The Fixed String of Elementary Cellular Automata

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The concepts of a fixed string and a skew fixed string greatly affect a cellular automaton's evolution if its initial configuration contains one or both. Interesting properties are proved using the theory of symbolic dynamics and formal languages. After that, a natural classification of elementary cellular automata is introduced and corresponding classification tables are provided.

1. Introduction

Cellular automata (CAs), originating from von Neumann's work on formalizing the self-reproductivity phenomena in living systems, are classes of mathematical systems consisting of a regular lattice of sites and characterized by discreteness (in space, time, and state values), determinism, and local interaction [1]. CAs have been widely used to model a variety of dynamical systems in physics, biology, chemistry, and computer science [2]. Though their local rules are simple, CAs can display a rich and complex evolution [3].

A one-dimensional CA consists of a double infinite line of sites whose values are taken from an alphabet, that is, a finite set of symbols $A_k = \{0, 1, \ldots, k-1\}$. The symbols of each site update synchronously according to a function of the values of the neighboring sites at the previous time step. The general form of a one-dimensional CA is given by

$$f : A_k^{2r+1} \longrightarrow A_k,$$

$$x_i^{t+1} = f\left(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t\right),$$

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where x_i^t denotes the value of site i at time t, f represents the local rule defining the automaton, and r is a non-negative integer specifying the radius of the rule. Therefore, f can induce a function $G : A_k^Z \longrightarrow A_k^Z$,

$$(G(x))_i = f(x_{i-r}\, x_{i-r+1} \ldots x_i \ldots x_{i+r})$$

where $x = \cdots x_{-2}\, x_{-1}\, x_0\, x_1\, x_2 \cdots \in A_k^Z$ is a double infinite symbol sequence. We call x the *configuration* and G the *global rule* of a CA. The simplest CAs are those with alphabet $k = 2$ and $r = 1$, which were named by Wolfram as *elementary* CAs [4, 5].

In this paper, we outline several influencing factors on the evolution of elementary cellular automata (ECAs), and introduce the useful conceptions of fixed string and skew fixed string. Then, some properties of the concepts are discussed. Finally, we give a simple classification of all ECAs.

This paper is organized as follows. In Section 2, we outline some useful notations that are used later. Fixed string and skew fixed string are introduced and discussed in Sections 3 and 4, respectively. In Section 5, we categorize all ECAs into four classes, and detailed characteristics and properties of each class are discussed. A generalized conception of fixed string (skew fixed string) is also put forward in Section 5.

2. Notations

An ECA is a one-dimensional CA with binary states $A = \{0, 1\}$ and $r = 1$. There are a total of $2^{2^3} = 256$ ECAs. For convenience, the 256 ECAs can be numbered by a non-negative integer among 0 to 255 [4]. Let ϵ be the empty string, that is, the string consisting of zero symbols. We use f as the local rule of the CA and extend its domain to A^* as follows:

$$c_1\, c_2 \ldots c_m \mapsto \begin{cases} \epsilon, & (m \leq 2); \\ f\,(c_1\, c_2\, c_3)\, f(c_2\, c_3\, c_4) \cdots f(c_{m-2}\, c_{m-1}\, c_m), & (m \geq 2\, r + 1). \end{cases}$$

where $c_i \in A(1 \leq i \leq m)$. That is to say, any string (finite configuration) can be mapped by f.

The set consisting of all finite strings over an alphabet set A is denoted by A^*. Any subset of A^* is called a *formal language* (or language) over A. Refer to [6, 7] for formal language theory and some important results concerning regular expressions and regular languages that appear later in this paper. Some theorems will be expressed using formal language theory.

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$|x|$ is denoted as the length of the string, that is, the number of all symbols of x; for example, if $x = a_1 a_2 \ldots a_n$, the length of x is $|x| = n$. Of course, the length of the empty string ϵ is zero. Clearly, $|x| - |f(x)| = 2$, if $|x| \geq 2$. An operator π on nonempty strings is defined in this paper as the operator where πx $(x \pi)$ is the string obtained from x by removing its first (last) symbol.

3. Fixed String

The influencing area of one site in a configuration gets increasingly larger as an ECA evolves. The influencing area can form into an isosceles triangle by the local rule (see Figure 1(a)). In general, there is the public part between the two triangular areas for two sites. The interaction property can lead to complicated behaviors of the ECA despite the fact that the rule is simple and local. On the other hand, the site in the evolutive configuration has the reversal triangle in which each line can determine it (see Figure 1(b)). However, both the influencing area and determinable area can be changed if the given ECA has some special properties. The examples can be found through the concept of a fixed string that some authors call wall, partition, or barrier [8, 9].

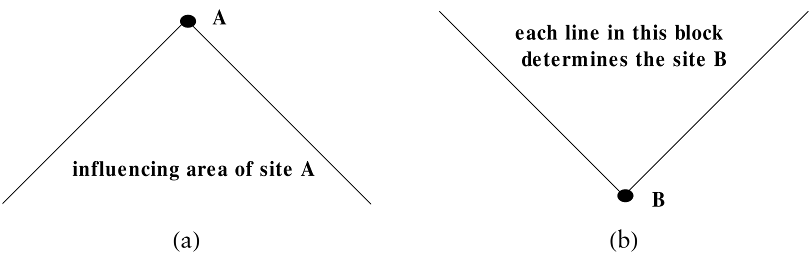

Figure 1. (a) The influencing area and (b) the determinable area of a site in a configuration.

