Inducing Class 4 Behavior on the Basis of Lattice Analysis

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A recipe is proposed for preparing class 4 cellular automata that uses lattice analysis based on rough set approximation. Superimposing class 2-like transitions on an intermediate layer of class 3 automata results in class 4 behavior.

1. Introduction

Since the conception of the idea of complex systems in the field of statistical mechanics, it has been expected that the notion of complexity can be reduced to a set of simple rules applicable at a microscopic scale. Cellular automata (CAs) revealing diverse complex behavior are also used as tools for understanding complexity generated by simple rules. Wolfram has classified all CAs with respect to their patterns of time development. Namely, class 1 as stable homogeneous patterns, class 2 as local periodic patterns, and class 3 as chaotic patterns. In this regard, class 4 is characterized by both a local periodic pattern and a chaotic pattern that can interact with each other. Class 4 is found at the critical area between classes 2 and 3 in a specific rule space [1, 2]. Turing machines can also be implemented by using a "glider" [3], which characterizes class 4 rules. A glider can be used as a device for logical gates, shifts, delays, and reflections [4, 5]. Whether simple or complex, behaviors can be reduced to transition rules, and even class 4 behavior is not an exception from this generalization.

Class 4 automata are considered to exhibit an essential feature of biological systems in that a system is divided into stably periodic parts which can interact with each other. Robust but dynamic biological systems in nature are assumed to be hierarchical [6, 7], while class 4 automata have no hierarchical structure. The notion of hierarchy can be rather problematic. Physicists regard hierarchy as consisting of a layer that has its own time constant (e.g., [8]). Different layers cannot influence each other due to differences between them in terms of the order of the time constant, and therefore the hierarchy can be maintained in a stable structure. In this sense, if the boundary between lay-

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ers is explicitly defined, hierarchy is found not in function but in structure. On the other hand, if there is a strong interaction between layers, it can break the boundary between layers as well as the hierarchy itself. Hierarchy involving interactions between layers might be in logical conflict.

Specifically, in examining a biochemical network or an immune system, functional hierarchy cannot be found in the interaction between the layers since the loss of the functional boundary allows the interaction. On the other hand, structural hierarchy is found to be robust. There is an explicit double meaning between structure and function in looking for hierarchy. Naturally, the question arises whether observers can freely move between the two meanings. As hierarchical systems are self-organizing systems, the boundary between layers has to be perpetually broken and reconstructed in order to be maintained in a robust state. This self-organizing process concerns not only structure, but also function. As a result, the boundary between structure and function is ambiguous and vague, and therefore the boundary between layers is destined to be ambiguous and vague. The notion of hierarchy itself is self-referential and ambiguous, and these properties have been recently reflected in the term "heterarchy" [9].

Here, we clarify the notion: (*i*) *Heterarchy* is a hierarchy consisting of different layers, and (*ii*) the boundary between different layers cannot be controlled. Thus, an observer observing a phenomenon at a focal level cannot determine the layer exerting the strongest influence on the focal level. This definition is strongly connected with the notion of emergence [2, pp. 737-750]. As the computation proceeds, such as applying elementary CA (ECA) rule 30 to an initial condition of just one seed of state 1, a random or complex binary sequence is generated. According to Wolfram in [2], complexity emerges from just one seed (simple matter). However, this statement implies a double meaning. On one hand, the initial condition is assumed to be one seed in a vacuum from a macroscopic perspective, and so the emergence of complexity can be observed. On the other hand, the initial condition is assumed to be a binary sequence consisting of 0 only, with the exception of a central 1 from a microscopic perspective, and then the ECA rule can be applied to the initial condition. It appears that only if an observer has a limited knowledge of the initial condition can emergence can be admitted. In this sense then, is emergence of complexity just an illusion?

I think that the notion of emergence inevitably involves a double meaning. As well, it needs both microscopic and macroscopic perspectives. Indeed, it needs an uncontrollable boundary between the microscopic and macroscopic perspectives. This is the only way to champion the notion of emergence. Emergence as an illusion can hold if observers can adopt both perspectives and freely move between the microscopic and macroscopic perspectives by their own will. In contrast, if the boundary between microscopic and macroscopic perspectives is uncontrollable and observers cannot determine where (i.e., in

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which perspective) they are, it cannot be said that emergence is merely an illusion. This observation can champion the notion of emergence.

If a heterarchical structure is implemented in ECAs, the question arises whether class 4 and/or more complex behaviors can emerge in such a system. If the answer is positive, it can be said that complex patterns can emerge in the form of class 4 behavior. We now propose a model for class 4 behavior as heterarchy that is constructed as follows. First, we analyze the macroscopic properties of ECAs by using an algebraic structure based on lattice analysis. Next, a virtual algebraic structure that characterizes class 1 or 2 ECAs that cannot be derived from class 3 rules is planned to be implemented in a class 3 rule. Then, we look for modifications in the rules in order to reveal such a virtual algebraic structure. This results in the superposition of the middle layer transition, which can mediate the microscopic ECA transition and the virtual algebraic structure. Finally, we obtain a recipe for producing class 3 ECAs that exhibit class 4 behavior. The resulting system consists of a microscopic layer defined through an ECA rule, a macroscopic layer assumed as a global property shown as a lattice structure, and a middle layer transition that mediates the microscopic and macroscopic layers. Since there is a conflict between the middle layer transition and the ECA rule, the transition for a triplet is not a deterministic rule. In this sense, the resulting system contains an uncontrollable boundary between microscopic and macroscopic layers.

