

Cellular Automata Complexity Threshold and Classification: A Geometric Perspective

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This paper presents the results of mathematical experiments on the so-called “orientation vector.” It looks at complexity in terms of three perspectives: Wolfram, Langton, and Chua. Critically, we consider Chua’s geometrical complexity index and a complexity-based classification of elementary cellular automata. Ideas in terms of solutions for ordinary differential equations and complexity measurements are proposed to the research community for discussion.

1. Introduction

During the Wolfram Science Summer School 2012, I worked on the project “Investigation of the Relation between Nonlinear Dynamical Systems and Elementary Cellular Automata.” The project dealt with Chua’s [1] geometrical representation of elementary cellular automata (ECAs) and led to these thoughts about the complexity threshold and measurement comparisons.

From an application point of view, Wolfram [2] shows a huge diversity for mathematical modeling of snowflakes, vegetation, seashells, and other natural forms in terms of ECAs. These mathematical models are explicitly considered as discrete dynamical systems and information-processing systems. Dynamical systems analysis in phase space (state space) proposes the following three behaviors [3].

- *Homogeneous dynamics (fixed point).* These tend to stabilize as a single point in the phase space that is fixed over time. The initial conditions significantly determine the long-term behavior; similar initial conditions give the same phase plane trajectories.

- *Periodic (limit cycle)*. These tend to stabilize as a closed path in the phase space. The initial conditions significantly determine the long-term behavior; similar initial conditions give the same phase plane trajectories.
- *Aperiodic (chaotic, strange attractors)*. These never seem to stabilize; however, the subspace in which the state path trajectory moves is restricted to a bounded manifold. This manifold often possesses complex, detailed structure. In general, similar initial states do not produce similar state path trajectories, making them insensitive to initial conditions [4].

One of the main questions about ECAs is if they can be used as mathematical models for ordinary differential equations (ODEs). There is more than one answer to that question. We compare the proposed criterion inspired by Chua's cubical representations for ECAs [1] and the other two criteria.

2. Wolfram's Classification

Wolfram's classification scheme [2] is phenomenological; that is, the ECA space is separated into four classes by visually inspecting space-time diagrams to evaluate their qualitative behavior and complexity (Figure 1):

- *Class I*. Steady state, representing homogeneous dynamics.
- *Class II*. Repetitive cycles, representing periodic dynamics.
- *Class III*. Random-like behavior, representing chaotic patterns.
- *Class IV*. Complex behavior, representing complex patterns with local structures that move through space and time.

Wolfram conjectures that class IV cellular automata (CAs) are computationally universal.

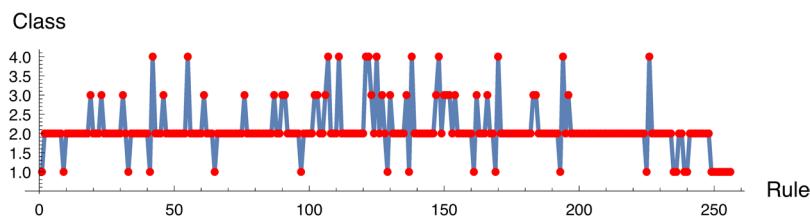


Figure 1. Wolfram's classification for ECAs.

3. Langton's λ Parameter

In 1990, Langton [5] discussed the question, under what conditions will physical systems support the basic operations of information transmission, storage, and modification constituting the capacity to support computation? His answer was, systems near a continuous (second-order) phase transition. These systems that are near criticality, between an ordered ("solid") and chaotic ("liquid") state, are especially capable of computations. In other words, ECA rules represent (some part of) the physical world. The initial configuration itself constitutes the computer, the program, and the data. Langton translates the original question into, when is it possible to adopt this view to understand ECA dynamics?

Lambda values range between 0 and 1 and are calculated as the fraction of ECA rules that lead to a new state with a live cell, expressed as a decimal. This means that the more ECA rules that lead to life, the larger the λ value. An ECA space starting from $\lambda = 0$ is fixed (ordered, cold, and predictable). By increasing λ , the system changes from ordered to what is called periodic (predictable but recurring), then it changes to complex (where the best life simulations come about), and finally, above this, the systems are chaotic (messy, unpredictable, and hot).

Wolfram's classification and Langton's λ parameter (Figure 2) correspond as follows. CAs with very low λ values are very likely to be class I. CAs with very high λ values are very likely to be class III. As λ increases, we expect a transition from class I to class II to class IV to class III. But sharp transitions are not always obtained, and sometimes we might go backward. Furthermore, class IV CAs are rare.