Definition 1. Let x be a nonempty string. We call x a *fixed string* of a CA with map f if $f(a\,x\,b) = x$, for every a and $b \in A = \{0, 1\}$. If an ECA has a fixed string, we also call it fixed.

If a fixed string exists in a configuration, an upright cylinder will form like a wall that remains fixed as the configuration evolves, whatever the states of other sites in its neighborhood may change into. Furthermore, the wall divides the evolution area of the configuration into two parts that are not mutually affected. So, the influencing and determinable areas of the site between the two fixed strings have both changed (see Figure 2). If two fixed strings exist in one configuration,

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the influencing and determinable areas will both become a band (see Figure 3). Since the string in the band will evolve like a finite CA with fixed boundary conditions, it will become eventually periodic, showing that the fixed string can greatly influence the evolution of a CA. To describe the details, we first define the vertical map of an ECA.

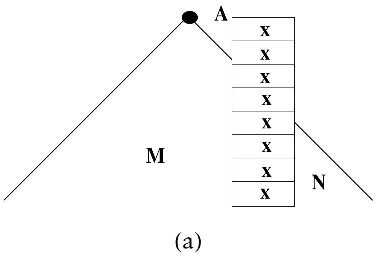

Figure 2. (a) The influencing area and (b) the determinable area of a site in a configuration has changed, if x is a fixed string.

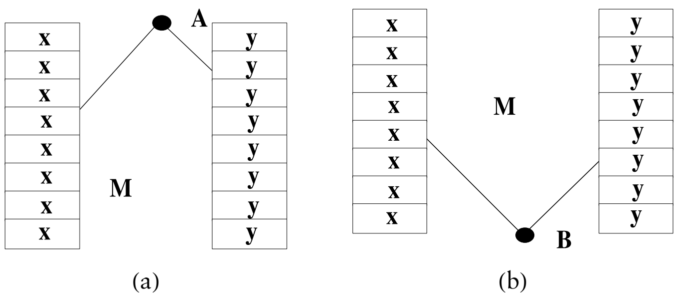

Figure 3. (a) The influencing area and (b) the determinable area of a site in a configuration both become bands when x and y are both fixed strings.

Definition 2. Let a be the last symbol of a fixed string x and b the first symbol of another fixed string y. Then the *vertical map* $V_{a\,b}$ of an ECA with local rule f between x and y is defined as $V_{a\,b}(z) = f(a\,z\,b)$.

Then, Theorem 1 holds.

Theorem 1. Let $V_{a\,b}$ be a vertical map of an ECA and $z_{i+1} = V_{a\,b}(z_i)\,(i = 1, 2, \ldots)$. Then the sequence $\{z_i\}_{i=1}^{\infty}$ is eventually periodic.

Now we will discuss some properties of fixed strings.

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Definition 3. For any $b \in A$, if a string $a_1 a_2$ with length 2 satisfies $f(b a_1 a_2) = a_1 (f(a_1 a_2 b) = a_2)$, then we say that $a_1 a_2$ is *left* (*right*) fixed.

According to Definition 1, Lemmas 1 and 2 hold.

Lemma 1. $a_1 a_2 \in A^2$ is a fixed string if and only if $a_1 a_2$ is both left and right fixed.

By Lemma 1, the string 00 is fixed if and only if the local rule of the ECA satisfies $000, 001, 100 \to 0$, that is, $f(000) = f(001) = f(100) = 0$.

Lemma 2. Let $|x| > 2$; x is fixed if and only if x satisfies the following three conditions:

1. the prefix of x with length 2 is left fixed;
2. the suffix of x with length 2 is right fixed; and
3. $f(x) = \pi x \pi$.

Lemmas 1 and 2 give us an easy way to check whether the given string is fixed or not.

Lemma 3. If 00 is the prefix (suffix) of some fixed string, 00 is also fixed.

Proof. We suppose that the fixed string $x = 00\,y$. Then, the local rule of an ECA should satisfy $000 \to 0$ and $100 \to 0$ because 00 is left fixed. If $y = 0^n$ $(n \geq 0)$, 00 must be right fixed. Therefore, 00 is fixed. If $y = 0^n\,1\,z$, the rule should satisfy $001 \to 0$. Then, 00 is fixed as well. □

Similarly, Lemmas 4 and 5 also hold.

Lemma 4. If 11 is the prefix (suffix) of some fixed string, 11 is also fixed.

Lemma 5. If 01 is the prefix and suffix of some fixed string, 01 is also fixed. The same is true for the string 10.

Theorem 2. If fixed strings exist for an ECA, then the length of the shortest fixed string (S-FS) for this ECA is not more than 4.

Proof. Let x be a fixed string and $|x| > 4$. Using Lemmas 4 and 5, we may suppose $x = 01\,y\,10$, where $|y| > 0$. Therefore, 01 is left fixed, 10 is right fixed, and $f(01\,y\,10) = 1\,y\,1$. If 0 is the prefix or suffix of y, then $f(010) = 1$, showing that 010 is fixed. If 1 is both the prefix and suffix of y, then $f(011) = 1$ and $f(110) = 1$. The string 0110 leads to be fixed. □

Theorem 2 gives us an easy and effective method for checking whether a given ECA has fixed strings or not.

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Theorem 3. For a given ECA, we let $F = \{x \mid x \text{ is a fixed string}\}$; then F is a regular language.

The result shows that the set consisting of all the fixed strings forms a regular language that is the simplest one in Chomsky's hierarchy [6]. To prove Theorem 3 we need two tools from the theory of formal languages: the natural equivalence relation R_L and the Myhill–Nerode theorem [6].