2. Lattices Derived by Double Indiscernibility

In this paper, we propose a method for revealing algebraic properties of an ECA by using a lattice driven by rough set analysis and how to make CAs equipped with the transition of the intermediate layer by using a lattice modification. Figure 1 shows a rough scheme of our recipe. A lattice (represented by a Hasse diagram in Figure 1) that is obtained by the transition rule of an ECA can reveal some algebraic property of the ECA. Since the behavior of ECAs in terms of classes 1 through 4 can be roughly compared to the algebraic property of a lattice, it is possible to design and make class 4 automata by using a lattice driven by rough set analysis.

First, we prepare a mathematical tool for the analysis of the algebraic properties of ECAs. The algebraic structure of an ECA is analyzed by using an equivalence relation derived through a transition. We review our recent work on constructing lattices by using two equivalence relations derived from two different maps. The construction is based on a rough set [10-14]. First, we review the idea of rough set.

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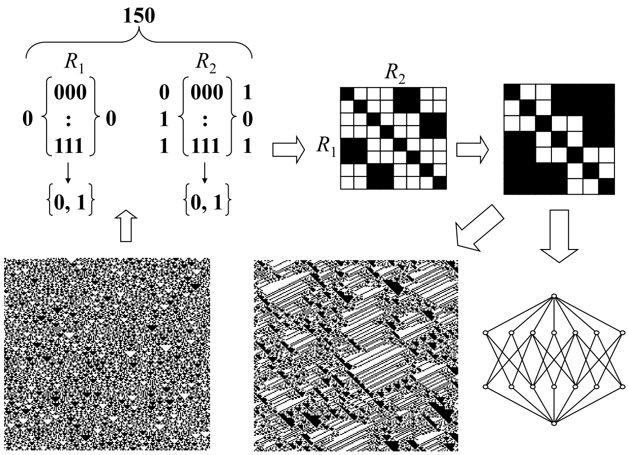

Figure 1. Our scenario for making a class 4 CA from a class 3 CA by using a lattice analysis driven by rough set analysis. First, a class 3 ECA (here, rule 150) is chosen and the transition rule is divided into two rules R_1 and R_2 dependent on the local boundary. Since two equivalence relations are obtained by R_1 and R_2, the binary relation between the two equivalence relations is obtained as shown in the matrix (top, center). The binary relation can represent a lattice revealing the property of an ECA. By modifying a binary relation to reveal a more distributive lattice (bottom, left), the underlying transition is also modified that can show class 4-like behavior (bottom, center).

Given a universal set U, let $R \subseteq U \times U$ be an equivalence relation on U. For $X \subseteq U$, we define the R-upper and R-lower approximations of X, which are denoted as $R^*(X)$ and $R_*(X)$, respectively, as

$$R^*(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}$$
$$R_*(X) = \{x \in U \mid [x]_R \subseteq X\},$$

where $[x]_R$ is an equivalence class of R such that $[x]_R = \{y \in U \mid xRy\}$. Since both $R^*(X)$ and $R_*(X)$ are approximations for a set X, they are called "rough sets".

It is clear that the upper and lower approximations form a Galois connection [14] if a single equivalence relation is given. In fact, given a universal set U and an equivalence relation $R \subseteq U \times U$, for a subset $X, Y \subseteq U$ and a Galois connection $R^*(X) \subseteq Y \Leftrightarrow X \subseteq R_*(Y)$ is formed. This leads to fixed point duality expressed as $R^*(X) = X \Leftrightarrow R_*(X) = X$. From the duality, it is shown that a partially ordered set, such as $\langle P; \subseteq \rangle$ with $P = \{X \subseteq U \mid R_*(X) = X\}$, is a set lattice, and that $\langle Q; \subseteq \rangle$ with $Q = \{X \subseteq U \mid R^*(X) = X\}$ is also a set lattice. In fact, it is straightforward to verify that join and meet in $\langle P; \subseteq \rangle$ and $\langle Q; \subseteq \rangle$ can be defined as union and intersection, respectively.

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To estimate the role of the difference between the upper and lower approximations, the composition between them is introduced. Then, it can be verified that $\langle P; \subseteq\rangle$ with $P = \{X \subseteq U \mid R_*(R^*(X)) = X\}$ is a set lattice. Similarly, $\langle Q; \subseteq\rangle$ with $Q = \{X \subseteq U \mid R^*(R_*(X)) = X\}$ is also a set lattice. Thus, the composition of the lower and upper approximations is reduced to a single approximation. Even if objects are recognized depending on the approximation based on an equivalence relation, the structure of the lattice is invariant; namely, it is a Boolean lattice. To introduce diversity into the lattice structure, it is necessary to break the Galois connection derived from the single equivalence relation [15, 16].

Thus, we introduce two equivalence relations. Given a universal set U, R and $S \subseteq U \times U$ are defined as different equivalence relations. The operations T and S are defined as $T = S_* R^*$ and $K = R^* S_*$. Then, for X, $Y \subseteq U$, $T(T(X)) = T(X)$ and $K(K(X)) = K(X)$. By using this operator, we can construct a lattice from a partially ordered set $\langle \boldsymbol{L_T}; \subseteq\rangle$ with $\boldsymbol{L_T} = \{X \subseteq U \mid T(X) = X\}$, or $\langle \boldsymbol{L_K}; \subseteq\rangle$ with $\boldsymbol{L_K} = \{X \subseteq U \mid K(X) = X\}$. Meet and join for these partially ordered sets are defined for X, and $Y \in \boldsymbol{L_T}$ as follows:

$$X \bigwedge Y = T(X \bigcap Y), \qquad X \bigvee Y = T(X \bigcup Y).$$

Similarly, for X and $Y \in \boldsymbol{L_K}$, $X \bigwedge Y = K(X \bigcap Y)$ and $X \bigvee Y = K(X \bigcup Y)$.

