Moreover, for an arbitrary state of the ECA, which cells of u_t are active and which are passive is undecidable. Therefore, it can never be claimed that the number of 1s and 0s of an arbitrary initial state of any ECA is conserved, since the automaton is updated asynchronously. Hence the proof. □

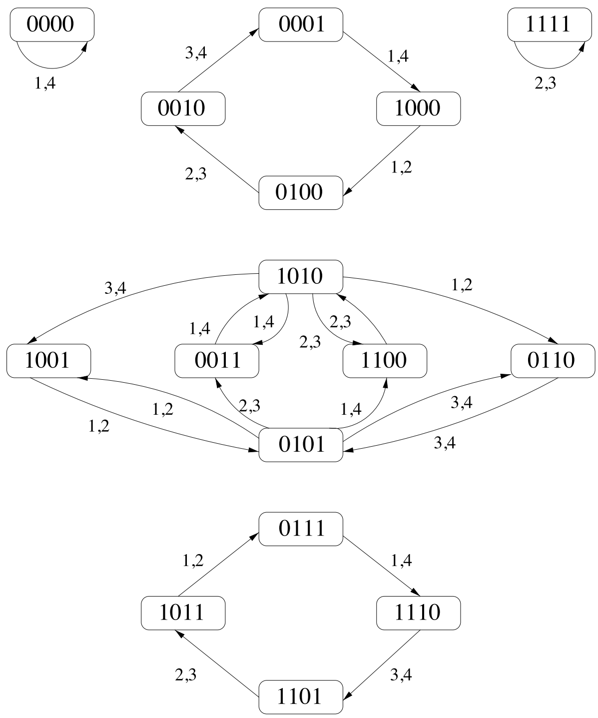

Figure 2. State transition diagram of ECA 184 with four cells and with the periodic boundary condition. The ECA maintains the number conservation property.

Theorem 1, therefore, warns us that ECAs, in general, are not number conserving under asynchronous update. However, we have observed that some ECAs, updated asynchronously, sometimes show the number conservation property. Consider the state transitions of ECA 184, shown in Figure 2. All the states of four-cell ECA 184 are covered here, and the number of 1s and 0s of each initial state is conserved for an update pattern. In fact, we can identify more than one update pattern in Figure 2 against some of the initial states like 1010,

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0101, and so forth. In all cases, however, the number conservation property of ECA 184 is followed. Note that for a different update pattern ECA 184 may not be number conserving (see Figure 1). To better understand asynchronous update, we now ask the following question.

Question 2. Can we claim that an ECA (except ECA 204) conserves the number of 1s and 0s of any initial state during its evolution for at least one update pattern?

We continue our study with this question and receive an affirmative answer. ECA 184 in Figure 2 is an example answer to Question 2, because there is at least one update pattern for each initial state for which the number of 1s and 0s is conserved. However, the existence of one update pattern, U_Q, implies that there is a probability of observing updates of ECAs according to U_Q. That is, if an ECA conserves the number of 1s and 0s of an initial state for one update pattern, then there is a probability that the ECA follows the number conservation property for the initial state under asynchronous update. The probability may be very small, but it is nonzero.

We now redefine the number-conserving ECAs under asynchronous update that will facilitate this study.

Definition 2. An ECA under asynchronous update is called a number-conserving ECA if the number of 1s and 0s of any initial state is conserved during the ECA's evolution for at least one update pattern.

Therefore, ECA 184 is number conserving (see Figure 2). There are as many as 64 ECAs (including ECA 204) that are number conserving according to Definition 2. To show an ECA as number conserving for "at least one update pattern," what we do in this paper is to show that there exists a next state of each of the possible ECA states, which we get by updating one or more cells (active or passive), and which conserves the numbers. The existence of a next state of each state proves in turn the existence of an update pattern. Theorem 2 guides us to get the number-conserving ECAs. Before going to the theorem, we next report a finding, which is used by Theorems 2 and 4.

Lemma 1. The RMT sequence of any state of an ECA is composed of RMTs of one or more sets from the following six RMT sets: {0}, {7}, {2, 5}, {1, 2, 4}, {3, 5, 6}, and {1, 3, 4, 6}.