Definition 4. If a language $L \subset A^*$ is given, where A is the alphabet, then a *natural equivalence relation* R_L is introduced by L into A^*: for $x, y \in A^*$, $x R_L y$ holds if and only if for any $z \in A^*$, $x z \in L$ exactly when $y z \in L$.

Myhill–Nerode theorem. The language L is regular if and only if R_L is of finite index, that is, the number of equivalence classes of R_L is finite.

Proof of Theorem 3. We divide the A^* into the following eight sets: $A_1 = \{\epsilon\}$, $A_2 = \{0\}$, $A_3 = \{1\}$, $A_4 = \{|x| \geq 2 \mid f(x) \neq \pi x \pi$ or the prefix of x with length 2 is not left fixed$\}$, $A_5 = \{|x| \geq 2 \mid f(x) = \pi x \pi$, the prefix of x with length 2 is left fixed and 00 is the suffix of $x\}$, $A_6 = \{|x| \geq 2 \mid f(x) = \pi x \pi$, the prefix of x with length 2 is left fixed and 01 is the suffix of $x\}$, $A_7 = \{|x| \geq 2 \mid f(x) = \pi x \pi$, the prefix of x with length 2 is left fixed and 10 is the suffix of $x\}$, $A_8 = \{|x| \geq 2 \mid f(x) = \pi x \pi$, the prefix of x with length 2 is left fixed and 11 is the suffix of $x\}$. We will prove that any two strings from the same set are equivalent. It is trivial for A_1, A_2, and A_3 because they are single element sets. Let $x, y \in A_4$; for any z, the strings $x z$ and $y z$ are both not fixed according to the definition of A_4. So all the strings in A_4 belong to one equivalence class. Let $x, y \in A_5$; for any z, if $x z$ is fixed, then $f(x z) = \pi x z \pi$ and the suffix of $x z$ with length 2 is right fixed. Therefore, $f(y z) = \pi y z \pi$ and the suffix of $y z$ with length 2 is also right fixed; hence, $y z$ is also fixed. If $y z$ is fixed, $x z$ would also be fixed by a similar method. Therefore, all the strings in A_5 also belong to one equivalence class, as do A_6, A_7, and A_8 by a similar discussion. Hence, the number of equivalence classes of R_F is at most 8, which shows that the language F is regular. □

Definition 5. Let $x_1 = y_1 w$ and $x_2 = w y_2$ be two strings, where w can be the empty string, while x_1 and y_1 are nonempty strings. Then, string $y_1 w y_2$ is called a *generalized combination*.

There may be more than one generalized combination of the string x and y. For example, if $x = 010 = y$, all the generalized combinations of x and y are 01010 and 010010. Lemma 6 is self-evident.

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Lemma 6. Let x and y be two fixed strings of one ECA; then any generalized combination of x and y is also fixed.

Now we define the generalized combination of finite strings.

Definition 6. Let $x_1, x_2, \ldots x_n$ be nonempty strings that can be written as

$$x_i = w_i\, y_i\, w_{i+1}, \qquad (i = 1, 2, \ldots, n).$$

Then $x = w_1\, y_1\, w_2\, y_2\, w_3 \ldots w_n\, y_n\, w_{n+1}$ is a generalized combination of $x_1, x_2, \ldots x_n$.

Definition 7. Let a language L be a set $L = \{x \mid x$ is a nonempty string and is a generalized combination of any finite strings coming from the strings $y_1, y_2, \ldots, y_n\}$; then we call $y_1, y_2, \ldots, y_n$ the *generators* (GEs) of language L and L is denoted by $(y_1 + y_2 + \cdots + y_n)^{\ddagger}$.

For example, if $L = \{0^n \mid n > 0\}$ and its regular expression is $0^{\dagger}$, it only has one GE, that is, 0. If $L = \{0^n \mid n > 1\}$ and its regular expression is $00^{\dagger}$, then 00 is the only GE.

Theorem 4. Let $x_1, x_2, \ldots, x_n$ be fixed strings; then any element in $(x_1 + x_2 + \cdots + x_n)^{\ddagger}$ is also fixed.

It is convenient to express the fixed string set by GEs. Examples are given in Section 5.

4. Skew Fixed String

Another kind of string that also influences the evolution of an ECA is the skew fixed string.

Definition 8. Let x be a nonempty string. If $f(a\, b\, x) = x$, for every a and $b \in A$, we call x a *left skew* fixed string of a CA with map f. If $f(x\, a\, b) = x$, for every a and $b \in A$, we call x a *right skew* fixed string of a CA with map f. If an ECA has a left (right) skew fixed string, we also call it left (right) skew fixed or simply skew fixed.

When a configuration with a skew fixed string evolves, the skew string will form into a skew wall (Figure 4). Clearly, the skew fixed string also affects the influencing and determinable areas. Typical cases are shown in Figure 5. The property of a skew fixed string is simpler than a fixed string's.

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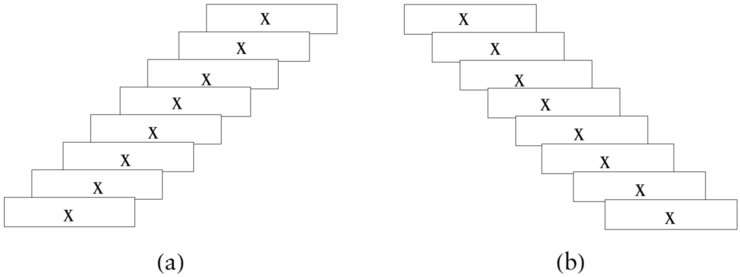

Figure 4. (a) Left skew fixed string forms a skew wall. (b) Right skew fixed string forms a skew wall.