Since the two equivalence relations are independent of each other, the Galois connection such that $R^*(X) \subseteq Y \Leftrightarrow X \subseteq S_*(Y)$ no longer holds. Despite the double indiscernibility, the Galois connection holds if subsets are chosen from a collection of fixed points such that $\boldsymbol{L_T} = \{X \subseteq U \mid T(X) = X\}$. It is revealed that a collection of fixed points with respect to T or K can constitute a stable structure.

Figure 2 shows an example of a lattice defined as $\langle \boldsymbol{L_T}; \subseteq\rangle$ with $\boldsymbol{L_T} = \{X \subseteq U \mid T(X) = X\}$. Given two equivalence relations, a collection of fixed points with respect to T constitutes a lattice. It is clear that there exists a lattice isomorphism between $\boldsymbol{L_T}$ and $\boldsymbol{L_K}$. In fact, it is possible to show that a map $\phi : \boldsymbol{L_T} \to \boldsymbol{L_K}$ is a lattice homomorphism, where for $X \in \boldsymbol{L_T}$, $\phi(X) = R^*(X)$, $\boldsymbol{L_T} = \{X \subseteq U \mid T(X) = X\}$, and $\boldsymbol{L_K} = \{X \subseteq U \mid K(X) = X\}$. It can also be verified that any lattice can be represented as a collection of fixed points with respect to operator T or K. This is a representation theorem [16].

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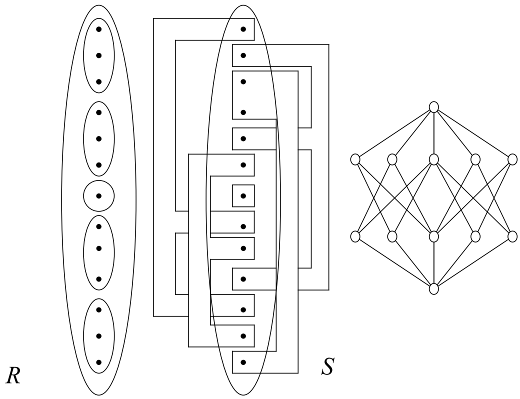

Figure 2. Hasse diagram of a lattice (right) defined as a collection where $T(X) = X$. The equivalence classes of R are represented as loops, and those of S are represented as polygons (left).

3. Lattice for Automata

Svozil proposes a method for constructing a lattice for a given automaton and shows that the Moore automaton revealing computational complementarity is expressed as an orthocomplemented lattice [17, 18]. An automaton is defined by the transition of an internal state $\delta : Q \times \Sigma \to Q$ and an output function $f : Q \to O$, where Σ is a finite set of input alphabets, Q is a finite set of states, and O is a finite set of output symbols. Computational complementarity found in some specific automata is defined as follows: there exists an automaton such that any pair of its states is distinguishable, while there are no experiments (i.e., giving an input sequence) that can determine what state the automaton had been in at the beginning of the experiment. In other words, an automaton has the property of computational complementarity if internal states cannot be distinguished from each other by any sequence of input symbols.

Moore automata are known as examples that reveal computational complementarity, where $Q = \{1, 2, 3, 4\}$, $\Sigma = \{0, 1\}$, $O = \{0, 1\}$, the transitions are defined as $\delta_0(1) = \delta_0(3) = 4$, $\delta_0(2) = 1$, $\delta_0(4) = 2$, $\delta_1(1) = \delta_1(2) = 3$, $\delta_1(3) = 4$, $\delta_1(4) = 2$ and the output function is defined as $f(1) = f(2) = f(3) = 0$ and $f(4) = 1$. The transitions δ_0 and δ_1 represent transitions due to input 0 and 1, respectively.

Intrinsic propositional calculus [17] is expressed as a partition of states due to the transition so as to see computational complementarity. A partition under the input k is expressed as $\{\{a, b\}, \{c, d\}\}$ if and only if $\delta_k(a) = \delta_k(b)$ and $\delta_k(c) = \delta_k(d)$. In the mentioned Moore automaton, we obtain three partitions, namely $\{\{1, 3\}, \{2\}, \{4\}\}$,

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{{1, 2}, {3}, {4}}, and {{1, 2, 3}, {4}} that depend on the input symbols. The first partition is derived from δ_0 and the second one is derived from δ_1. The third partition is derived from the output function. In the first partition the states 1 and 3 cannot be distinguished, while 1 and 2 can be discriminated. The second partition shows the contrary case (the third condition is useless in this point of view). Since there is complementarity between distinction 1 from 2 and 1 from 3, states 1, 2, and 3 cannot be distinguished by giving any sequence of input symbols.

Each partition can reveal a set lattice (Boolean lattice) whose atoms are represented as elements of partitions. The method proposed by Svozil for constructing lattices is based on pasting Boolean lattices. A lattice for an automaton is constructed by collecting all elements of the set lattices that are derived from the partitions. Order is defined by inclusion existing in some lattices derived from the partitions. Although Svozil refers to his method as "intrinsic" propositional calculus, an observer who constructs a lattice knows all states of the automaton. There is also a double standard. Thus, the observer can deal with each partition depending on the other partitions and can paste them. An intrinsic observer is, however, destined to know the states of the automaton only through a partition depending on the input [19-21]. In other words, an observer must be faced with an uncontrollable boundary between the internal stance at which the internal state can be seen and the external stance at which the internal state cannot be seen. Therefore, we cannot assume that an observer can prepare the states of the automaton, and two experiments dependent on the input cannot be conducted in a parallel fashion. This shows that two partitions cannot be distributed and must be ordered in a sequence.

