The Mandelbrot Set with Partial Memory

Ramón Alonso-Sanz
Universidad Politécnica de Madrid, ETSIA (Estadística, GSC)
C. Universitaria. 28040, Madrid, Spain
ramon.alonso@upm.es

An exploratory study is made on the effect of partial memory of past iterations on the dynamics of the quadratic map defining the Mandelbrot set.

1. Background: Memory in Discrete Maps

Memory can be embedded in conventional finite difference equations, (often referred to as *discrete dynamical systems*): $z_{T+1} = f(z_T)$ by means of $z_{T+1} = f(m_T)$, with m_T being an average value of past states: $m_T = m(z_1, \ldots, z_T)$ [1].

We will consider first the effect of average memory with geometric decay:

$$m_T = \frac{z_T + \sum_{t=1}^{T-1} \alpha^{T-t} z_t}{1 + \sum_{t=1}^{T-1} \alpha^{T-t}} \equiv \frac{\omega(T)}{\Omega(T)} = \frac{z_T + \alpha\omega(T-1)}{\Omega(T)}. \tag{1}$$

The choice of the memory factor α tunes the intensity of memory: from the ahistoric model with $\alpha = 0$ to equally weighted records or *full* memory with $\alpha = 1$. Note in equation (1) that only an additional number (ω) needs to be stored to implement this kind of memory, to be referred to as α-memory.

Limited trailing memory would remember only the last τ time steps. Thus $m_T = m(z_\top, \ldots, z_T)$, with $\top = \max(1, T - \tau + 1)$. In the short-term $\tau = 2$ case (referred to as $\tau = 2$ α-memory), it is

$$m_T = \frac{z_T + \alpha\, z_{T-1}}{1 + \alpha}, \quad 0 \le \alpha \le 1. \tag{2}$$

We will consider also memory of only the last two states implemented in the general form

$$m_T = (1 - \epsilon)\, z_T + \epsilon\, z_{T-1}, \quad 0 \le \epsilon \le 1. \tag{3}$$

If $\epsilon \le 1/2$, the models of equations (2) and (3) are interchangeable according to $\epsilon = \alpha/(1 + \alpha)$. But levels of the ϵ parameter over 0.5 (meaning a higher contribution of the past than of the present state)

https://doi.org/10.25088/ComplexSystems.23.3.227

generate dynamics uncovered with $\tau = 2$ α-memory. This kind of memory implementation will be referred to as ϵ-memory.

2. The Mandelbrot Set with Partial Memory

The complex quadratic map (z, $c \in$ C):

$$z_{T+1} = z_T^2 + c \tag{4}$$

is equivalent to the two-dimensional real map ($z = x + y\,i$, $c = a + b\,i$):

$$x_{T+1} = x_T^2 - y_T^2 + a \tag{5a}$$

$$y_{T+1} = 2\,x_T\,y_T + b. \tag{5b}$$

The effect of endowing memory in both equations (5) (i.e., $z_{T+1} = m_T^2 + c$) has been explored in a previous study [2]. Here the equations (5) are to be partially endowed with memory, either in the x coordinate ($\overline{x}$):

$$x_{T+1} = \overline{x}_T^2 - y_T^2 + a$$
$$y_{T+1} = 2\,\overline{x}_T\,y_T + b,$$

or in the y coordinate ($\overline{y}$):

$$x_{T+1} = x_T^2 - \overline{y}_T^2 + a$$
$$y_{T+1} = 2\,x_T\,\overline{y}_T + b.$$

The Mandelbrot set is the set of all c for which the iteration of equation (4), starting from $z = 0$, does not diverge to infinity [3–6]. The Mandelbrot set **M** is a compact set, contained in the closed disk of radius 2 around the origin. In fact, a point c belongs to the Mandelbrot set if and only if $|z_T| < 2$ for all T.