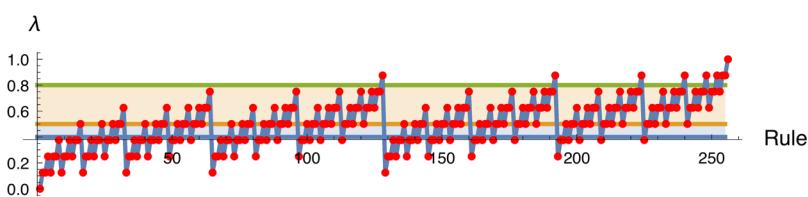


Figure 2. Langton's λ parameters for ECA.

4. Chua's Complexity Index κ

Chua [1] introduced an idea for measuring complexity he called the complexity index κ . This measurement is based on the geometrical analysis of separability, that is, the minimum number of separating

planes. We have six real variables $\{z_2, z_1, z_0; b_1, b_2, b_3\}$ and two integers ± 1 . By making substitutions, configurations are generated for those variables assigned to the ECA space (Figure 3). We make some statistical studies on those outputs, finding more acceptable configurations for $\{z_2, z_1, z_0; b_1, b_2, b_3\}$ and two integers ± 1 , that is, the minimum number of κ . This leads to the complexity index of a local rule, which characterizes the geometrical structure of the corresponding Boolean cube, namely, the minimum number of parallel planes necessary to separate the colored vertices. Hence, all linearly separable rules have a complexity index of $\kappa = 1$.

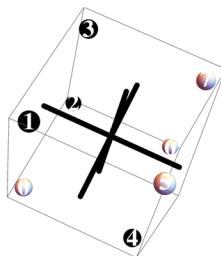


Figure 3. Rule 30 represented geometrically.

4.1 Vertex Projection on an Orientation Vector and Number of Transitions

By making a projection for each vertex on the orientation vector as defined by Chua, we get a cube as shown in Figure 4.

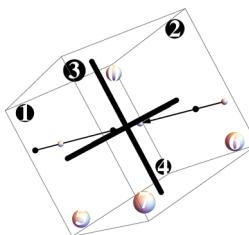


Figure 4. Rule 30 projection over the orientation vector $\{1, 3, -3\}$.

This can lead to a representation in two dimensions with the orientation vector as a line and vertices as white or black dots. The idea of linear separability is one of the most important concepts in artificial neural networks (ANNs). Particularly in single-layer ANNs (perceptrons), that can also be used as a solution for the classification prob-

lem in machine learning. Chua [1] introduces a way to express an ODE as an ECA and then make use of that in ANN applications.

The notion of linear separability assumes that there is a pattern set X . This set is divided into subsets X_1, X_2, \dots, X_R , respectively. If a linear machine can classify the patterns from X_i as belonging to class i , for $i = 1, 2, \dots, R$, then the pattern sets are linearly separable. Using this property of the linear discriminant functions, the notion of linear separability can be stated more formally. If R linear functions of x exist such that:

$$\begin{aligned} g_i(x) &> g_j(x) \text{ for all } x \in X_i, i = 1, 2, \dots, R; \\ j &= 1, 2, \dots, R, i \neq j, \end{aligned}$$

then the pattern sets X_i are linearly separable.

Geometrically, linear separability can be extended in n dimensions to be the hyperplane separating the two sets of points. By counting the alternations of vertex colors on the vector, that is, using the two-dimensional line to check for color alternations, we can indicate whether a CA is linearly separable or not.

4.2 Definition of κ

Chua [1] defines the complexity index κ as the minimum number of parallel hyperplanes that can separate the bipolar vertices. This index is an integer [0, 1, 2, 3].

The ECA space can be separated into these four classes using the complexity index κ (Figure 5):

- $\kappa = 0$, only the two rules 0 and 255, because there is no need for a separation hyperplane.
- $\kappa = 1$, the linearly separable rules that need only one separation hyperplane.
- $\kappa = 2$, the linearly separable rules that need two separation hyperplanes.
- $\kappa = 3$, the linearly separable rules that need three separation hyperplanes.

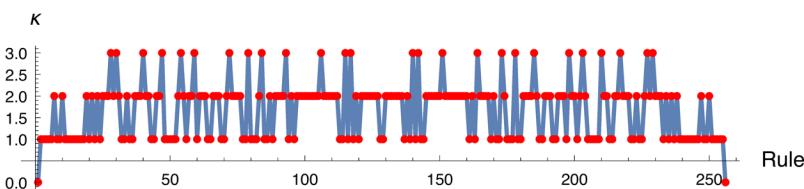


Figure 5. Using Chua's complexity index to separate ECAs into four classes.