Proof. We can identify six homogeneous states in the state space of an ECA: **0**, **1**, **01**, **001**, **011**, and **0011**. The RMT sequences of these six homogeneous states are: ⟨000 …⟩ (for **0**), ⟨777 …⟩ (for **1**), ⟨25 252 …⟩ (for **01**), ⟨1 241 241 …⟩ (for **001**), ⟨3 653 653 …⟩ (for **011**), and ⟨136 413 641 …⟩ (for **0011**). Note that the sequences ⟨2525 …⟩ and ⟨5252 …⟩ are equivalent in the periodic boundary condition. Similarly, ⟨124 124 …⟩, ⟨241 241 …⟩, and ⟨412 412 …⟩ are equivalent.

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Here, each of the six RMT sequences is unique. That is, no sequence is the subsequence of any other. It is already mentioned in Section 2 that RMTs in an RMT sequence follow a relation, which is shown in Table 2. From Table 2, however, only these six unique sequences can be identified. An arbitrary RMT sequence, therefore, is composed of these unique sequences. That is, any RMT sequence (of an ECA state) can be decomposed into subsequences of the above types. For example, RMT sequence ⟨12 400 … 01 241 …⟩ (of state 01 000 … 0100 … 0) contains RMTs of two of the above sequences: ⟨00 … 00⟩ and ⟨12 412 …⟩. No RMT sequence can be found that cannot be decomposed into the above sequences. Therefore, the RMT sequence of an arbitrary state contains one or more sets of the six RMT sets: {0}, {7}, {2, 5}, {1, 2, 4}, {3, 5, 6}, and {1, 3, 4, 6}. □

Theorem 2. An ECA conserves the number of 1s and 0s of any initial state during its evolution for at least one update pattern if and only if RMT 0 and RMT 7 of the ECA are passive.

Proof. To prove this theorem, we show that for each state of an ECA there is a next state with the same number of 1s and 0s, if and only if RMT 0 and RMT 7 of the ECA are passive.

According to Lemma 1, the RMT sequence of an arbitrary state *Q* contains RMTs of one or more sets from the six RMT sets: {0}, {7}, {2, 5}, {1, 2, 4}, {3, 5, 6}, and {1, 3, 4, 6}. Now if any RMT in the sequence is passive, then by updating the cell corresponding to the passive RMT we can get *Q* as the next state of *Q*. In this case, therefore, the number of 1s and 0s is conserved in the next state of *Q*.

Now consider the possibility that all the RMTs of the sequence are active. In this situation, the following cases are possible.

Case i. Assume the sequence contains RMTs 2 and 5 (and they are active). If two cells acting on RMTs 2 and 5 are updated, we get the next state of *Q* where the number of 1s and 0s is conserved. Here, the cell with RMT 2 becomes 0 from 1 and the cell with RMT 5 becomes 1 from 0.

Case ii. Consider if the sequence contains RMTs 1, 2, and 4 (and they are active). If a cell acting on RMT 2 and another cell acting on RMT 1 or 4 are updated, then the next state of *Q* follows the number conservation property.

Case iii. Let us assume the sequence contains RMTs 3, 5, and 6 (and they are active). With similar logic as stated above, we get that ECAs can follow the conservation property if the proper cells are updated.

Case iv. If RMTs 1, 3, 4, and 6 are present in the sequence (and they are active), and if a cell with RMT 1 or 4 and another cell with

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RMT 3 or 6 are selected for update, then we observe that the next state of Q follows the number conservation property.

Therefore, whether RMTs are active or passive in an RMT sequence of a state does not matter if the sequence contains RMTs 2 and 5; RMTs 1, 2, and 4; RMTs 3, 5, and 6; or RMTs 1, 3, 4, and 6. It matters for two states only: **0** and **1**. Hence, if RMT 0 and RMT 7 of an ECA are passive, the ECA is number conserving for at least one update pattern. On the other hand, if an ECA is number conserving for at least one update pattern, then RMT 0 and RMT 7 of the ECA are passive. Hence the proof. □

There are 64 ECAs with RMT 0 and RMT 7 as passive. For the sake of completeness, we list out these ECA rules in Table 3. These ECAs are also referred to as *double-quiescent* ECAs [8]. We shall use this terminology in our future reference.