Figures 1, 3, 5, 7, 8, and 10 show the effect of partial memory on the Mandelbrot set, based on the escape time, with simulations run up to $T = 100$. Thus, red color indicates the Mandelbrot set, whereas blank/blue colors indicate an odd/even number of iterations up to divergence (LSM method), precisely up to $|z|$ reaching the breakout value 8.0. The boundary of the contour curves in these figures can be interpreted as equipotential curves. The region of the complex plane shown in the figures is $[-4.0, 2.6]\times[-3.3, 3.3]$. The pictures shown in the figures remain with memory symmetrical around the real axis, as in the conventional scenario.

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2.1 α-memory

Figures 1 and 3 show the effect of partial α-memory on the Mandelbrot set.

As a result induced by the inertial effect that α-memory exerts, $\mathbf{M}$ (and its intricate neighborhood) grows as α increases in both figures. The area of the Mandelbrot set has been estimated close to 1.5066 [7]. In simulations up to $T = 1000$ in a lattice with high resolution (1001×1001), the area estimated by pixel counting grows progressively in Figure 1 as: 1.5134, 1.8992, 2.3768, 2.9276, 3.5308, 4.0877, 4.3484, 4.5878, 5.0148, 5.3262, 5.4696, 5.6398. In the simulations just described, the estimations by pixel counting of the real component of the center of gravity evolve as: -0.2902, -0.3861, -0.4839, -0.6184, -0.7007, -0.7705, -0.8036, -0.8352, -0.8894, -0.9313, -0.9701. In Figure 3, the area grows progressively as: 1.5134, 1.8992, 2.3768, 2.9276, 3.5308, 4.0877, 4.3484, 4.5878, 5.0148, 5.3262, 5.4696, 5.6398; and the real component of the center of gravity evolves as: -0.2908, -0.3635, -0.4318, -0.5006, -0.5555, -0.5887, -0.6075, -0.6240, -0.6436, -0.6410, -0.6214, -0.6145.

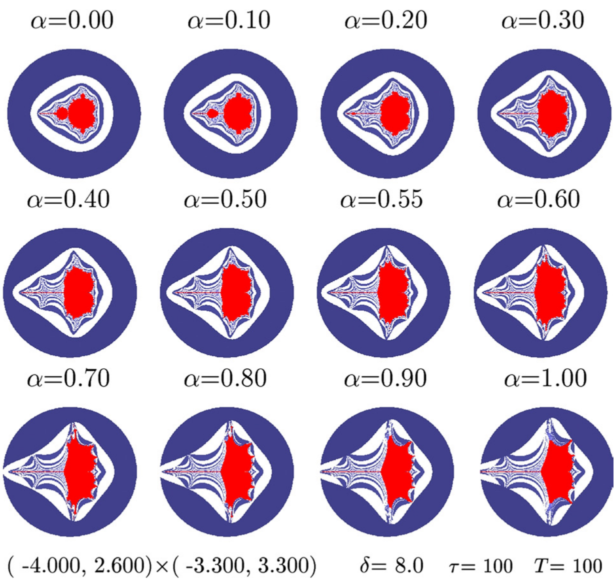

Figure 1. The Mandelbrot set with α-memory in the x coordinate.

Figure 2 shows the orbits of the real line of $\mathbb{M}$ juxtaposed to its boundary when α-memory in the x coordinate is implemented. The last 50 values of 1000 iteration runs have been plotted.