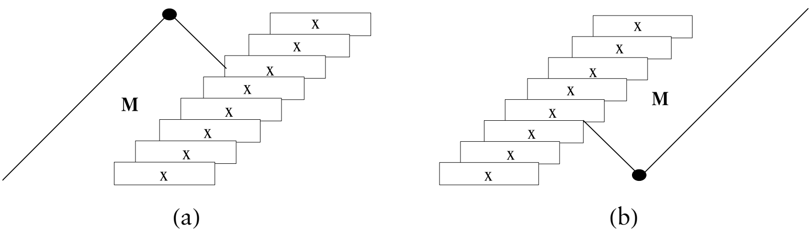

Figure 5. (a) The influencing area and (b) the determinable area when the initial configuration contains a left skew fixed string.

Theorem 5. If an ECA has skew fixed strings, then the length of the shortest string is 1.

Proof. First, note that if $f(x) = y$, the equation $f(x\pi) = y\pi$ holds. Therefore, if x is a left skew string, all the nonempty prefixes of x are left skew fixed strings. □

It is easy to judge whether an ECA has skew fixed strings or not by Theorem 5. Similar to the fixed string, Theorems 6 and 7 about the skew fixed string also hold.

Theorem 6. For a given ECA, we let $S = \{x \mid x$ is a left (right) fixed string$\}$; then S is a regular language.

Theorem 7. Let $x_1, x_2, \ldots, x_n$ be skew fixed strings; then any element in $(x_1 + x_2 + \cdots + x_n)^{\ddagger}$ is also skew fixed.

If a is left skew fixed, so is $a\,a$. Every string at the right side of $a\,a$ will evolve as shown in Figure 6. For convenience, we define the left skew map of an ECA.

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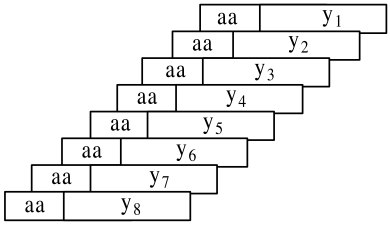

Figure 6. The evolution of string y_1 with a left side that is a left skew fixed string $a\,a$.

Definition 9. If an ECA has a left skew fixed string a with local rule f, the *left skew map* of f is defined as $L_{a\,a}(y) = f(a\,a\,y)$ for any $y \in A^*$.

Similarly, we can define a right skew map $R_{a\,a}$ for an ECA with the right skew fixed string $a\,a$. According to the definition, we have $L_{a\,a}(y_i) = y_{i+1}$ as shown in Figure 6.

Theorem 8. Let an ECA have a left skew fixed string a with local rule f. For any $y_0 \in A^*$, the sequence $\{y_i\}_{i=0}^{\infty}$ that is generated by $L_{a\,a}(y_i) = y_{i+1}$, $(i = 0, 1, \ldots)$ is eventually periodic and the period is one.

Proof. We prove the second part because the first part is obvious. We assume that $a = 0$ is the left skew fixed string, which means that

$$000,\ 010,\ 100,\ 110 \to 0.$$

Let $|y_0| = n$. Now we use mathematical induction on n.

1. When $n = 0$, that is, $y_0 = \epsilon$, the result holds.

2. Suppose that when $n = k$, the result holds; then when $n = k + 1$, let $y_0 = z_0\,b_0$, where $|z_0| = k$ and $|b_0| = 1$. By induction, assume that there exists an N, such that $z_N = z_{N+1} = \cdots$, when $i \geqslant N$. Denote $c_1\,c_2$ as the suffix of $00\,z_N$. If $b_N = 0$, which is skew fixed, the $y_N = y_{N+1} = \cdots$ holds, meaning that the sequence $\{y_i\}$ is eventually fixed. If $b_N = 1$, we will have $y_N = y_{N+1} = \cdots$ when $f(c_1\,c_2\,1) = 1$ or $y_{N+1} = y_{N+2} = \cdots$ when $f(c_1\,c_2\,1) = 0$. These both show the result is correct. Hence, we have completed the proof. □

5. A Simple Classification of Elementary Cellular Automata

The problem of classifying CAs is a basic one among the 20 problems that were posed by Wolfram in [10]. Using the concepts of fixed and skew string, we give a natural classification of the ECAs in Table 1.

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Class A: Fixed and Skew Fixed	Class B: Skew Fixed Only	Class C: Fixed Only
0, 8, 32, 40, 128, 136, 160, 168	2, 10, 34, 42, 130, 138, 162, 170	4, 5, 12, 13, 28, 29, 36, 44, 72, 73, 76, 77, 78, 94, 104, 108, 132, 140, 156, 164, 172, 200, 204, 232

Class D: Neither Fixed nor Skew Fixed
1, 3, 6, 7, 9, 11, 14, 15, 18, 19, 22, 23, 24, 25, 26, 27, 30, 33, 35, 37, 38, 41, 43, 45, 46, 50, 51, 54, 56, 57, 58, 60, 62, 74, 90, 105, 106, 110, 122, 126, 134, 142, 146, 150, 152, 154, 178, 184

Table 1. The classification of the ECAs.