128	130	132	134	136	138	140	142	144	146
148	150	152	154	156	158	160	162	164	166
168	170	172	174	176	178	180	182	184	186
188	190	192	194	196	198	200	202	204	206
208	210	212	214	216	218	220	222	224	226
228	230	232	234	236	238	240	242	244	246
248	250	252	254						

Table 3. List of number-conserving ECAs (i.e., they conserve the number of 1s and 0s of any initial state for at least one update pattern).

Therefore, these ECAs of Table 3 conserve the number of 1s and 0s of each initial state for at least one update pattern. Figure 3 shows the transition of each state of ECA 222 with four cells. Here, the ECA with each initial state follows the number conservation property (Figure 3). In fact, the next state of each state is the state itself; that is, all transitions made in this figure are passive. Figure 4 shows another example of number conservation with ECA 228. Note that not all the transitions of Figure 4 are passive. However, many of the ECAs of Table 3 show the number conservation property with passive transitions only.

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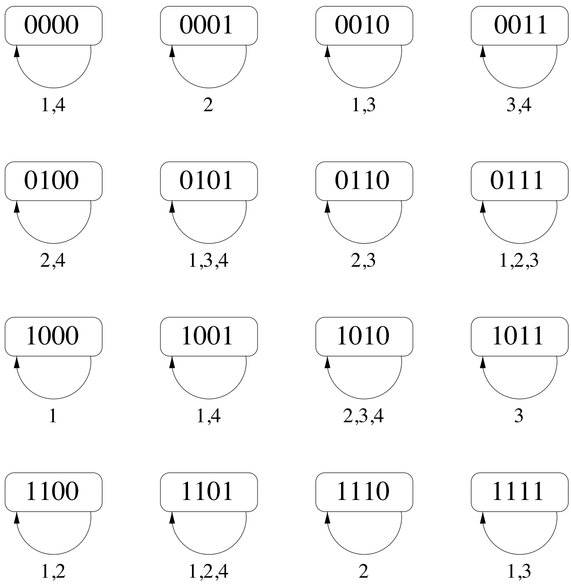

Figure 3. State transition diagram of ECA 222 with four cells. The ECA maintains the number conservation property.

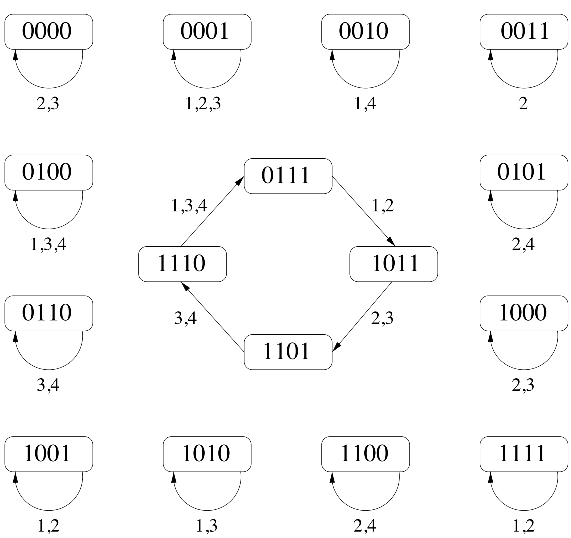

Figure 4. State transition diagram of ECA 228 with four cells. The ECA maintains the number conservation property.

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4. Number Conservation of Elementary Cellular Automata with Active Transitions

A state is repeated in the next time step if all transitions of the ECA are passive. That is, no change in the ECA's state is observed. The case where only passive transitions of an ECA are observed is a trivial one. However, there are some ECAs in Table 3 that show the number conservation property with active transitions also. ECA 228 can be declared number conserving because each possible state is repeated with some passive transitions (like Figure 3). But the ECA can also conceive active transitions. This can be verified with initial state 0111 and update pattern ⟨{1, 2}, {2, 3}, {3, 4}, {1, 3, 4}⟩ (see Figure 4). It is, however, not possible to allow active transitions for ECA 222 if we want to show its number conservation property. To distinguish these two types of ECAs, we continue our study with the following question.

Question 3. Can we get a double-quiescent ECA that conserves the number of 1s and 0s of at least one initial state with some active transitions?

To get an answer to this question, we present the following theorem. This theorem guides us to exclude some ECAs that are number conserving only with passive transitions.