By adopting this idea, two partitions can be regarded as two kinds of equivalence relations (i.e., the elements of a partition form an equivalence class). Regarding the Moore automaton, we believe that the equivalence relations S and R are defined as {{1, 3}, {2}, {4}} and {{1, 2}, {3}, {4}}, respectively. Since the two experiments (partitions) must be ordered in a sequence, the lattice derived from the Moore automaton can be obtained as a collection of fixed points, such as $\boldsymbol{L}_T = \{X \subseteq X \mid T(X) = X\}$. The lattice is simply a distributive lattice (Heyting algebra) and does not exhibit the properties of an orthocomplemented lattice obtained with the construction provided by Svozil.

Conversely, if experiments with an automaton must be ordered in a sequence, and if the lattice is obtained only from the composition of two partitions, an orthocomplemented lattice reveals a more complicated automaton. Figure 3 shows an example of an automaton that reveals an orthocomplemented lattice. The one represented as loops shows the transition δ_0, and the other partition, represented as arrows, shows the transition δ_1. These partitions have the following remarkable features: (*i*) For any a in $\delta_0(Q)$ and $\delta_1(Q)$, there exists x in Q such that $\delta_0(x) = \delta_1(x) = a$ and (*ii*) if $\delta_0(x) = a$ and $\delta_1(x) = b$, then

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there exists y in Q such that $\delta_0(y) = b$ and $\delta_1(y) = a$. It can be shown that the automaton satisfying these features reveals a complemented lattice, where the two equivalence relations R and S are defined by δ_0 and δ_1 such that $R = \{\langle x, y\rangle \in Q \times Q \mid \delta_0(x) = \delta_0(y)\}$ and $S = \{\langle x, y\rangle \in Q \times Q \mid \delta_1(x) = \delta_1(y)\}$. This recipe for constructing a lattice is now applied to ECAs.

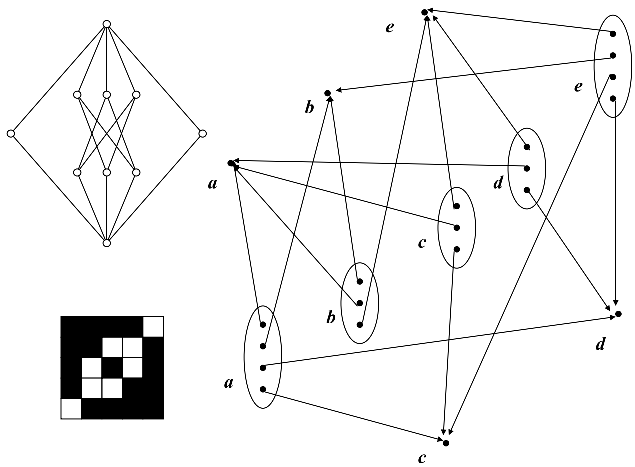

Figure 3. Hasse diagram of a lattice (upper left) defined as a collection where $T(X) = X$ for an automaton defined by a transition (right). In the transition, the partition represented as loops is the equivalence relation R, and the other partition represented as arrows is the other equivalence relation S. The binary relation between two quasi-sets divided by each equivalence class is shown as a matrix.

4. Elementary Cellular Automata and Lattices

ECAs are analyzed by using a lattice derived from a rough set in order to grasp the macroscopic properties of their behaviors. The transition rule of an ECA $f_r : \{0, 1\}^3 \to \{0, 1\}$ such that $c_i^{t+1} = f_r\left(c_{i-1}^t, c_i^t, c_{i+1}^t\right)$ is numbered by r following Wolfram. Since a lattice derived from a rough set consists of equivalence classes, a set of states is divided into equivalence classes in terms of a transition. A set of states $\{0, 1\}$ in an ECA is too small to construct a lattice consisting of equivalence classes. Thus, we replace the state space with a set of triplets of $\{0, 1\}$, such as $\{(0, 0, 0), (0, 0, 1), \ldots, (1, 1, 1)\}$. Given an ECA with a transition $c_i^{t+1} = f_r\left(c_{i-1}^t, c_i^t, c_{i+1}^t\right)$, it is regarded as the following automata:

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$$Q = \{(0, 0, 0),\ (0, 0, 1),\ \ldots,\ (1, 1, 1)\},$$
$$\Sigma = \{(0, 0),\ (0, 1),\ (1, 0),\ (1, 1)\}.$$

For any internal state $q^t = (q_0, q_1, q_2) \in Q$ and input symbol $p^t = (p_0, p_1) \in \Sigma$, $q^{t+1} \in Q$ is determined by the transition $\delta_r : Q \times \Sigma \to Q$ such that

$$q^{t+1} = \delta_r(q^t; p^t) = \delta_r((q_0, q_1, q_2), (p_0, p_1)) = \\ (f_r(p_0, q_0, q_1), f_r(q_0, q_1, q_2), f_r(q_1, q_2, p_1)).$$

We define a pair of equivalence relations $R_{a\,b}$ and $R_{\Sigma-a\,b}$ derived from transition δ_r. Given an input pair $(a, b) \in \Sigma$, $P = Q \times \Sigma^+$, $\Sigma^+ = \Sigma - \{(a, b)\}$ is defined, and $R_{a\,b}$ and $R_{\Sigma-a\,b}$ are defined as subsets of $P \times P$. The equivalence relation $R_{a\,b}$ is defined as

$$R_{a\,b} = \\ \{((q, (c, d)),\ (s, (e, f))) \in P \times P \mid \delta_r(q, (a, b)) = \delta_r(s, (a, b)), \\ q, s \in Q, (c, d), (e, f) \in \Sigma^+\}.$$

The other equivalence relation, namely $R_{\Sigma-a\,b}$, is expressed as

$$R_{\Sigma-a\,b} = \\ \{((q, (c, d)), (s, (e, f))) \in P \times P \mid \delta_r(q, (c, d)) =_{\text{ex}} \delta_r(s, (c, d)), \\ q, s \in Q, (c, d), (e, f) \in \Sigma^+\}.$$

where the equivalence relation $=_{\text{ex}} \subseteq Q \times Q$, which cannot be uniquely determined, is defined under the following conditions.