4.3 Parameters that Influence κ

We found that κ is influenced by some parameters in the equivalent ODE of an ECA. Focusing only on the influence of the orientation vector, we found some odd values for κ , such as $\kappa = 4$. In other words, varying the orientation vector led to variations in κ . We also found that κ is sensitive to that variation.

Part of the study was stepping from $[-4, 4]$ with various step values to generate ODE parameters and then check the corresponding ECA. Additionally, the number of transitions is calculated, to indicate the linear separability; its minimum number represents the value of κ . Figure 6 shows the rules versus the number of transitions. Those transitions need to be refined; that is, we need to calculate the minimum numbers that converge to be candidate κ values.

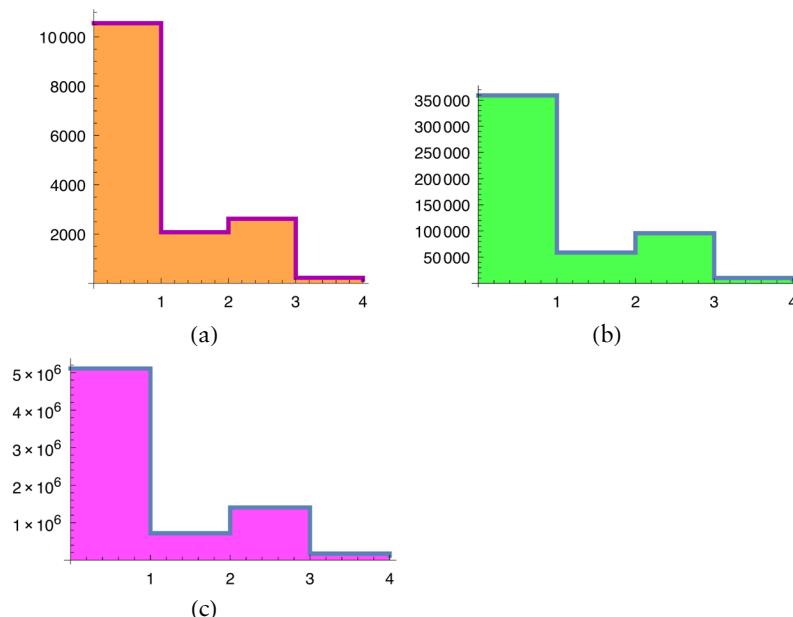


Figure 6. Configurations versus number of transitions κ for (a) second step = 2, (b) second step = 1, and (c) third step = 0.6.

We started by eliminating the $\kappa = 4$ configuration values and then taking the minimum values of κ in each step. We were surprised to find typical results (Figure 7). That means we got a true κ and a large number of values for the orientation vector.

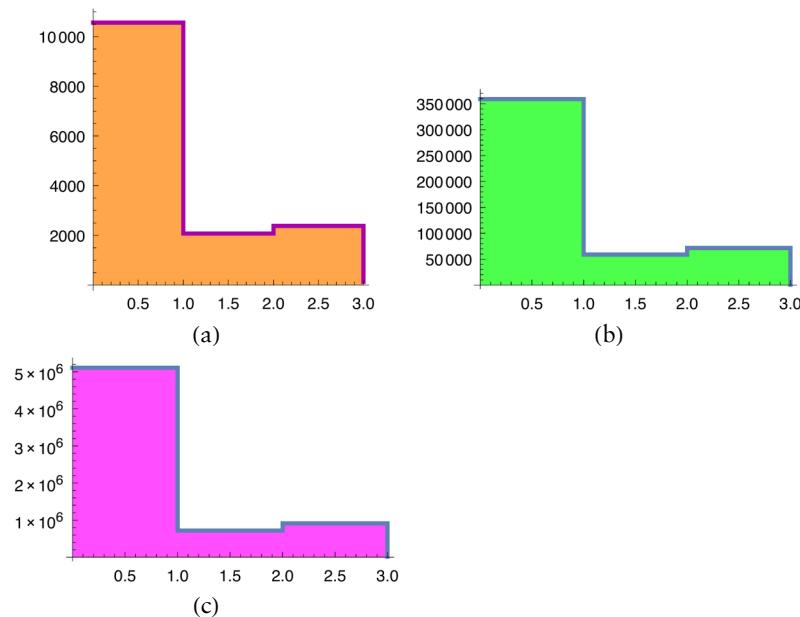
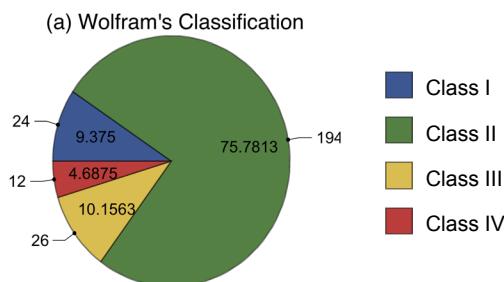


Figure 7. Configurations versus number of transitions κ for (a) second step = 2, (b) second step = 1, and (c) third step = 0.6.