Theorem 3. A double-quiescent ECA that allows active transitions does not follow the number conservation property for any initial state if and only if RMTs 1, 4, and 5 or RMTs 2, 3, and 6 of the ECA are passive.

Proof. Let us first consider that RMTs 1, 4, and 5 of a double-quiescent ECA are passive. An active transition in any state of the ECA implies that a cell with state 1 becomes 0, because only RMT 2, 3, or 6 can be active. However, no cell with state 0 can become 1, since RMTs 1, 4, and 5 are passive. Therefore, the next state of the ECA state cannot conserve the number of 1s and 0s if we allow any active transition. A similar result is obtained if RMTs 2, 3, and 6 of a double-quiescent ECA are passive. That is, the ECAs that allow active transitions do not follow the number conservation property if RMTs 1, 4, and 5 or RMTs 2, 3, and 6 of the ECA are passive.

Now consider that no active transition is possible in a double-quiescent ECA that conserves the number of 1s and 0s of any initial state during its evolution. This implies: (*i*) all the RMTs are passive; or (*ii*) RMTs 1, 4, and 5 are passive (because in this case if a transition occurs from 1 to 0, there is no option to make a transition from 0 to 1); or (*iii*) RMTs 2, 3, and 6 are passive (if a transition occurs from 0 to 1, there is no way to get a transition from 1 to 0). That is, the ECAs that allow active transitions do not follow the number conserva-

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tion property if RMTs 1, 4, and 5 or RMTs 2, 3, and 6 of the ECA are passive. Hence the proof. □

There are 15 double-quiescent ECA rules in which RMTs 1, 4, and 5 or RMTs 2, 3, and 6 are passive. These rules are 128, 132, 136, 140, 192, 196, 200, 204, 206, 220, 222, 236, 238, 252, and 254. The rest of the number-conserving ECA rules are noted in Table 4. We can get some active transitions in each of the ECAs in Table 4 for some initial states when the ECAs maintain the number conservation property. For example, ECA 228 in Figure 4 follows the property with initial state 0111 and update pattern ⟨{1, 2}, {2, 3}, {3, 4}, {1, 3, 4}⟩. Here, some transitions are active. However, many of the transitions of Figure 4 are passive. In fact, it is not possible to get many active transitions for ECA 228 with an arbitrary initial state, where the ECA has to maintain the number conservation property. In contrast with this scenario, all transitions (except 0000 and 1111) of ECA 184 in Figure 2 are active. There are some ECAs in Table 4 that are capable of showing the number conservation property with all initial states (except **0** and **1** states) and with active transitions only. For these ECAs, therefore, a change can always be observed in the ECA state during their evolutions. As a final goal of this characterization, we extract such ECAs from Table 4. So we continue our study after asking the following question.

130	134	138	142	144	146	148	150	152	154
156	158	160	162	164	166	168	170	172	174
176	178	180	182	184	186	188	190	194	198
202	208	210	212	214	216	218	224	226	228
230	232	234	240	242	244	246	248	250	

Table 4. List of number-conserving ECAs that allow some active transitions.

Question 4. Can we get a double-quiescent ECA that conserves the number of 1s and 0s of any initial state except **0** and **1** with only active transitions?

As an answer to this question, we present Theorem 4.

Theorem 4. A double-quiescent ECA conserves the number of 1s and 0s of any initial state except **0** and **1** with only active transitions if and only if RMTs 2 and 5, and 1 or 4, and 3 or 6 are active.

Proof. Let us first assume that RMTs 2 and 5, and 1 or 4, and 3 or 6 of a double-quiescent ECA are active. Since it is double-quiescent, RMTs 0 and 7 of the ECA are passive. We now show that for each state (except **0** and **1**), there exists a next state with the same number of 1s and 0s and with active transitions only.