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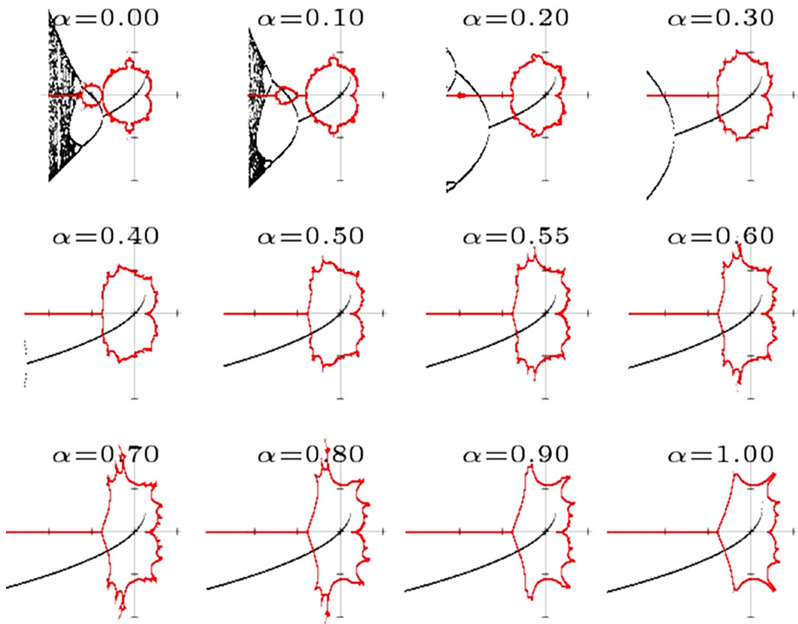

Figure 2. Orbits of the purely real numbers in $\mathbb{M}$ with α-memory in the x coordinate.

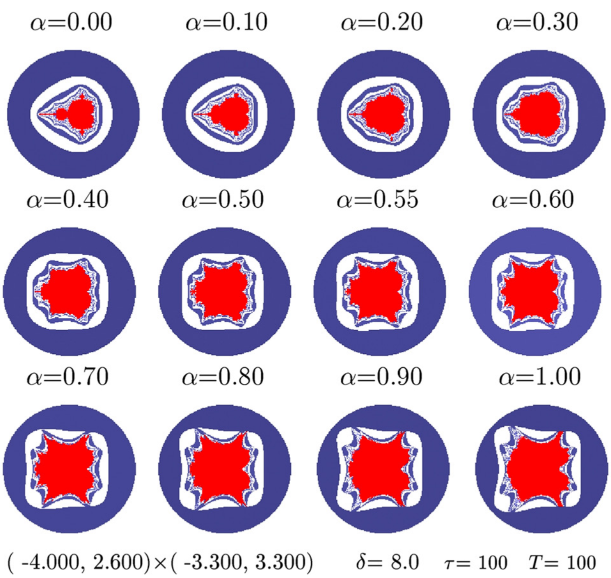

Figure 3. The Mandelbrot set with α-memory in the y coordinate.

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Unlimited trailing α-memory in x expands the fixed-point region in Figure 2, so that with $\alpha \geq 0.5$, it occupies the whole *spine* of $\mathbb{M}$. As a result, the rich behavior regarding the long-term dynamics emerging with low memory in the left part of $\mathbb{M}$ tends to vanish as memory increases. The *arc* that marks the convergence to a fixed point in the orbits of the real line is extended (or *left*-enlarged) with α-memory in the real component in Figure 2, though not very much altered in the form given in the ahistoric formulation, that is, $z = z^2 + c$. Contrary to what happens in the purely real negative region, it is remarkable that the length of the interval of positive purely real numbers in the Mandelbrot set is not significantly augmented by α-memory. Thus, the *cusp* $c = 0.25$ (the highest real c allowed to be maintained in $\mathbb{R}$) increases to a non-negligible extent only with high memory charge; the maximum being $c = 0.334$ with $\alpha = 1.0$.

Figure 4 shows the orbits of the imaginary line of $\mathbb{M}$ with α-memory in the y coordinate juxtaposed to the boundary of $\mathbb{M}$. These figures show also the trajectory of $c = i$. As a rule, the points in $\mathbb{M}$ with *real*$(c) = 0$ converge to a fixed point, regardless of the type and degree of memory implemented. The real part of the (complex) fixed point is shown in green, and its imaginary part in blue, in Figure 4. Memory seems to allow a further enlargement of the ahistoric graph-