5. Observations

5.1 Major Class

Using the number of partitions from each criterion, we can introduce a comparison. It is clear that the largest class from Wolfram is class II, from Chua $\kappa = 2$, and from Langton $\lambda = 0.5$ or $0.375 \geq \lambda \geq 0.625$ as in Figure 8.



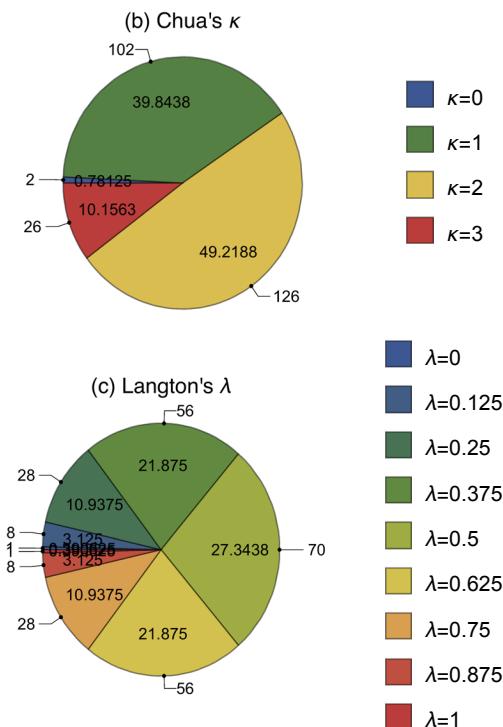


Figure 8. (a) Wolfram's classification. (b) Chua complexity index. (c) Langton's lambda.

5.2 Computational Universality

The 12 rules in Wolfram's class IV are computationally universal. For all computationally universal rules, Chua's index $\kappa = 2$ and $\lambda = 0.5$ is most likely with 50%, as shown in Figure 9.

Rule	κ	λ	Rule	κ	λ
41	2	0.375	124	2	0.625
54	2	0.5	137	2	0.375
106	2	0.5	147	2	0.5
110	2	0.625	169	2	0.5
120	2	0.5	193	2	0.375
121	2	0.625	225	2	0.5

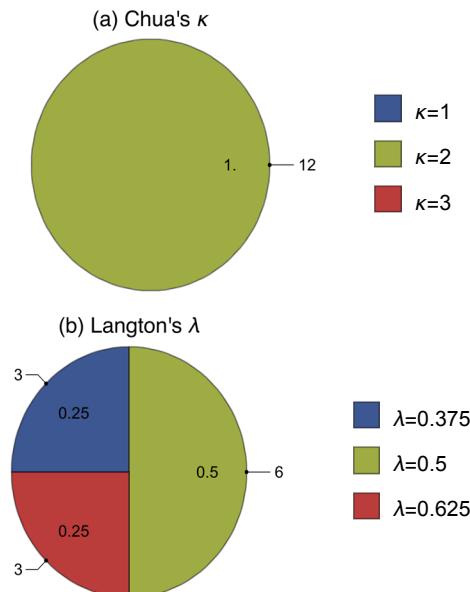


Figure 9. (a) Chua's complexity index κ for universal computation. (b) Langton's λ for computationally universal ECAs.

5.3 Linear Separability

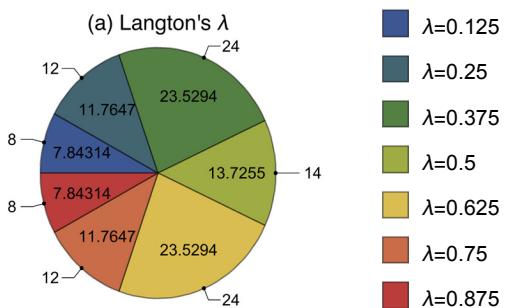
Linear separability is a vital concept in terms of Chua's definition of complexity because it is this set of rules that needs only one line/plane to separate the bicolor projected orientation vector. There are 102 linearly separable rules with a Chua complexity index of $\kappa = 1$. Linear separability is distributed over the λ space with a symmetrical distribution as shown in Figure 10. Wolfram's class II gives the largest set of linearly separable rules with about 82%; it is clear that both classes III and IV are missing from Figure 10.