Class A contains the ECAs that are both fixed and skew fixed. Class B contains the ECAs that only have skew fixed strings. Class C contains the ECAs that only have fixed strings. Class D contains the ECAs that are neither fixed nor skew fixed. By Theorems 2 and 5, it is easy to judge which class any ECA belongs to. There are 256 ECAs, and many of them are mutually equivalent [5]. So the number 256 can be reduced to 88. We now discuss each class.

5.1 Class A: Fixed and Skew Fixed

There are eight ECAs in Class A and each one has both fixed and skew fixed strings. Their typical evolution can be seen in Figure 7.

(a) (b)

Figure 7. The evolution of ECAs (a) 160 and (b) 168, which both have fixed and left skew fixed strings.

By Theorems 3 and 6, all the fixed strings and skew fixed strings are easy to find and each of the two sets can be expressed by a regular expression or generalized combinations of some GEs (see Table 2 for detailed information). All of the results in Table 2 should be proved rigorously. Proposition 1 helps us to know why the information contained in Table 2 is correct.

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Rule Number	S-FS	GE of FS	All the FS	S-SFS	GE of SFS	All the SFS
0	0	0	$0^\dagger$	0	0	$0^\dagger$
8	0	0	$0^\dagger$	0	0	$0^\dagger$
32	00	00	$00^\dagger$	0	0	$0^\dagger$
40	00	00	$00^\dagger$	0	0	$0^\dagger$
128	0	0	$0^\dagger$	0	0	$0^\dagger$
136	0	0	$0^\dagger$	0	0	$0^\dagger$
160	00	00	$00^\dagger$	0	0	$0^\dagger$
168	00	00	$00^\dagger$	0	0	$0^\dagger$

Table 2. The fixed strings (FS) and skew fixed strings (SFS) of the eight ECAs that comprise Class A.

Proposition 1. Suppose that an ECA has fixed strings and skew fixed strings and $a \in A$ is the S-FS; then (1) $a\,a$ is fixed, (2) $a^\dagger$ is the regular expression of all the skew fixed strings, and (3) either $a^\dagger$ or $a\,a^\dagger$ is the regular expression of all the fixed strings.

Proof. We might as well let the ECAs have left skew fixed strings. If $|x| = n \quad (n \geq 1)$ is the fixed string, we will prove that $x = a^n$. Let $y = a^n$ be the left skew string. Both x and y appear in the initial configuration (see Figure 8). Several time steps later, the fixed string x and skew fixed string y will meet. However, they both should remain fixed and this leads to $x = y = a^n$. By Lemmas 3 and 4, we know $a\,a$ must be a fixed string. Therefore, if a is the S-FS, all the fixed strings can be written as $a^\dagger$; if $a\,a$ is the S-FS, all the fixed strings can be written as $a\,a^\dagger$. □

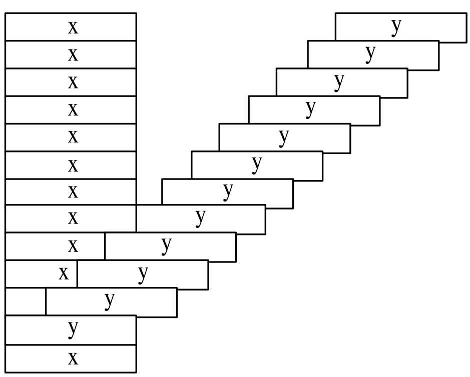

Figure 8. If x is a fixed string and y is a skew fixed string, they will influence each other.

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Proposition 2. (1) ECAs 0 and 8 will eventually evolve to 0^∞ for any initial configuration.

(2) ECAs 128 and 136 will eventually evolve to 1^∞ if the initial configuration is 1^∞ and to 0^∞ otherwise.

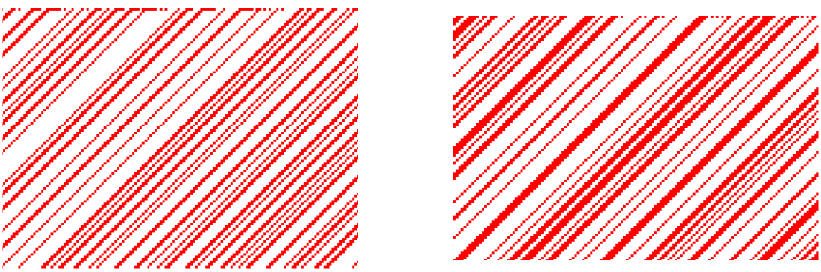
(3) ECAs 32 and 40 will eventually evolve to $(01)^\infty$ if the initial configuration is $(01)^\infty$ and to 0^∞ otherwise.

(4) ECAs 160 and 168 will eventually evolve to $(01)^\infty$ if the initial configuration is $(01)^\infty$, to 1^∞ if the initial configuration is 1^∞, and to 0^∞ otherwise.

Proposition 2 says that all the ECAs in Class A will evolve to a spatially homogeneous state from almost every initial configuration, that is, every site is in the same state, which is exactly the Class I in Wolfram's classification scheme.

5.2 Class B: Skew Fixed Only

There are eight ECAs in Class B and each one is skew fixed. Their typical evolution can be seen in Figure 9. Detailed information about Class B is listed in Table 3. The proofs of the results in Table 3 are easy so we just prove the result for ECA 138 as our typical example.



(a) (b)

Figure 9. The evolution of ECAs (a) 10 and (b) 138, which only have left skew fixed strings.