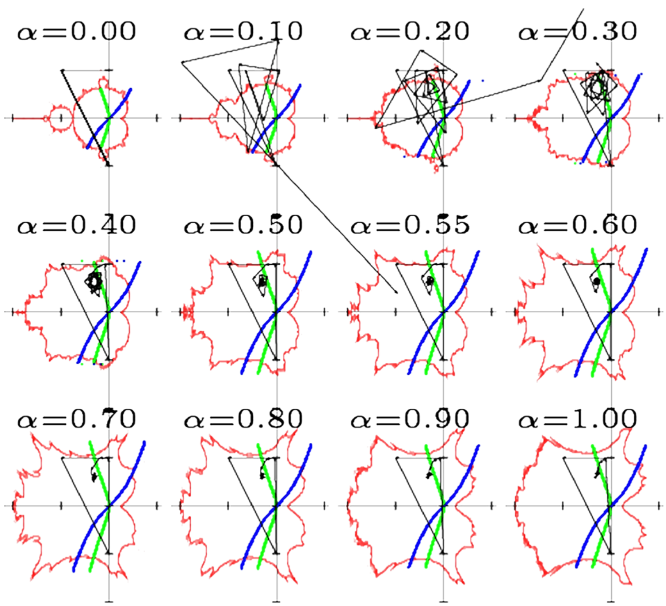

Figure 4. Orbits of the purely imaginary numbers in $\mathbb{M}$ and trajectory of $c = i$ with α-memory in the y coordinate.

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ics of both components of the fixed point, with no significant change in their trends. The (Misiurewicz) point $c = i$ generates in the conventional ahistoric model a period-two $(-1 + i, -i)$ cycle from $T = 2$. Consequently, it belongs to the $\mathbb{M}$ set. But it appears as the tip of a filament, which the simulations presented here are not fine-grained enough to appreciate. The trajectory of $c = i$ with α-memory in the imaginary part is shown in Figure 4 up to $T = 25$, or up to early divergence, which comprises the cases of $\alpha = 0.1$, with divergence at $T = 12$; and $\alpha = 0.2$, with divergence at $T = 24$. The simulation $\alpha = 0.3$ diverges at $T = 35$. These are scenarios in which the filament of $\mathbb{M}$ that reaches i in the ahistoric formulation does not exist, and $\mathbb{M}$ itself has not expanded enough to cover the i point.

2.2 $\tau = 2$ α-memory

Figures 5 and 7 show the effect of partial $\tau = 2$ α-memory on the Mandelbrot set. Accordingly with the limitation of the memory charge to only two time steps, a smaller increase in the area of $\mathbb{M}$ is achieved in Figures 5 through 7 compared to that already reported with unlimited trailing memory in Figures 1 through 3. Thus, in Figure 5 with full memory, the area only grows to 1.8929, and the center of gravity is only displaced to -0.3153. The equivalent parameters in Figure 7 with full memory are 2.7068 for the area and -0.3388 for the center of gravity. Incidentally, the shape of $\mathbb{M}$ in Figure 5 tends to somehow resemble that of the leaf of the *Gingko biloba* tree. A stylized *Gingko biloba* leaf is achieved in [8] by means of:

$$x_{T+1} = x_T^2 - y_T^2 + a \tag{6a}$$

$$y_{T+1} = 2\, x_T^2\, y_T + b, \tag{6b}$$

thus replacing the second-order term in equation (5b) by the third-order term $2\, x_T^2\, y_T$ in equation (6b).

The effect of $\tau = 2$ α-memory on the purely real numbers in Figure 6 resembles that of unlimited α-memory in Figure 2 up to $\alpha = 0.6$, but, unexpectedly, with higher values of the memory factor, a period-doubling regime and even a strongly (albeit narrow) chaotic region appear.