Rule	Class	λ
1	1	0.125
2	2	0.125
3	2	0.25
4	2	0.125
5	2	0.25
7	2	0.375
8	2	0.125
10	2	0.25
11	2	0.375

Rule	Class	λ
12	2	0.25
13	2	0.375
14	2	0.375
15	2	0.5
16	2	0.125
17	2	0.25
19	3	0.375
21	2	0.375
23	3	0.5

Rule	Class	λ
31	3	0.625
32	2	0.125
34	2	0.25
35	2	0.375
42	4	0.375
43	2	0.5
47	2	0.625
48	2	0.25
49	2	0.375

Rule	Class	λ
50	2	0.375
51	2	0.5
55	4	0.625
59	2	0.625
63	2	0.75
64	2	0.125
68	2	0.25
69	2	0.375
76	3	0.375
77	2	0.5
79	2	0.625
80	2	0.25
81	2	0.375
84	2	0.375
85	2	0.5
87	3	0.625
93	2	0.625
95	2	0.75
112	2	0.375
113	2	0.5
115	2	0.625
117	2	0.625
119	2	0.75
127	3	0.875
128	2	0.125
136	3	0.25
138	4	0.375
140	2	0.375
142	2	0.5
143	2	0.625
160	2	0.25
162	3	0.375
168	2	0.375
170	4	0.5
171	2	0.625
174	2	0.625
175	2	0.75
176	2	0.375
178	2	0.5
179	2	0.625
186	2	0.625
187	2	0.75
191	2	0.875
192	2	0.25
196	3	0.375
200	2	0.375
204	2	0.5
205	2	0.625
206	2	0.625
207	2	0.75
208	2	0.375
212	2	0.5
213	2	0.625
220	2	0.625
221	2	0.75
223	2	0.875
224	2	0.375
232	2	0.5
234	2	0.625
236	1	0.625
238	2	0.75
239	1	0.875
240	1	0.5
241	2	0.625
242	2	0.625
243	2	0.75
244	2	0.625
245	2	0.75
247	2	0.875
248	2	0.625
250	1	0.75
251	1	0.875
252	1	0.75
253	1	0.875
254	1	0.875



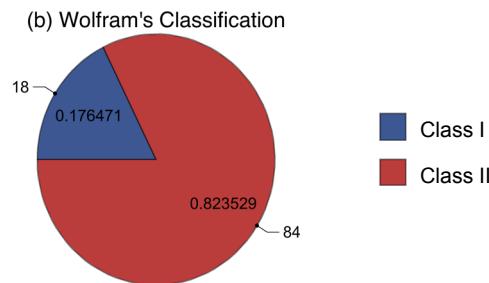


Figure 10. (a) Langton’s λ for linearly separable ECAs. (b) Wolfram’s classification for linearly separable ECAs.

■ 5.4 Complexity Threshold

The complexity threshold is related to the chosen system and will vary over the three systems in this paper. For our purposes, the ECA is the system; that is, Wolfram’s point of view is computation, Langton’s point of view is information, and Chua’s point of view is ODE or dynamical systems. The complexity threshold from Wolfram is class III, from Langton it is the area of λ , and from Chua it is $\kappa = 2$.

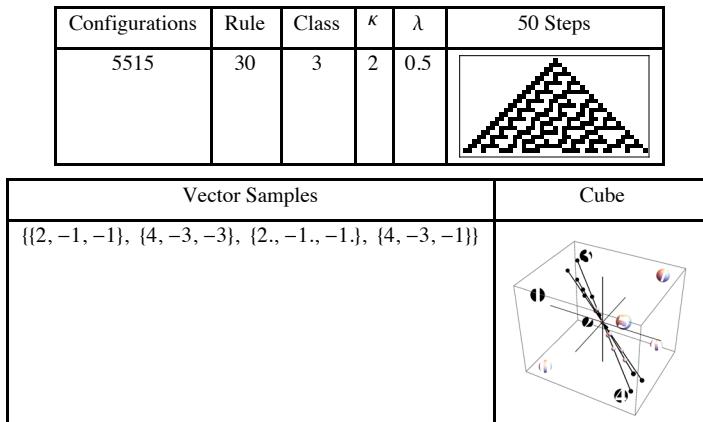
■ 5.5 Rules 0 and 255

Rules 0 and 255 were the most commonly found configurations in the experiment of stepping the orientation vector. Notice that the number of configurations for these rules is not affected before/after the filtration because no separating planes are needed. Also observe the large percentage of rule 255 configurations (more than 63%). When step = 2, we lose some rules after the filtration process, leaving only 180 rules with valid configurations.