1. $q = s \Rightarrow q =_{\text{ex}} s$,
2. $(\delta_r(q, (c, d)), \delta_r(s, (e, f))) \notin =_{\text{ex}}$, for $(q, (c, d)), (s, (e, f)) \in R_{a\,b}$ such that $\delta_r(q, (c, d)) \neq \delta_r(s, (e, f))$ with $(c, d), (e, f) \in \Sigma^+$,
3. $(\delta_r(q, (c, d)), \delta_r(s, (a, b))) \notin =_{\text{ex}}$, for $(q, (c, d)), (s, (e, f)) \in R_{a\,b}$ such that $\delta_r(q, (c, d)) \neq \delta_r(s, (a, b))$ with $(c, d) \in \Sigma^+$,
4. $(\delta_r(q, (a, b)), \delta_r(w, (a, b))) \notin =_{\text{ex}}$, for $(q, (c, d)), (w, (e, f)) \notin R_{a\,b}$,
5. $P / R_{a\,b} \cong Q / =_{\text{ex}}$.

Finally, we can construct a lattice as a collection of fixed points with respect to the upper and lower approximations such that $\boldsymbol{L}_T = \{X \subseteq P \mid R_{a\,b^*} R^*_{\Sigma-a\,b}(X) = X\}$.

We show the case of f_{90} such that

$$f_{90}(0, 0, 0) = f_{90}(1, 0, 1) = f_{90}(0, 1, 0) = f_{90}(1, 1, 1) = 0$$

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and

$$f_{90}(0, 0, 1) = f_{90}(1, 0, 0) = f_{90}(0, 1, 1) = f_{90}(1, 1, 0) = 1.$$

The transition δ_{90} for input (0, 0) is expressed as

$$\begin{aligned}
\delta_{90}((0, 0, 0), (0, 0)) &= \left(f_{90}(0, 0, 0),\ f_{90}(0, 0, 0),\ f_{90}(0, 0, 0)\right) = (0, 0, 0),\\
\delta_{90}((0, 0, 1), (0, 0)) &= \left(f_{90}(0, 0, 0),\ f_{90}(0, 0, 1),\ f_{90}(0, 1, 0)\right) = (0, 1, 0),\\
\delta_{90}((0, 1, 0), (0, 0)) &= \left(f_{90}(0, 0, 1),\ f_{90}(0, 1, 0),\ f_{90}(1, 0, 0)\right) = (1, 0, 1),\\
&\vdots\\
\delta_{90}((1, 1, 0), (0, 0)) &= \left(f_{90}(0, 1, 1),\ f_{90}(1, 1, 0),\ f_{90}(1, 0, 0)\right) = (1, 1, 1),\\
\delta_{90}((1, 1, 1), (0, 0)) &= \left(f_{90}(0, 1, 1),\ f_{90}(1, 1, 1),\ f_{90}(1, 1, 0)\right) = (1, 0, 1).
\end{aligned}$$

A quotient set divided by equivalence relation R_{00} is expressed as

$$\begin{aligned}
P \big/ R_{00} = \big\{&\{(0, (c, d)), (5, (c, d)) \mid (c, d) \in \Sigma^{+}\},\\
&\{(2, (c, d)), (7, (c, d)) \mid (c, d) \in \Sigma^{+}\},\\
&\{(1, (c, d)), (4, (c, d)) \mid (c, d) \in \Sigma^{+}\},\\
&\{(3, (c, d)), (6, (c, d)) \mid (c, d) \in \Sigma^{+}\}\big\},
\end{aligned}$$

where a triplet $(a, b, c) \in Q$ is expressed as a decimal number by $(abc)_2$. It is easy to see $((0, (c, d)), (5, (c, d))) \in R_{00}$ since $\delta_{90}(0, (0, 0)) = \delta_{90}(5, (0, 0)) = 0$. Similarly,

$$\begin{aligned}
&\delta_{90}(0, (0, 1)) = \delta_{90}(5, (0, 1)) = 1,\\
&\delta_{90}(0, (1, 0)) = \delta_{90}(5, (1, 0)) = 4, \text{ and}\\
&\delta_{90}(0, (1, 1)) = \delta_{90}(5, (1, 1)) = 5,
\end{aligned}$$

condition 2 states that 1, 4, and 5 must be distinguished from each other, and condition 3 states that $0 =_{\text{ex}} 1$, $0 =_{\text{ex}} 4$, and $0 =_{\text{ex}} 5$ are not allowed. From condition 4, each output dependent on input (0, 0) must be distinguished from other such output. Thus, one of the equivalence relations $=_{\text{ex}}$ satisfying conditions 1 through 5 gives a quotient set $Q /=_{\text{ex}} = \{\{0, 3\}, \{2, 1\}, \{5, 6\}, \{4, 7\}\}$.