The convergence to a fixed point is also the rule with respect to the trajectories of purely real numbers in Figure 6, as in Figure 2. The convergence to a fixed point is also reached in the dynamics with memory of the numbers in $\mathbb{M}$ with $real(c) = imag(c)$ and with $real(c) = -imag(c)$. Thus, it is then arguable that the points in the *leaf-like* region remain, with memory converging to a fixed point. The

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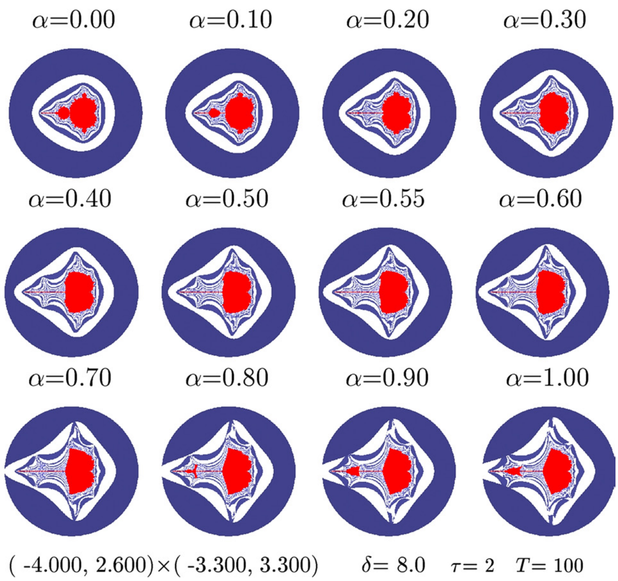

Figure 5. The Mandelbrot set with $\tau = 2$ α-memory in the x coordinate.

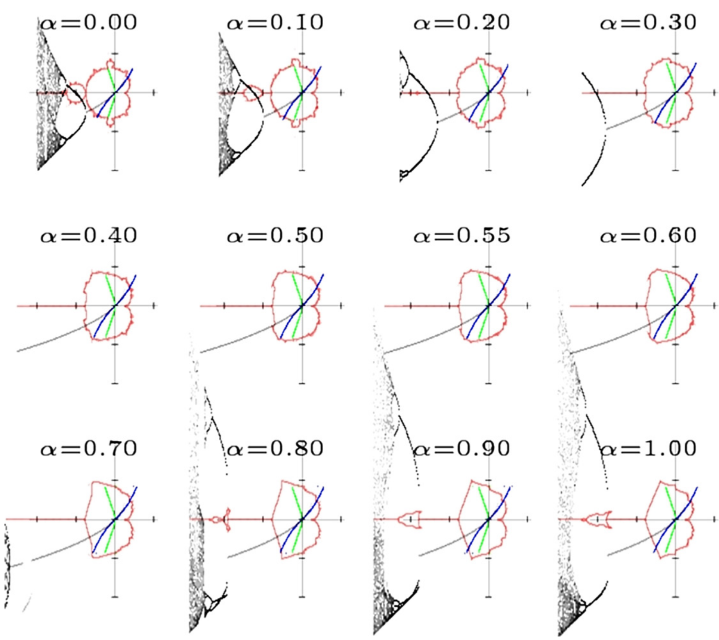

Figure 6. Orbits of the purely real and imaginary numbers in $\mathbb{M}$ with $\tau = 2$ α-memory in the x coordinate.

trajectory of $c = i$ with $\tau = 2$ α-memory soon diverges in every scenario of Figure 6 (at $T = 7$, with $\alpha = 0.1, 0.2$; at $T = 8$, with $\alpha = 0.3$ to 0.7; at $T = 12$, with $\alpha = 0.8$ to 1.0), whereas with memory in the imaginary part, Figure 4 shows divergence only for low values of the memory charge (at $T = 12$ with $\alpha = 0.1$, at $T = 20$ with $\alpha = 0.2$, at $T = 18$ with $\alpha = 0.3$, at $T = 32$ with $\alpha = 0.4$, and at $T = 20$ with $\alpha = 0.50$).