Proof of the result of ECA 138 in Table 3. The local rule of ECA 138 is defined as 001, 011, 111 $\to$ 1, others $\to$ 0. On the one hand, it is easy to verify that 0 and 001^n $(n > 0)$ are all left skew fixed. On the other hand, we will show any left fixed string $|x| > 1$ can be written as the generalized combination of strings 0 and 001^n $(n > 0)$, that is to say, 00 is a prefix of x and 101 is not a substring of x. Note that all the prefixes of a left skew fixed string are also left skew fixed. So, 00 must be the prefix of x if $|x| > 1$. Because 101 $\to$ 0, 101 cannot appear in the string x. □

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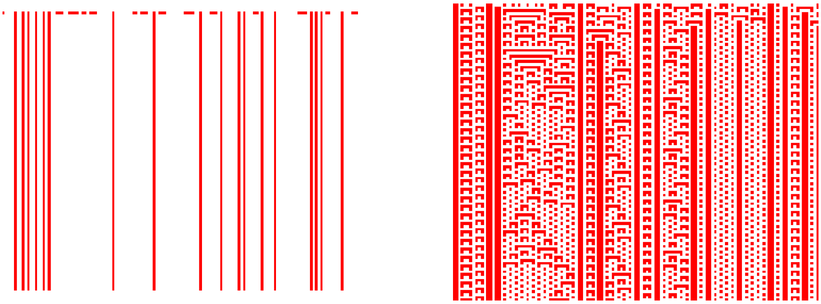
Rule Number	S-SFS	GEs	All the SFS
2	0	0, 001	$0^{\dagger}\left(010^{\dagger}\right)^{*}(\epsilon+01)$
10	0	0, 001, 0011	$0^{\dagger}\left(0\,(1+11)\,0^{\dagger}\right)^{*}(\epsilon+01+011)$
34	0	0, 01	$0^{\dagger}\left(10^{\dagger}\right)^{*}(\epsilon+1)$
42	0	0, 01, 011	$0^{\dagger}\left((1+11)\,0^{\dagger}\right)^{*}(\epsilon+1+11)$
130	0	0, 001	$0^{\dagger}\left(010^{\dagger}\right)^{*}(\epsilon+01)$
138	0	0, 001^{n} $(n>0)$	$0^{\dagger}\left(01^{\dagger}\,0^{\dagger}\right)^{*}(\epsilon+01)$
162	0	0,01	$0^{\dagger}\left(10^{\dagger}\right)^{*}(\epsilon+1)$
170	0,1	0, 1	$(0+1)^{\dagger}$

Table 3. The skew fixed strings (SFS) of the eight ECAs that comprise Class B.

By Theorem 8, we know the evolution of every configuration for any ECA in Class B will eventually be a (left) shift.

5.3 Class C: Fixed Only

There are 24 ECAs in Class C and each one is not skew fixed but fixed. Their typical evolution can be seen in Figure 10. The S-FS of ECAs in Class C can be seen in Table 4, in which the GEs and regular expressions are also listed. Theoretically, all these results should be proved rigorously one by one. However, these proofs are easy and therefore we just select the proof for ECA 73 as our typical example.



(a) (b)

Figure 10. The evolution of ECAs (a) 4 and (b) 73, which only have fixed strings.

Rule Number	S-FS	GEs	Regular Expression of FS
4	0	0, 010	$0^{\dagger}\left(10^{\dagger}\right)^{*}$
5	010	010	$010\,(10+010)^{*}$
12	0	0, 01	$(0+01)^{\dagger}$
13	01	01, 010	$01\,(0+01)^{*}$
28	01	01	$(01)^{\dagger}$
29	01	01	$(01)^{\dagger}$
36	00	00, 00100	$00^{\dagger}\left(100^{\dagger}\right)^{*}$
44	00	00, 001	$00^{\dagger}\left(1+00^{\dagger}\right)^{*}$
72	0	0, 0110	$0^{\dagger}\left(110^{\dagger}\right)^{*}$
73	0110	0110	$0110\,(0110+110)^{*}$
76	0	0, 01, 10	$(0+01+10+101)^{\dagger}$
77	01,10	01, 10	$(01+10+010+101)^{\dagger}$
78	10	10, 101	$(10+101)^{\dagger}$
94	101	101	$101\,(101+01)^{*}$
104	00	00, 001100	$00^{\dagger}\left(1100^{\dagger}\right)^{*}$
108	00	00, 100, 001	$(00+100+001+000+0001+1000+1001)^{\dagger}$
132	0	0, 010	$0^{\dagger}\left(10^{\dagger}\right)^{*}$
140	0	0, 01	$(0+01)^{\dagger}$
156	01	01	$(01)^{\dagger}$
164	00	00, 00100	$00^{\dagger}\left(100^{\dagger}\right)^{*}$
172	00	00, 001	$00^{\dagger}\left(100^{\dagger}\right)^{*}(\epsilon+1)$
200	0	0, 11	$(0+11+111)^{\dagger}$
204	0,1	0, 1	$(0+1)^{\dagger}$
232	00,11	00,11	$\left(00^{\dagger}+11^{\dagger}\right)^{\dagger}$

Table 4. The fixed strings (FS) of the 24 ECAs that comprise Class C.