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According to Lemma 1, the RMT sequence of an arbitrary state contains the RMTs of one or more sets from six sets: {0}, {7}, {2, 5}, {1, 2, 4}, {3, 5, 6}, and {1, 3, 4, 6}. Now if the RMT sequence contains RMTs 2 and 5, and two cells acting on RMTs 2 and 5 are selected to update, the next state conserves the number of 1s and 0s. This is because the cell acting on RMT 2 becomes 0 from 1, and the cell acting on RMT 5 becomes 1 from 0. Now if the RMT sequence contains RMT 2 and RMT 1 or 4, and two cells acting on RMT 2 and RMT 1 or 4 are updated, then the next state also conserves the numbers. If RMT 3 or 6, and RMT 5 are present in the RMT sequence, and two cells acting on RMT 3 or 6, and RMT 5 are updated, the number of 1s in the next state remains unaltered. We can get a similar result if RMT 1 or 4, and RMT 3 or 6 are active. Therefore, we can always get a next state with only active transitions and with the same number of 1s and 0s of an arbitrary state except **0** and **1** of a double-quiescent ECA if RMTs 2 and 5, and 1 or 4, and 3 or 6 of the ECA are active.

Now to prove the "only if" part we consider that a double-quiescent ECA conserves the number of 1s and 0s of all but **0** and **1** as an initial state with only active transitions. In this case, to get a next state with the same number of 1s and 0s: (*i*) RMTs 2 and 5 are to be active for state **01** (RMT sequence contains RMTs 2 and 5); (*ii*) RMT 2 and RMT 1 or 4 are to be active for state **001** (RMT sequence contains RMTs 1, 2, and 4); (*iii*) RMT 5 and RMT 3 or 6 are to be active for state **011** (RMT sequence contains RMTs 3, 5, and 6); and (*iv*) RMT 3 or 6 and RMT 1 or 4 are to be active for state **0011** (RMT sequence contains RMTs 1, 3, 4, and 6). Combining all four cases, we get that RMTs 2 and 5, and 1 or 4, and 3 or 6 of the ECA are to be active. Since the RMT sequence of any state except **0** and **1** contains at least one set of RMTs from these four RMT sets: {2, 5}, {1, 2, 4}, {3, 5, 6}, and {1, 3, 4, 6} (Lemma 1), we can get a next state with only active transitions and with the same number of 1s and 0s of that state under the above conditions of the RMTs. Hence the proof. □

We can get nine double-quiescent ECAs from Theorem 4. Those ECA rules are 162, 170, 176, 178, 184, 186, 226, 240, and 242. These nine ECAs are capable of showing the number conservation property with any initial state except **1** and **0** and with active transitions only. Figure 2 is an example having this property. Note that all the transitions except **1** and **0** states' transitions are active.

5. Update of Number-Conserving Elementary Cellular Automata

In Sections 3 and 4, we have identified ECAs that follow the number conservation property under different conditions. Sometimes we have

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allowed any type of transitions, active or passive; sometimes we restrict ourselves to only active transitions. The primary condition of getting the number-conserving ECAs is that they follow the number conservation property for at least one update pattern. Practically, however, we can get many update patterns against an arbitrary initial state for these ECAs. In this section we develop an update scheme for those number-conserving ECAs.

It is very trivial to consider only active or only passive transitions in an evolution of any ECA. Here we consider random update of ECA cells, which can trigger passive as well as active transitions. There are 15 ECAs in Table 3 that can be number conserving only with passive transitions. The ECAs in Table 4 can allow active transitions. However, the following two conditions are to be satisfied if we allow active transitions during getting the next state of an ECA state.

1. There are to be $2k$ active transitions, $k = 0, 1, 2, \ldots$.

2. k active cells become 0 from 1, and the rest of the k active cells become 1 from 0. That is, k active cells act on RMT 0, 1, 4, or 5, and another k active cells act on RMT 2, 3, 6, and 7.

In asynchronous update, any number of cells can be updated in each time step. It is impossible to guarantee that the two conditions are obeyed in every step of evolution. This is formally stated in Theorem 1. However, to observe the behavior of the ECAs identified in previous sections under asynchronous update, we develop a simulation program. We assume that each ECA cell is updated with probability α. The ECAs in the program evolve from arbitrary initial states with specific densities of 1s (*density*). The input to the simulation program is an ECA rule, α, and *density*.

We have experimented with each ECA in Table 4 under different values of α and *density* (we have excluded 15 ECAs from Table 3, because they are number conserving only with passive transitions). The results are interesting. Figure 5 shows a sample result of the experimentation. In this experimentation, the number of cells and α are 10^6 and 0.1, respectively. However, the densities of 1s in the initial states of these ECAs are different. Note that no remarkable fluctuations of the number of 1s of the initial states are found in Figure 5 for ECAs 150, 184, 214, and 240. It is found in experimentation that there are a number of ECAs whose fluctuations from the original number of 1s in the initial state are negligible.