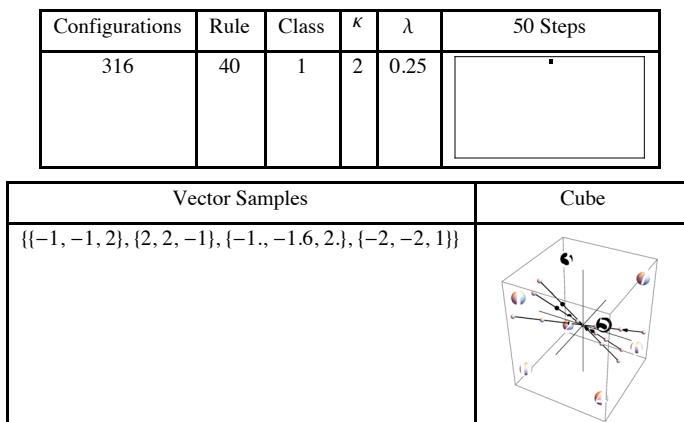
Step	Rule	Configurations	%
2	0	593	3.7952
1	0	12 729	2.39519
0.6	0	120 874	1.60533
2	255	9968	63.7952
1	255	346 296	65.1617
0.6	255	4 987 262	66.236

■ 5.6 Interesting Case Studies

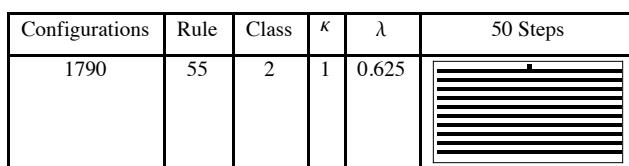
Rule 30. All Wolfram class III rules have $\kappa = 2$ except two cases: rule 105 and rule 150, with $\kappa = 3$.

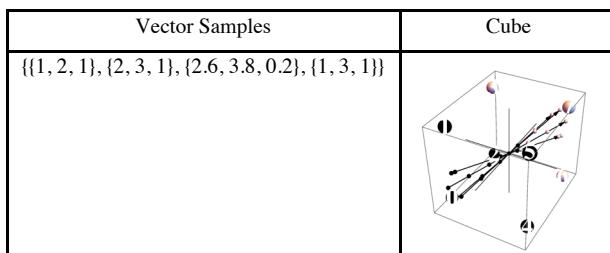


Rule 40. All of Wolfram's class I rules have $\kappa = 1$ except for rules 1 and 255, with $\kappa = 0$, and four cases (40, 96, 235, and 249) that have $\kappa = 2$. This is one of those cases.



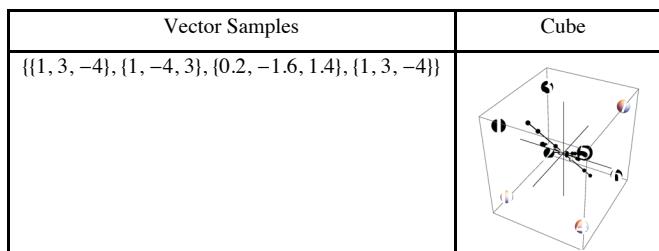
Rule 55. All linearly separable rules are in Wolfram classes I and II, indicating a correlation between linear separability and computational complexity. This is one case from Wolfram's class II with a relatively high λ .





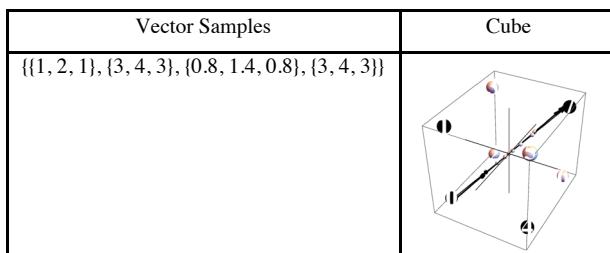
Rule 110. Wolfram's class IV rules always have $\kappa = 2$. Rule 110 is a clear example for this case.

Configurations	Rule	Class	κ	λ	50 Steps
3854	110	4	2	0.625	

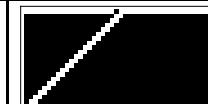


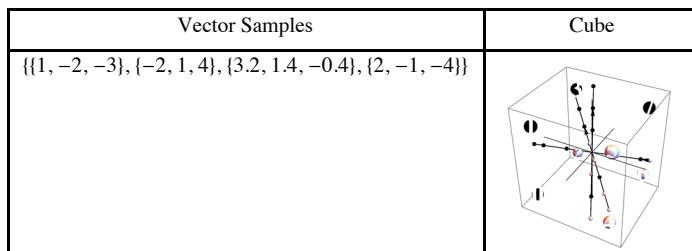
Rule 147. This is another clear example for class IV.

