Exact Calculation of Lyapunov Exponents and Spreading Rates for Rule 40

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In this paper we exactly calculate the Lyapunov exponents and two kinds of spreading rates for elementary cellular automaton (ECA) rule 40. One spreading rate, simply called the spreading rate, is defined by following A. Ilachinski [1], and the other one originally defined in this paper is called the strict spreading rate. For the Lyapunov exponents, we use M. A. Shereshevsky's definition [2]. For an arbitrarily given rational number between $1/2$ and 1, we specifically construct a configuration having the Lyapunov exponent equal to the rational number. For rule 40, the Lyapunov exponent is equal to the spreading rate but not necessarily to the strict spreading rate. For the strict spreading rate, it is proved that for an arbitrarily given real number between $1/2$ and 1, there exists a configuration having the strict spreading rate equal to the given real number. This theorem is proved by construction. These dynamical properties are observed on the set of configurations of a specific type. We formally prove that the Bernoulli measure of this set is 0, which is why these dynamical properties have not been observed in computer simulations.

1. Introductory Preliminaries

Using computer simulations, S. Wolfram [3] has classified cellular automata (CAs) into four classes. He defined class I as the set of CAs that generate space-time patterns that eventually die out for randomly given initial configurations.

This paper, a continuation of F. Ohi [4] in which we showed that elementary cellular automaton (ECA) rule 40 of Wolfram's class I has a Devaney chaotic dynamical subsystem, exactly evaluates the Lyapunov exponents, and evaluates two kinds of spreading rates for ECA rule 40. One spreading rate is defined by following A. Ilachinski [1] and is simply called the spreading rate. The other one is originally defined in this paper and is called the strict spreading rate. For the Lyapunov exponents we use M. A. Shereshevsky's definition [2].

It is shown that for an arbitrarily given rational number between $1/2$ and 1, there exists a configuration having the Lyapunov exponent equal to the rational number. For rule 40, the Lyapunov exponent is equal to the spreading rate, but not necessarily to the strict spreading rate. For the strict spreading rate, it is proved that for an ar-

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bitrarily given real number there exists a configuration having the strict spreading rate equal to the given real number.

These chaotic properties of rule 40 are shown on a set of configurations of a specific type. We also explain why these specific configurations could not be observed in computer simulations by showing that the Bernoulli probability measure of the set is 0.

An ECA is defined to be a tuple $(\{0, 1\}, g)$, where g is a mapping from $\{0, 1\}^3$ to $\{0, 1\}$ and is called a *local transition function*. An ECA is completely determined by g and we simply call it ECA g.

There exist $2^8 = 256$ ECAs and each of them has the unique rule number $\sum_{a,b,c} g(a, b, c)\, 2^{a\,2^2+b\,2+c}$. We write the local transition function having the rule number r as g_r. The local transition function g_{40} of rule 40 is given by Table 1.

(a, b, c)	$(1, 1, 1)$	$(1, 1, 0)$	$(1, 0, 1)$	$(1, 0, 0)$
$g_{40}(a, b, c)$	0	0	1	0
(a, b, c)	$(0, 1, 1)$	$(0, 1, 0)$	$(0, 0, 1)$	$(0, 0, 0)$
$g_{40}(a, b, c)$	1	0	0	0

Table 1.

It is difficult to determine that rule 40 has chaotic properties and diverse Lyapunov exponent values by observing space-time patterns generated by computer simulation for randomly given initial configurations. The patterns typically die out in a finite number of time steps.

An ECA g defines the global transition function $\boldsymbol{g}$ from $\mathcal{A} \equiv \{0, 1\}^{\mathbf{Z}}$ to $\mathcal{A}$ as

$$\boldsymbol{x} = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathcal{A},$$
$$(\boldsymbol{g}(\boldsymbol{x}))_i = g(x_{i-1}, x_i, x_{i+1}), \; i \in \mathbf{Z}.$$

We usually use a bold letter to show the corresponding global transition function. An element of $\mathcal{A}$ is called a *configuration*. $\mathbf{Z}$ is the set of all integers.

The left- and right-shift transformations are written as $\sigma_L : \mathcal{A} \to \mathcal{A}$ and $\sigma_R : \mathcal{A} \to \mathcal{A}$, respectively.