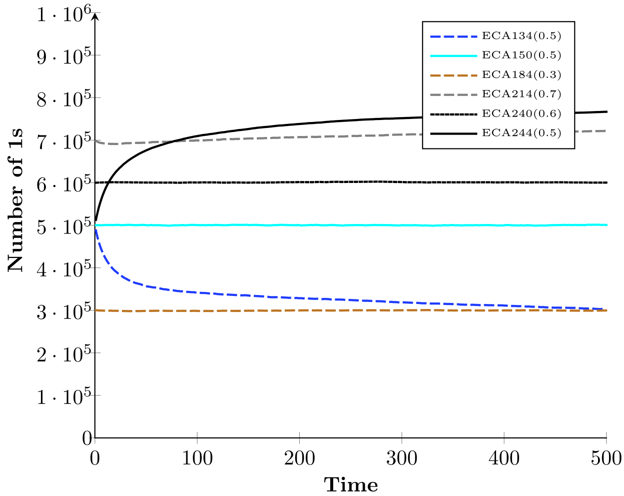

Figure 5. Fluctuation of 1s with the evolutions of six ECAs. For each ECA, the number of cells and α are 10^6 and 0.1, respectively. The density of 1s in the initial state of each ECA is noted within the bracket.

To make the ECAs number conserving, however, we put a little attention on the cells to be updated asynchronously in each step. The following steps are incorporated in the simulation program before generating each of the next states.

1. Prepare a set S_{01} of cells to note the active cells of the ECA that will transit from 0 to 1.
2. Prepare another set S_{10} of cells to note the active cells of the ECA that will transit from 1 to 0.
3. If $|S_{01}| > |S_{10}|$, then remove some cells arbitrarily from S_{01} so that $|S_{01}| = |S_{10}|$. If $|S_{01}| < S_{10}|$, then remove some cells arbitrarily from S_{10} to make $|S_{01}| = |S_{10}|$.

We have simulated the procedure and experimented with different ECAs. Updates of the cells of ECAs are always asynchronous in nature. Figure 6 shows space-time diagrams of ECAs 134, 150, 184, and 240. The size of each ECA in Figure 6 is six, and the ECAs start evolving from the same initial state: 101100. The green and white boxes in the figure correspond to cells with states 1 and 0, respectively. The cells selected for update in each step are marked with a special pattern. Note that the transitions of marked cells are active as well as passive.

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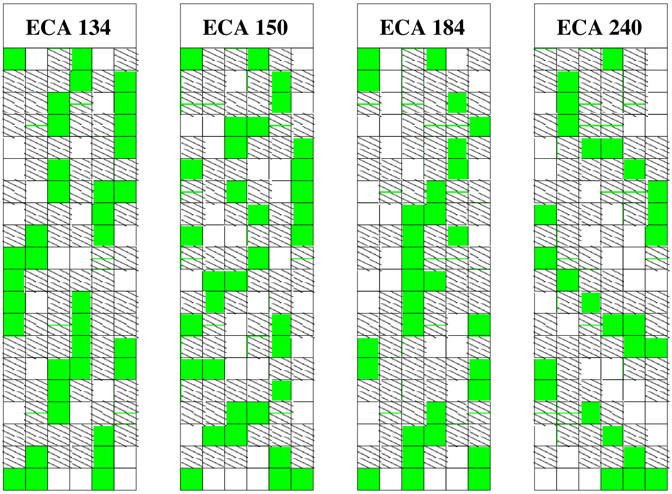

Figure 6. Evolution of ECAs 134, 150, 184, and 240. The ECAs evolve from the same initial state. The shaded cells in each step are updated.

6. Conclusion

This paper has shown that even under asynchronous update some (64) elementary cellular automata (ECAs) are capable of showing the number conservation property. Out of 64 such ECAs, some ECAs can be number conserving with active or passive transitions, while some can be number conserving with active transitions only. Finally, an (asynchronous) update scheme for such number-conserving ECAs is reported.

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