Configurations	Rule	Class	κ	λ	50 Steps
4091	147	4	2	0.5	

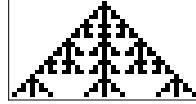


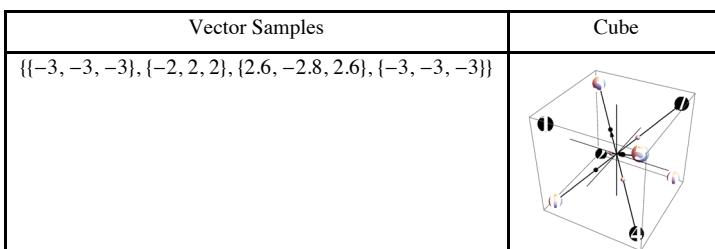
Rule 139. All rules with $\kappa = 3, \lambda = 0.5$ are in Wolfram's class II except 105 and 150; here is an example for this case.

Configurations	Rule	Class	κ	λ	50 Steps
429	139	2	3	0.5	

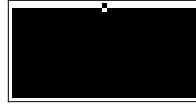


Rule 150. $\kappa = 3$ only in Wolfram's class III rules 105 and 150.

Configurations	Rule	Class	κ	λ	50 Steps
2601	150	3	3	0.5	



Rule 251. All very high λ value ($\lambda = 0.875$) and very low λ value ($\lambda = 0.125$) rules have $\kappa = 1$ and are in Wolfram's class I or II (equal distribution).

Configurations	Rule	Class	κ	λ	50 Steps
5215	251	1	1	0.875	

Vector Samples	Cube
$\{(-3, 3, -3), (-3, 3, -2), (-2.8, 2.6, -2.8), (-1, 2, -1)\}$	

6. Conclusion

This paper reports a detailed survey of various complexity measurement approaches. It should be clear that there is no measure of complexity appropriate to evolution in general, but that it is desirable to be clear about the language of representation, the type of difficulty, and the overall formulation to which each variety refers. At the very least, we should distinguish concepts like size, order, and variety from complexity. In this way, we can be clearer about the strength and context of any such use for analysis and communication.

Acknowledgments

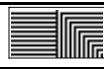
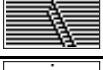
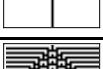
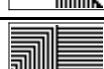
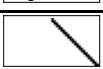
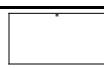
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Appendix

A. List of Classifications

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
0	1	0	0.		18	3	2	0.25	
1	2	1	0.125		19	2	1	0.375	
2	2	1	0.125		20	2	2	0.25	
3	2	1	0.25		21	2	1	0.375	
4	2	1	0.125		22	3	2	0.375	
5	2	1	0.25		23	2	1	0.5	
6	2	2	0.25		24	2	2	0.25	
7	2	1	0.375		25	2	2	0.375	
8	1	1	0.125		26	2	2	0.375	
9	2	2	0.25		27	2	3	0.5	
10	2	1	0.25		28	2	2	0.375	
11	2	1	0.375		29	2	3	0.5	
12	2	1	0.25		30	3	2	0.5	
13	2	1	0.375		31	2	1	0.625	
14	2	1	0.375		32	1	1	0.125	
15	2	1	0.5		33	2	2	0.25	
16	2	1	0.125		34	2	1	0.25	
17	2	1	0.25		35	2	1	0.375	

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
36	2	2	0.25		57	2	2	0.5	
37	2	2	0.375		58	2	3	0.5	
38	2	2	0.375		59	2	1	0.625	
39	2	3	0.5		60	3	2	0.5	
40	1	2	0.25		61	2	2	0.625	
41	4	2	0.375		62	2	2	0.625	
42	2	1	0.375		63	2	1	0.75	
43	2	1	0.5		64	1	1	0.125	
44	2	2	0.375		65	2	2	0.25	
45	3	2	0.5		66	2	2	0.25	
46	2	3	0.5		67	2	2	0.375	
47	2	1	0.625		68	2	1	0.25	
48	2	1	0.25		69	2	1	0.375	
49	2	1	0.375		70	2	2	0.375	
50	2	1	0.375		71	2	3	0.5	
51	2	1	0.5		72	2	2	0.25	
52	2	2	0.375		73	2	2	0.375	
53	2	3	0.5		74	2	2	0.375	
54	4	2	0.5		75	3	2	0.5	
55	2	1	0.625		76	2	1	0.375	
56	2	2	0.375		77	2	1	0.5	