Due to the equivalence relation $=_{\text{ex}}$, $P \big/ R_{\Sigma-00}$ is expressed as

$$\begin{aligned}
&P \big/ R_{\Sigma-00} =\\
&\quad \{\{(1, (0, 1)),\ (4, (0, 1))\,(3, (1, 0)),\ (6, (1, 0)),\ (2, (1, 1)),\\
&\qquad (7, (1, 1))\},\ \{(0, (0, 1)),\ (5, (0, 1))\,(2, (1, 0)),\ (7, (1, 0)),\\
&\qquad (3, (1, 1)),\ (6, (1, 1))\},\ \{(3, (0, 1)),\ (6, (0, 1))\,(1, (1, 0)),\\
&\qquad (4, (1, 0)),\ (0, (1, 1)),\ (5, (1, 1))\},\ \{(2, (0, 1)),\\
&\qquad (7, (0, 1))\,(0, (1, 0)),\ (5, (1, 0)),\ (1, (1, 1)),\ (4, (1, 1))\}\}.
\end{aligned}$$

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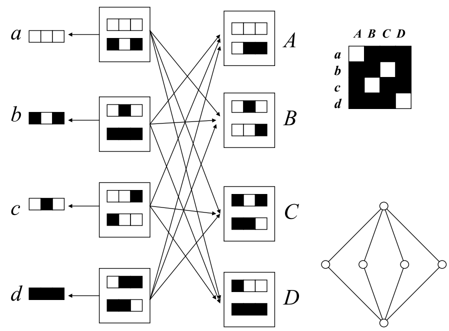

Figure 4. Construction of a lattice derived from the lower and upper approximations defined as ECA 90. The central box represents an element of $P \big/ R_{00}$, and $P \big/ R_{00} \cong \{a, b, c, d\}$. Here, $Q\ /{=_{\text{ex}}}\ \{\{0, 3\}, \{2, 1\}, \{5, 6\}, \{4, 7\}\} = \{A, B, C, D\}$. An element of $P \big/ R_{\Sigma-00}$ is represented as a collection of sources whose arrows reach an element of $Q\ /{=_{\text{ex}}}$ where input is omitted. An element $\{(1, (0, 1)), (4, (0, 1)), (3, (1, 0)), (6, (0, 1)), (2, (1, 1)), (7, (1, 1))\}$ in $P \big/ R_{\Sigma-00}$ is expressed as a collection of sources of arrows whose target is a box A, such as $\{2, 7, 1, 4, 3, 6\}$. The binary relation between $P \big/ R_{00}$ and $P \big/ R_{\Sigma-00}$ is represented in terms of the box in the upper-right corner. The derived lattice is represented as a Hasse diagram.

For the two equivalence relations R_{00} and $R_{\Sigma-00}$, a collection of fixed points such that $\boldsymbol{L}_T = \left\{X \subseteq P \,\middle|\, R_{00^*}\, R^*_{\Sigma-00}(X) = X\right\}$ is a lattice, as shown in Figure 4. It is clear that $\left\{(0, (c, d)),\ (5, (c, d)) \,\middle|\, (c, d) \in \Sigma^+\right\}$ is an element of $\boldsymbol{L}_T$:

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$$
\begin{aligned}
&R_{00^*}\, R^*_{\Sigma-00}\left(\left\{(0, (c, d)),\ (5, (c, d)) \,\middle|\, (c, d) \in \Sigma^+\right\}\right) = \\
&\quad R_{00^*}\left(\{(0, (0, 1)),\ (5, (0, 1))\,(2, (1, 0)),\ (7, (1, 0)),\right. \\
&\qquad (3, (1, 1)),\ (6, (1, 1)),\ (3, (0, 1)),\ (6, (0, 1))\,(1, (1, 0)), \\
&\qquad (4, (1, 0)),\ (0, (1, 1)),\ (5, (1, 1)),\ (2, (0, 1)),\ (7, (0, 1)) \\
&\qquad \left.(0, (1, 0)),\ (5, (1, 0)),\ (1, (1, 1)),\ (4, (1, 1))\}\right) = \\
&\left\{(0, (c, d)),\ (5, (c, d)) \,\middle|\, (c, d) \in \Sigma^+\right\}.
\end{aligned}
$$

We estimate all 256 ECAs in terms of a lattice derived from upper and lower approximation operators. More specifically, we examine whether a derived lattice is a distributive lattice. In our lattice analysis, each element of a lattice is a collection of binary sequences. If a pattern generated by an ECA is reduced to an atomistic view, the pat-

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tern is described as a combination of elementary subpatterns. Since the reduction is based on a distributive law such that $(a \wedge (b \vee c)) = (a \wedge b) \vee (a \wedge c)$, we believe that an ECA of class 1 or 2 can yield a distributive lattice in our lattice analysis.

Figure 5 shows the distributivity of all synchronous legal automata with respect to all pairs of equivalence relations dependent on the input. Each row represents a corresponding rule denoted by its Wolfram rule number, and each column represents a pair of two equivalence relations, where $R_{01}\,R_{00+10+11}$ is a pair of R_{01} and $R_{\Sigma-01}$. A filled cell indicates that the derived lattice is distributive, and a vacant cell indicates that it is nondistributive. Class 1 automata show distributive lattices independent of equivalence relations, and class 3 automata exhibit nondistributive lattices. Class 2 features intermediate properties. The results for class 4 rules, which are asymmetric, are also shown, and it is clear that they are similar to those of class 3.

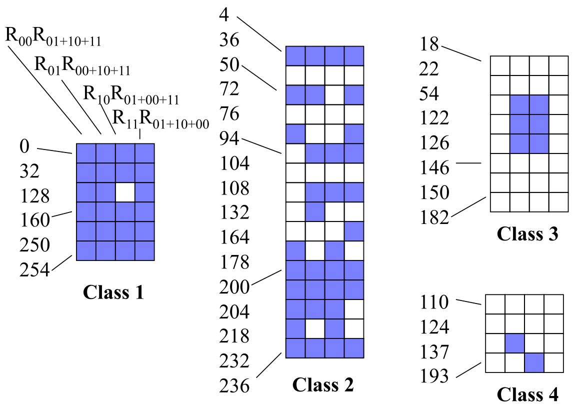

Figure 5. Distributivity of a lattice derived from upper and lower approximation operators. See the text.