Defining a metric d on $\mathcal{A}$ as

$$\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}, \; d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}},$$

we have a topological dynamical system $(\mathcal{A}, \boldsymbol{g})$ that defines an orbit of an arbitrarily given initial configuration $\boldsymbol{x} \in \mathcal{A}$ as

$$\boldsymbol{g}^0(\boldsymbol{x}) = \boldsymbol{x}, \; \boldsymbol{g}^{t+1}(\boldsymbol{x}) = \boldsymbol{g}\left(\boldsymbol{g}^t(\boldsymbol{x})\right), \; t \geq 0.$$

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The space-time pattern generated by $\boldsymbol{g}$ for an initial configuration $\boldsymbol{x} \in \mathcal{A}$ is the set $\{(t, \boldsymbol{g}^t(\boldsymbol{x})), t \geq 0\}$. Our general problem is to analyze the dynamical system and characterize the space-time patterns.

A dynamical system $(\mathcal{S}, \boldsymbol{g})$ is called a *dynamical subsystem* of $(\mathcal{A}, \boldsymbol{g})$ if $\mathcal{S} \subseteq \mathcal{A}$ and $\boldsymbol{g}(\mathcal{S}) \subseteq \mathcal{S}$. $(\mathcal{S}, d)$ is a metric subspace of $(\mathcal{A}, d)$.

A topological dynamical system $(\mathcal{S}, \boldsymbol{g})$ is called the *Devaney chaos* when it has a dense orbit and the class of all periodic configurations is dense in $\mathcal{S}$. See G. Cattaneo, et al. [5].

In this paper we use the notations in Section 1.1 for our rigorous examination of the space-time patterns generated by rule 40.

1.1 Notations

(1) For $\boldsymbol{\alpha}_i \in \{0, 1\}^{n_i}$, $\boldsymbol{\beta}_i \in \{0, 1\}^{m_i}$, $n_i \geq 1$, $m_i \geq 1$, $i \in \mathbf{Z}$, we define

$$(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)_{i=-\infty}^{+\infty} = \left(\ldots, \alpha_1^{-1}, \ldots, \alpha_{n_{-1}}^{-1}, \beta_1^{-1}, \ldots, \beta_{m_{-1}}^{-1}, \alpha_1^0, \ldots, \alpha_{n_0}^0, \beta_1^0, \ldots, \beta_{m_0}^0, \alpha_1^1, \ldots, \alpha_{n_1}^1, \beta_1^1, \ldots, \beta_{m_1}^1, \ldots\right),$$

where $\boldsymbol{\alpha}_i = \left(\alpha_1^i, \ldots, \alpha_{n_i}^i\right)$, $\boldsymbol{\beta}_i = \left(\beta_1^i, \ldots, \beta_{m_i}^i\right)$, $i \in \mathbf{Z}$.

(2) $\mathbf{0}$ means one of the three types $(\ldots, 0, 0, 0, \ldots)$, $(\ldots, 0, 0)$, and $(0, 0, \ldots)$. It is clear from the context which type of $\mathbf{0}$ is meant. We also use the terminology $\mathbf{0}_n = (\underbrace{0, \ldots, 0}_{n})$, $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all non-negative integers. $\mathbf{1}$ and $\mathbf{1}_n$ are interpreted similar to $\mathbf{0}$ and $\mathbf{0}_n$, respectively. When $n = 0$, the blocks are empty.

(3) We write the set of all configurations consisting of 01 or 011 as

$$\mathcal{S}_2 = \left\{ \left(0, \mathbf{1}_{p_i}\right)_{i=-\infty}^{\infty} \mid p_i = 1 \text{ or } 2 \right\} = \left\{ \left(\ldots, 0, \mathbf{1}_{p_{-1}}, 0, \mathbf{1}_{p_0}, 0, \mathbf{1}_{p_1}, \ldots\right) \mid p_i = 1 \text{ or } 2, \; i = \ldots, -1, 0, 1, \ldots \right\}.$$

2. Spreading Rates and Lyapunov Exponents

Following [2], for an ECA g the Lyapunov exponent of $\boldsymbol{x} \in \mathcal{A}$ is defined by the following procedure. For $s \in \mathbf{Z}$, let

$$W_s^+(\boldsymbol{x}) \equiv \{\boldsymbol{y} \in \mathcal{A} \mid \forall\, i \geq s,\; y_i = x_i\},$$

$$\tilde{\Lambda}_t^+(\boldsymbol{x}) \equiv \min\left\{s \mid \boldsymbol{g}^t\left(W_0^+(\boldsymbol{x})\right) \subset W_s^+\left(\boldsymbol{g}^t(\boldsymbol{x})\right)\right\},$$

$$\Lambda_t^+(\boldsymbol{x}) \equiv \max_{j \in \mathbf{Z}} \left\{\tilde{\Lambda}_t^+\left(\sigma_L^j\, \boldsymbol{x}\right)\right\},$$