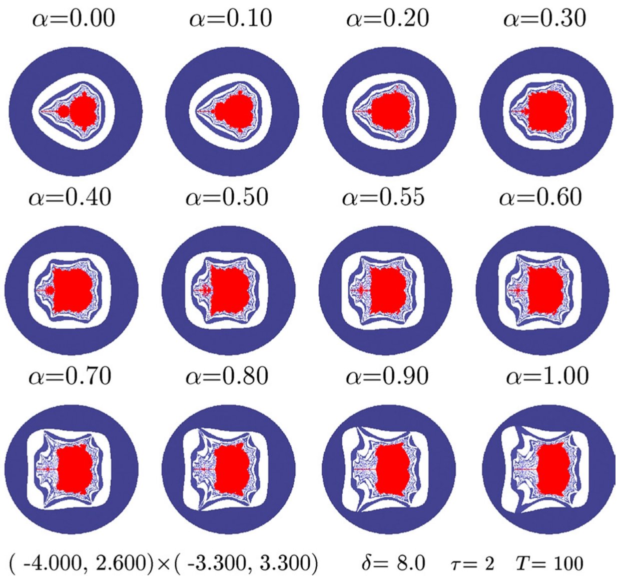

Figure 7. The Mandelbrot set with $\tau = 2$ α-memory in the y coordinate.

2.3 ϵ-memory

Figures 8 and 10 show the effect of partial ϵ-memory on the Mandelbrot set. Again, partial memory in the x coordinate generates in Figure 8, as in Figure 5, $\mathbb{M}$ sets resembling the leaf of the *Ginkgo biloba* tree.

The ϵ-type memory given in equation (3) tends to ignore the contribution of the last time step for very high values of ϵ. In the extreme case, if $\epsilon = 1$ it is $m_T = z_{T-1}$, so that every state of the ahistoric evolution is generated twice. In this case, the ahistoric $\mathbb{M}$ remains unaltered. This trend is reflected in a lower alteration of the area and position of the center of gravity of $\mathbb{M}$ in simulations with high ϵ compared to the corresponding values in the ahistoric model, that is, 1.5066 and 0.2868. Thus, the area of the Mandelbrot set with ϵ-memory in the real part decreases as ϵ grows in Figure 9 as 1.7452, 1.5964, 1.5281, 1.4677, 1.4440, 1.4322; and the center of gravity evolves as -0.2516,

-0.1765, -0.1664, -0.1695, -0.2021, -0.2411. The area of $\mathbb{M}$ with ϵ-memory in the imaginary part in Figure 10 decreases as 2.5516, 2.3524, 2.1102, 1.8797, 1.7038, 1.4636; and the position of its center of gravity moves as -0.2442, -0.2898, -0.1995, -0.1786, -0.1992, -0.1839.

The kind of ϵ-memory with $\epsilon \geq 0.5$ regarding the purely real numbers appears in Figure 9 as disruptive as envisaged with the higher memory charges in Figure 6. This particularly altering effect of ϵ-memory when $\epsilon > 0.5$ was already reported in the logistic map in [9, 10].

The trajectory of $c = i$ (and that of $c = -i$ due to symmetry) with ϵ-memory in the real part diverges in every scenario of Figure 9 (at $T = 7$, with $\alpha = 0.1, 0.2$; at $T = 8$, with $\alpha = 0.3$ to 0.7; at $T = 12$, with $\alpha = 0.8$ to 1.0), whereas with ϵ-memory in the imaginary part in Figure 10, convergence is achieved with $\epsilon = 0.55$ and $\epsilon = 0.60$. (Divergence occurs at $T = 41$ with $\epsilon = 0.65$, at $T = 32$ with $\epsilon = 0.70$, at $T = 13$ with $\epsilon = 0.80$, and at $T = 9$ with $\epsilon = 0.90$.)