Proof of the results of ECA 73 in Table 4. The local rule of ECA 73 is defined as

$$110,\ 011,\ 000 \to 1,\ \text{others} \to 0.$$

It is easy to verify that 0110 is the S-FS. Now, by Theorem 4, we will prove that any fixed string y can be written as the generalized combination of finite 0110s. To begin with, only 01 may be left fixed and 10 may be right fixed. Therefore, 01 should be its prefix and 10 the suffix. Then, because $000 \to 1$, $010 \to 0$, $111 \to 0$, that is, the

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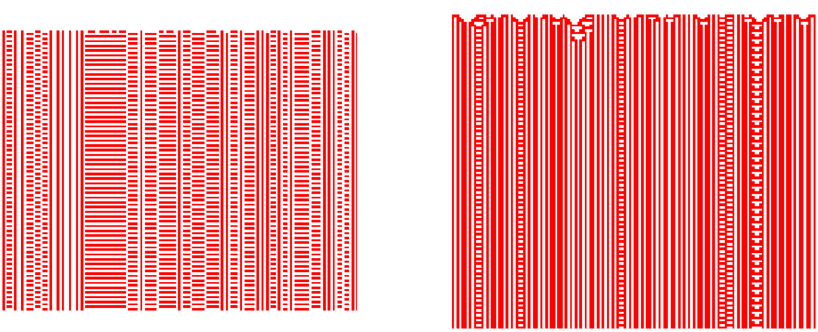
three strings do not satisfy $f(x) = \pi x \pi$, the three strings cannot appear in the fixed string y. Therefore, y must be written as the generalized combination of finite 0110s. □

By Theorem 1, any string between two fixed strings will eventually evolve to periodic sequences and the period may be 1 or the other positive integers. We can categorize Class C into three subclasses according to their different periods (see Table 5 for details).

Subclass C_1: Period=1	Subclass C_2: Period=1 or 2	Subclass C_3: Period=1, 2, 3, …
4, 12, 13, 36, 44, 72, 76, 77, 78, 104, 132, 140, 164, 172, 200, 204, 232	5, 28, 29 108, 156	94, 73

Table 5. The three subclasses of Class C.

Figure 10(a) shows the evolution of ECA 4, which is in Subclass C_1 and Figure 10(b) shows ECA 73, belonging to Subclass C_3. The typical evolution of an ECA Subclass C_2 can be seen in Figure 11(a). These proofs of the properties of Subclasses C_1 and C_2 are not difficult but are lengthy if we prove them one by one. Because of this, they are omitted. By computer search for ECA 94, we find that the strings 10100101 and 10111101 on the vertical map are periodic with period 2, while strings 1011001101, 1010000101, and 1011111101 have period 3. It can be proved that all the periods are 1, 2, 3, and 6. Similarly, for ECA 73, the period of strings 011000110 and 011010110 is 2, the period of string 0110111111110110 is 5, and the period of string $01100^9\,0110$ is 18. Clearly, the evolution is more complicated than for ECA 94. Knowing whether or not the set consisting of all the periods is finite is still an open question.



(a) (b)

Figure 11. The evolution of ECAs (a) 5 and (b) 94, which belong to Classes C_2 and C_3, respectively.

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An interesting study on ECA 73 is to predict the appearance of the fixed string 0110 (see Table 6). This is equivalent to studying the preimages of 0110. For a given string $x \in A^*$, let $P_k(x) = \{y \mid f^k(y) = x\}$ be the set containing all the k-step preimages of x. Let $n_k(x) = \#(P_k(x))$ be the cardinality of $P_k(x)$. Using computer search, we get $n_k(0110)$ for $k = 1, 2, \ldots, 14$.

k	1	2	3	4	5	6	7
n_k	5	25	103	426	1764	7180	29 606
n_{k+1} / n_k	5	4.12	4.1359	4.1408	4.0703	4.1234	4.0746

k	8	9	10	11
n_k	120 632	488 419	1 987 973	8 019 142
n_{k+1} / n_k	4.0488	4.0702	4.0338	4.0303

k	12	13	14
n_k	32 319 505	130 031 847	523 421 555
n_{k+1} / n_k	4.0233	4.0253	

Table 6. The number of 0110 k-step preimages for ECA 73.

The set $P_k(0110)$ is rather complex if $k \geq 3$. Because 0110 is fixed, the set $\{x_1\, 0110\, x_2 \mid x_1, x_2 \in A^k\}$ must be the subset of $P_k(0110)$. Therefore, $n_k(0110) \geq 4^k$. This leads to $\lim_{k \to \infty} n_{k+1} / n_k = 4$. The string $10^{4\,k}\,1$ will evolve to 0110 after $2\,k - 1$ steps. Table 7 gives the $c_k = n_1\left(10^{4\,k}\,1\right)$ found through computer search.

k	1	2	3	4	5	6
c_k	5	14	40	122	373	1147
c_{k+1} / c_k	2.8	2.8571429	3.05	3.0573770	3.0750670	3.0775937

k	7	8	9	10	11
c_k	3530	10869	33470	103072	317418
c_{k+1} / c_k	3.0790368	3.0794001	3.0795339	3.0795754	3.0795890

k	12	13	14	15
c_k	977517	3010355	9270674	28549925
c_{k+1} / c_k	3.0795935	3.0795949	3.0795954	3.0795956

Table 7. The number of $10^{4\,k}\,1$ preimages for ECA 73.

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We can prove that $\lim_{k\to\infty} n_1\left(10^{4\,(k+1)}\,1\right) / n_1\left(10^{4\,k}\,1\right) = \lambda^4$, where $\lambda = \left(\left(9 - \sqrt{69}\right) / 18\right)^{1/3} + \left(\left(9 + \sqrt{69}\right) / 18\right)^{1/3}$ is the real root of the equation $\lambda^3 = \lambda + 1$.

The double fixed string 01100110 is strange, because its k-step preimage set $P_k(01100110)$ is exactly $\left\{x_1\, 01100110\, x_2 \mid x_1, x_2 \in A^k\right\}$, which means the fixed string 01100110 cannot be produced. Certainly, $n_k(01100110) = 4^k$.