Figure 6 shows examples of Hasse diagrams of lattices derived from ECAs. The Hasse diagram of rule 18 (class 3) contains a pentagon sublattice (shaded elements), indicating that the lattice is nondistributive. From the results of the estimation with respect to distributivity, we conclude that the lattice of class 3 automata showing mixing subpatterns is a nondistributive lattice, as well as that if the patterns are stable, then the distributivity is increased.

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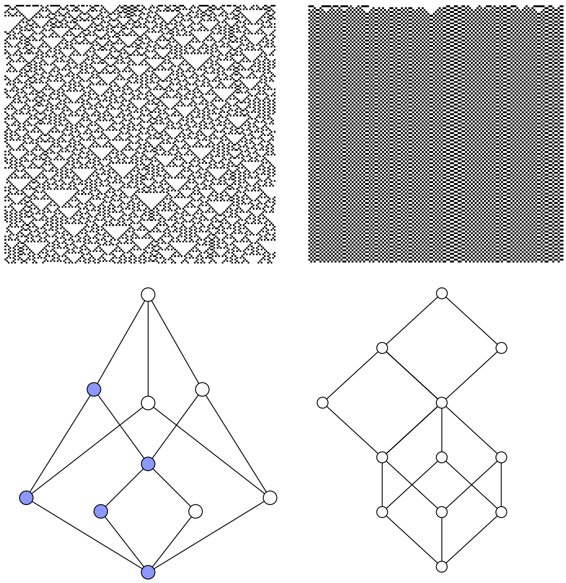

Figure 6. Two examples of Hasse diagrams of lattices derived from an ECA. The left and right ECAs correspond to rule 18, exhibiting a class 3 pattern, and rule 50, exhibiting a class 2 pattern.

5. Inducing Class 4 Behavior by Introducing a Discrepancy between Microscopic and Macroscopic Properties

As mentioned in Section 1, we propose a model of class 4 automata based on the dynamical negotiation between microscopic and macroscopic layers. Since an ECA is based on local deterministic rules, the macroscopic phase is excluded from the process of pattern development. While the macroscopic properties of an ECA can be described in terms of a lattice derived from upper and lower approximations, as mentioned in Section 4, there is no discrepancy between the ECA rules and the properties exhibited by a lattice. We introduce a virtual macroscopic property independent of the ECA rules and superimpose it on the ECA, which results in a discrepancy between microscopic and macroscopic properties. We estimate whether such a recipe induces class 4 behavior.

Given a class 3 ECA rule, first we prepare a lattice derived from upper and lower approximations, which is expected to behave as a nondistributive lattice. Then, we search for a lattice characterized by greater distributivity (i.e., the distributive law holds for more elements), as well as for a minor modification to allow for the implementation of such a lattice. The minor modification is implemented as a transition for binary strings larger than triplets and reveals a middle

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layer between the macroscopic layer described by the lattice and the microscopic layer implemented by the ECA rule.

We present a recipe for preparing a class 4 automaton from a class 3 automaton using rule 146 (Figure 7). Also, for the construction of the relation between R_{00} and $R_{\Sigma-00}$ of rule 90, we prepare a binary relation between P/R_{00} and $P/R_{\Sigma-00}$, as shown in the matrix in the upper right corner in Figure 7, where $P/R_{00} \cong \{a, b, c, d, e\}$ and $P/R_{\Sigma-00} \cong \{A, B, C, D, E\}$. A set $L_T = \left\{X \subseteq P \,\middle|\, R_{00^*} R^*_{\Sigma-00}(X) = X\right\}$ reveals a nondistributive lattice containing a sublattice of the pentagon. As mentioned earlier, a nondistributive lattice can reveal class 3 behavior.

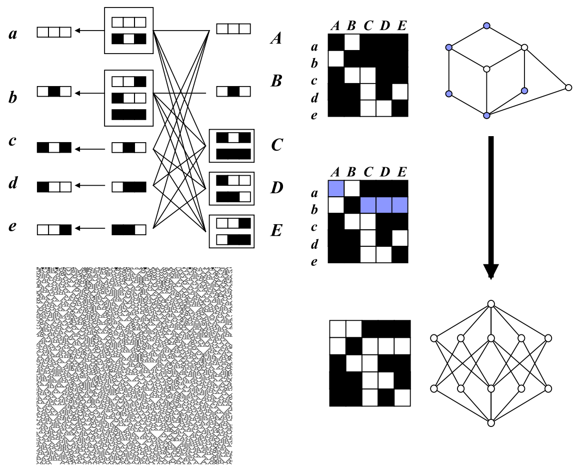

Figure 7. Transition of rule 146 for triplets under input 00 (*a* through *d*) and under 01, 10, and 11 (*A* through *D*) (upper left) and time development (lower left). The matrix in the upper panel represents the binary relation between P/R_{00} and $P/R_{\Sigma-00}$ revealing the macroscopic properties of rule 146. A nondistributive lattice is shown on the right. The matrix and the Hasse diagram in the lower panel represent the virtual relation and its corresponding lattice.