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
78	2	3	0.5		99	2	2	0.5	
79	2	1	0.625		100	2	2	0.375	
80	2	1	0.25		101	3	2	0.5	
81	2	1	0.375		102	3	2	0.5	
82	2	2	0.375		103	2	2	0.625	
83	2	3	0.5		104	2	2	0.375	
84	2	1	0.375		105	3	3	0.5	
85	2	1	0.5		106	4	2	0.5	
86	3	2	0.5		107	2	2	0.625	
87	2	1	0.625		108	2	2	0.5	
88	2	2	0.375		109	2	2	0.625	
89	3	2	0.5		110	4	2	0.625	
90	3	2	0.5		111	2	2	0.75	
91	2	2	0.625		112	2	1	0.375	
92	2	3	0.5		113	2	1	0.5	
93	2	1	0.625		114	2	3	0.5	
94	2	2	0.625		115	2	1	0.625	
95	2	1	0.75		116	2	3	0.5	
96	1	2	0.25		117	2	1	0.625	
97	2	2	0.375		118	2	2	0.625	
98	2	2	0.375		119	2	1	0.75	

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
120	4	2	0.5		141	2	3	0.5	
121	4	2	0.625		142	2	1	0.5	
122	3	2	0.625		143	2	1	0.625	
123	2	2	0.75		144	2	2	0.25	
124	4	2	0.625		145	2	2	0.375	
125	2	2	0.75		146	3	2	0.375	
126	3	2	0.75		147	4	2	0.5	
127	2	1	0.875		148	2	2	0.375	
128	1	1	0.125		149	3	2	0.5	
129	3	2	0.25		150	3	3	0.5	
130	2	2	0.25		151	3	2	0.625	
131	2	2	0.375		152	2	2	0.375	
132	2	2	0.25		153	3	2	0.5	
133	2	2	0.375		154	2	2	0.5	
134	2	2	0.375		155	2	2	0.625	
135	3	2	0.5		156	2	2	0.5	
136	1	1	0.25		157	2	2	0.625	
137	4	2	0.375		158	2	2	0.625	
138	2	1	0.375		159	2	2	0.75	
139	2	3	0.5		160	1	1	0.25	
140	2	1	0.375		161	3	2	0.375	

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
162	2	1	0.375		183	3	2	0.75	
163	2	3	0.5		184	2	3	0.5	
164	2	2	0.375		185	2	2	0.625	
165	3	2	0.5		186	2	1	0.625	
166	2	2	0.5		187	2	1	0.75	
167	2	2	0.625		188	2	2	0.625	
168	1	1	0.375		189	2	2	0.75	
169	4	2	0.5		190	2	2	0.75	
170	2	1	0.5		191	2	1	0.875	
171	2	1	0.625		192	1	1	0.25	
172	2	3	0.5		193	4	2	0.375	
173	2	2	0.625		194	2	2	0.375	
174	2	1	0.625		195	3	2	0.5	
175	2	1	0.75		196	2	1	0.375	
176	2	1	0.375		197	2	3	0.5	
177	2	3	0.5		198	2	2	0.5	
178	2	1	0.5		199	2	2	0.625	
179	2	1	0.625		200	2	1	0.375	
180	2	2	0.5		201	2	2	0.5	
181	2	2	0.625		202	2	3	0.5	
182	3	2	0.625		203	2	2	0.625	

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
204	2	1	0.5		225	4	2	0.5	
205	2	1	0.625		226	2	3	0.5	
206	2	1	0.625		227	2	2	0.625	
207	2	1	0.75		228	2	3	0.5	
208	2	1	0.375		229	2	2	0.625	
209	2	3	0.5		230	2	2	0.625	
210	2	2	0.5		231	2	2	0.75	
211	2	2	0.625		232	2	1	0.5	
212	2	1	0.5		233	2	2	0.625	
213	2	1	0.625		234	1	1	0.625	
214	2	2	0.625		235	1	2	0.75	
215	2	2	0.75		236	2	1	0.625	
216	2	3	0.5		237	2	2	0.75	
217	2	2	0.625		238	1	1	0.75	
218	2	2	0.625		239	1	1	0.875	
219	2	2	0.75		240	2	1	0.5	
220	2	1	0.625		241	2	1	0.625	
221	2	1	0.75		242	2	1	0.625	
222	2	2	0.75		243	2	1	0.75	
223	2	1	0.875		244	2	1	0.625	
224	1	1	0.375		245	2	1	0.75	

Rule	Class	κ	λ	Sample	Rule	Class	κ	λ	Sample
246	2	2	0.75		251	1	1	0.875	
247	2	1	0.875		252	1	1	0.75	
248	1	1	0.625		253	1	1	0.875	
249	1	2	0.75		254	1	1	0.875	
250	1	1	0.75		255	1	0	1.	

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