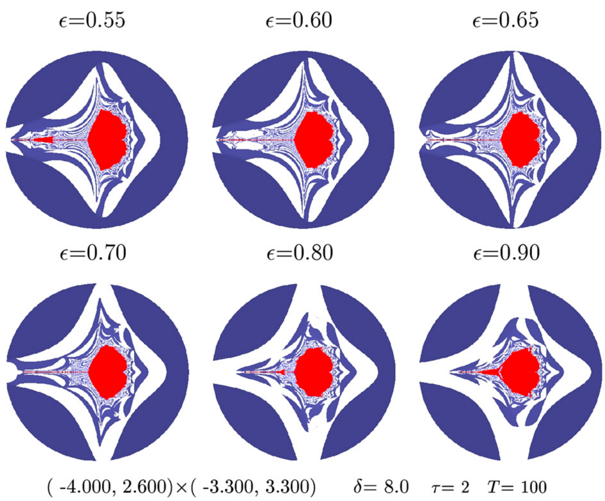

Figure 8. The Mandelbrot set with ϵ-memory in the x coordinate.

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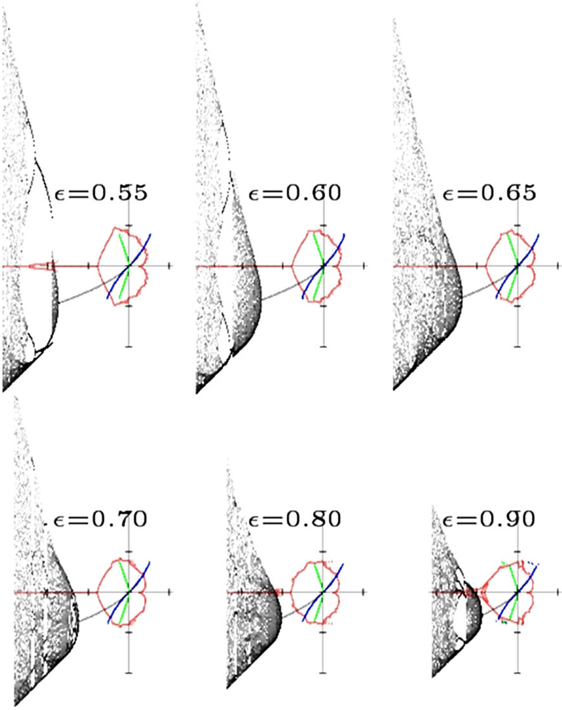

Figure 9. Orbits of the purely real and imaginary numbers in $\mathbb{M}$ with ϵ-memory in the x coordinate.

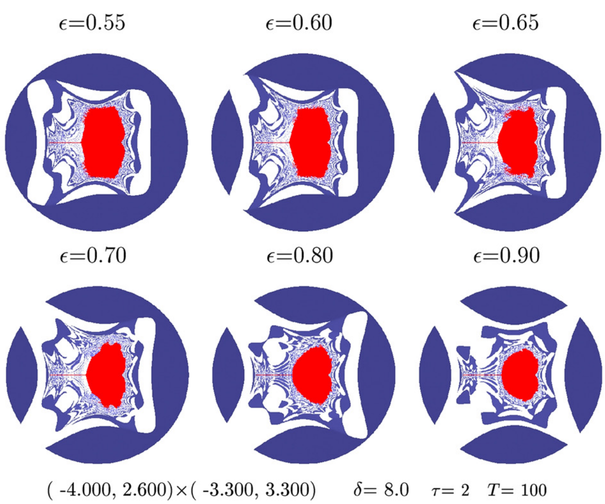

Figure 10. The Mandelbrot set with ϵ-memory in the y coordinate.

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3. Future Work

Further study of the effect of memory on complex quadratic maps is due, not only regarding a more detailed scrutiny of geometry [11] but also with respect to its general mathematical foundation. Fractional calculus has been applied in the analysis of discrete maps with memory [12]. Maybe it can help in this respect.

Higher-degree complex maps as well as quaternion iterative maps [13] endowed with memory deserve particular study. The 16 possible partial memory schemes in the quaternion context will likely induce interesting behaviors.

Popularizing the study of the effect of memory embedded in discrete dynamical systems by providing interactive tools (e.g., Java applets) on the internet is a challenging task. Such applets are almost required by current researchers.

Acknowledgment

This work was supported by the Spanish Project MTM2012-39101-C02-01.

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