Our search for a lattice with greater distributivity resulted in the following operation. If the binary relation J between P/R_{00} and $P/R_{\Sigma-00}$ is modified by erasing certain filled cells in the matrix in the upper-right corner, represented by the shaded (not black) cells in the middle matrix, then a modular orthomodular lattice is revealed, as shown by the Hasse diagram in the lower-right corner of Figure 7. Regarding the implementation of the operation, since the top row where (a, A), (a, C), (a, D), and $(a, E) \in J$ indicates that

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$$\delta_{146}(q, (0, 0)) = 0 \text{ and}$$
$$\delta_{146}(q, (c, d)) = 0, 5, 7, 4, 6, 1, \text{ and } 3, \text{ for } (c, d) \in \Sigma^{+},$$

the operation of erasing (a, A) from J can be expressed as $\delta_{146}(q, (c, d)) \neq 0$. Since $q = 0, 5$ one of the possible operations is expressed as

$$\delta_{146}(q, (0, 1)) = 1, \quad \delta_{146}(q, (1, 0)) = 4, \quad \delta_{146}(q, (1, 1)) = 5$$

which can reveal $\delta_{146}(q, (c, d)) \neq 0$. The operation of erasing (b, C), (b, D), $(b, E) \in J$ is also implemented as

$$\delta_{146}(q, (c, d)) = 2, \text{ for } (c, d) \in \Sigma^{+}, \ q = 1, 4, 7.$$

Those operations are defined as transitions for triplets and are inconsistent with the original ECA rule 146. They represent the transition for the intermediate layer, which is neither a microscopic ECA rule nor a macroscopic modular orthomodular lattice. The transition of the intermediate layer can be implemented as $\delta_{146}(q, (c, d)) \neq 0$ with $q = 0, 5$, or as $\delta_{146}(q, (c, d)) = 2$, $q = 1, 4, 7$. The former operation is

$$\left(\Sigma_{j=-2}^{2} c_{i+j}^{t}\right) = 11 \text{ or } 1 \Rightarrow c_{i+1}^{t+1} = 1,$$
$$\left(\Sigma_{j=-2}^{2} c_{i+j}^{t}\right) = 26 \text{ or } 16 \Rightarrow c_{i-1}^{t+1} = 1,$$
$$\left(\Sigma_{j=-2}^{2} c_{i+j}^{t}\right) = 27 \text{ or } 17 \Rightarrow c_{i-1}^{t+1} = 1, \ c_{i+1}^{t+1} = 1.$$

Thus, the dynamics of a class 4 automaton originating from rule 146 are described as follows: (*i*) the transition of rule 146 is applied to a binary sequence, (*ii*) the transition of the intermediate layer is applied to a binary sequence if the neighbors with radii = 2 satisfy the conditions for the intermediate layer.

Figure 8 shows the development of such dynamics with both class 3 behaviors and periodic patterns featuring gliders.

We apply the same recipe to other class 3 ECAs, in which elements of the binary relation between P / R_{00} and $P / R_{\Sigma-00}$ are erased and added. This results in class 4 behavior, while the dynamics contain a discrepancy between the microscopic and macroscopic phases (Figure 9). This is especially true for rule 54 where there are six possibilities to realize a virtual lattice. All six intermediate transitions resulting from the six possibilities exhibit class 4 behavior.

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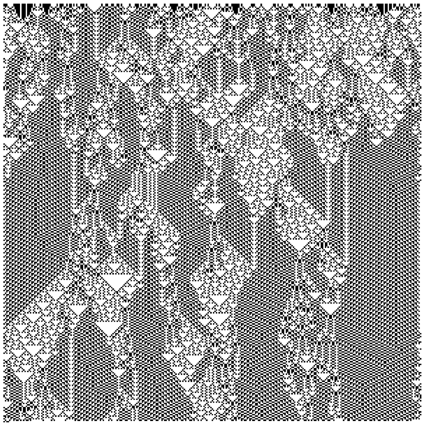

Figure 8. Time development of rule 146 with the transition for the intermediate layer.

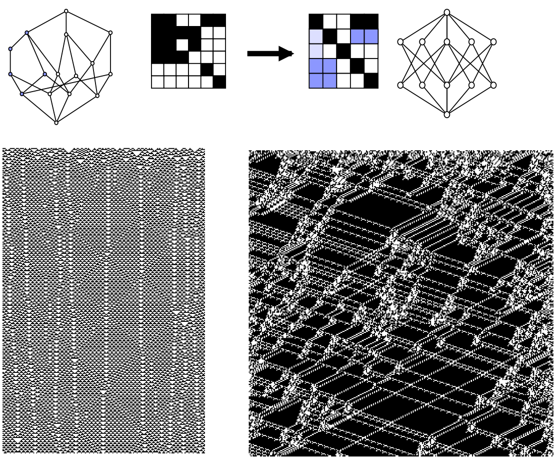

Figure 9. Time development of rule 54 with the transition for the intermediate layer. Some elements of the binary relation between P / R_{00} and $P / R_{\Sigma-00}$ are erased (pale blue cells), and some are added (blue cells).

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6. Conclusion

We propose a recipe for preparing class 4 automata as heterarchies. Since the recipe is based on lattice analysis, first we review our lattice construction derived from a rough set. Next, we classify all elementary cellular automata (ECAs) with respect to our lattice analysis. An ECA transition is re-interpreted as a transition whose state is a triplet that gets divided into two transitions depending on the input. These two transitions result in two equivalence relations, and consequently it is possible to define approximation operators and a lattice of ECAs.

Estimating lattice analysis for ECAs, we obtain the result that classes 1 and 2 yield a distributive lattice and class 3 CAs yield a nondistributive lattice. This is consistent with our assumption that classes 1 and 2 can be reduced to combinations of simple local patterns (where the distributive law holds) while class 3 cannot be reduced in such a way, and the patterns are continuously mixed as time progresses. Subsequently, we construct class 4 automata by superimposing the transition for triplets on the class 3 transition. Since the transition for triplets is not concordant with the ECA transition, it introduces a discrepancy between the microscopic and macroscopic layers, which can result in class 4 behavior.

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