5.4 Class D: Neither Fixed nor Skew Fixed

There are 48 ECAs in Class D and each one has neither fixed strings nor skew fixed strings (see Table 1). Their typical evolution can be seen in Figure 12. From the evolution of these 48 ECAs, we find that there are still some simple ones, such as ECA 51, which are neither fixed strings nor skew strings. However, for any x, $f^2(x) = \pi\, x\, \pi$. So it has a very simple dynamic behavior, that is, every initial configuration will evolve to a periodic orbit with period 2. Another example is ECA 15; for any x, it satisfies $f^2(x) = \pi\, \pi\, x$. Its dynamic behavior is also simple (Figure 13). So we categorize Class D into several subclasses. Now we introduce a new definition whose special cases are fixed and skew fixed strings.

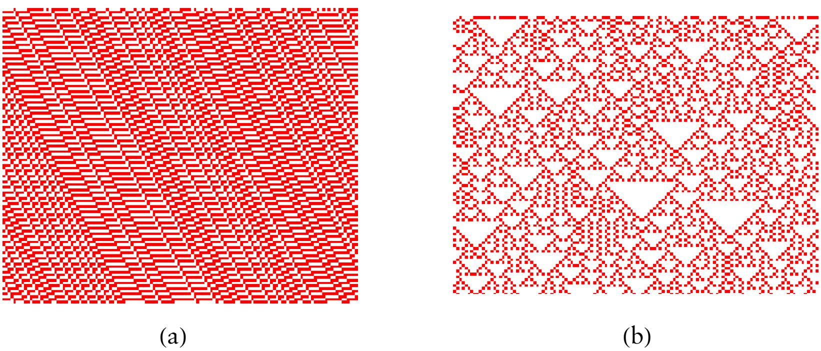

Figure 12. The evolution of ECAs (a) 27 and (b) 18, which are both in Class D.

Definition 10. Let x be a nonempty string and $-T \leqslant n \leqslant T$ both be integers. If for any $|x_1| = T - n$ and $|x_2| = T + n$, the equation $f^T(x_1\, x\, x_2) = x$ always holds. Then, x is called a *periodic* fixed string with period T and right shift n (or left shift $-n$, when $n < 0$), or we simply say x is a periodic fixed string with parameters (T, n).

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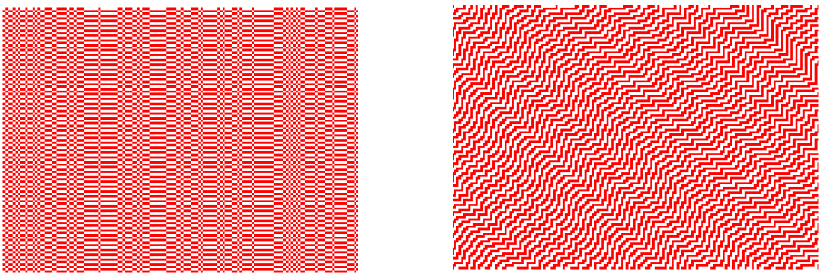
Without special explanations, the period T refers to the minimal one. Clearly, string x being periodic fixed with parameters (1, 0) means that x is fixed. String x being periodic fixed with parameters (1, 1) means that x is right skew fixed. For ECA 51, every nonempty string x is periodic fixed with parameters (2, 0), while for ECA 15, every nonempty string x is also periodic fixed but with parameters (2, 2). By the way, one can easily obtain many results that are similar to (skew) fixed ECA. Now we can divide Class D into two subclasses:

Subclass D_1: 1, 3, 7, 15, 19, 23, 27, 33, 38, 50, 51, 178.

Subclass D_2: 6, 9, 11, 14, 18, 22, 24, 25, 26, 30, 35, 37, 41, 43, 45, 46, 54, 56, 57, 58, 60, 62, 74, 90, 105, 106, 110, 122, 126, 134, 142, 146, 150, 152, 154, 184.



(a) (b)

Figure 13. The evolution of ECAs (a) 51 and (b) 15, which are both in Class D but have simple dynamic behaviors.

Each ECA in Subclass D_1 has periodic fixed strings with period 2, while those in Subclass D_2 do not. Detailed information about periodic fixed strings of Subclass D_1 found by computer search is given in Table 8. For any given ECA in Subclass D_1, the evolution will be eventually periodic if the initial configuration is randomly selected. By computer search, we have not found any periodic fixed string with period 2, 3, 4, 5, and 6 for ECAs in Subclass D_2. But we cannot claim that all the ECAs in Subclass D_2 do not have a periodic fixed string.

As far as the subclass D_2 is concerned, it is too big and we hope it can be divided further. Some of the ECAs have simple dynamic behaviors while others have chaotic behaviors, but we have not yet found a better mathematical warrant.

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Rule Number	Parameter	S-PFS	GEs
1	(2,0)	1	1,000
3	(2,1)	1	1,00
7	(2,1)	11	11,000
15	(2,2)	0,1	0,1
19	(2,0)	00,11	00,11
23	(2,0)	00,11	00,11
27	(2,1)	111	111, 0^n 111, 10 111, 100 111, etc.
33	(2,0)	000	000, 0001^n 000
38	(2,-2)	0	0,001,0011
50	(2,0)	01,10	01,10
51	(2,0)	0,1	0,1
178	(2,0)	01,10	01,10

Table 8. The shortest periodic fixed string (S-PFS) and GEs of Subclass D_1.

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