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$$W_s^-(x) \equiv \{y \in \mathcal{A} \mid \forall\, i \le s,\ y_i = x_i\},$$
$$\tilde{\Lambda}_t^-(x) \equiv \max\left\{s \mid g^t(W_0^-(x)) \subset W_s^-\left(g^t(x)\right)\right\},$$
$$\Lambda_t^-(x) \equiv \min_{j\in\mathbf{Z}}\left\{\tilde{\Lambda}_t^-\left(\sigma_L^j\, x\right)\right\},$$

where $\sigma_L^j = \sigma_R^{-j}$ for $j < 0$. When

$$\lim_{t\to\infty} \frac{\Lambda_t^+(x)}{t} \text{ and } \lim_{t\to\infty} \frac{\Lambda_t^-(x)}{t}$$

exist, we call them the *right* and *left Lyapunov exponents* of x, respectively.

For $x, y \in \mathcal{A}$, we set

$$DFR(x, y) \equiv \sup\{i \mid x_i \ne y_i\} \text{ and } DFL(x, y) \equiv \inf\{i \mid x_i \ne y_i\}.$$

$DFR(x, y)$ is the coordinate number of the rightmost different site between x and y. $DFL(x, y)$ similarly means the coordinate number of the leftmost different site. Lemma 1 relates the spreading rates and the Lyapunov exponents.

Lemma 1. For ECA g, configuration $x \in \mathcal{A}$ and $t \in \mathbf{N}$, we have

$$\max_{y\in W_0^+(x)} DFR\left(g^t(x), g^t(y)\right) = \min\left\{s \mid g^t\left(W_0^+(x)\right) \subset W_s^+\left(g^t(x)\right)\right\} - 1,$$
$$\min_{y\in W_0^-(x)} DFL\left(g^t(x), g^t(y)\right) = \max\left\{s \mid g^t(W_0^-(x)) \subset W_s^-\left(g^t(x)\right)\right\} + 1.$$

Following [1], the spreading rate of a configuration $x \in \mathcal{A}$ is defined by using the function $n_j : \mathcal{A} \to \mathcal{A}\ (j \in \mathbf{Z})$ defined as

$$x \in \mathcal{A},\ \left(n_j(x)\right)_i = \begin{cases} x_i, & i \ne j, \\ \overline{x_i}, & i = j. \end{cases}$$

n_j reverses the state of the j^{th} site in the configuration x. For ECA g, letting

$$\Gamma_t^+(x) \equiv \max_{j\in\mathbf{Z}}\left\{DFR\left(g^t(x), g^t\left(n_j(x)\right) - j\right\},\right.$$
$$\Gamma_t^-(x) \equiv \min_{j\in\mathbf{Z}}\left\{DFL\left(g^t(x), g^t\left(n_j(x)\right) - j\right\},\right.$$

when

$$\gamma^+(x) = \lim_{t\to\infty} \frac{\Gamma_t^+(x)}{t} \text{ and } \gamma^-(x) = \lim_{t\to\infty} \frac{\Gamma_t^-(x)}{t}$$

exist, we call them the *right* and *left spreading rates* of x, respectively.

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Theorem 1. For ECA g, at $x \in \mathcal{A}$ and $t \in \mathbf{N}$, we have

$$(1)\ \Lambda_t^+(x) = \max_{j \in \mathbf{Z}} \max_{y \in W_j^+(x)} \left\{ DFR\left(g^t(x), g^t(y)\right) - j \right\} + 1,$$

$$\Gamma_t^+(x) \leq \Lambda_t^+(x).$$

$$(2)\ \Lambda_t^-(x) = \min_{j \in \mathbf{Z}} \min_{y \in W_j^-(x)} \left\{ DFR\left(g^t(x), g^t(y)\right) - j \right\} - 1,$$

$$\Lambda_t^-(x) \leq \Gamma_t^-(x).$$

The spreading rate does not generally coincide with the Lyapunov exponent, but when the ECA is leftmost or rightmost permutive, they are generally equivalent. Rule 40 is neither rightmost nor leftmost permutive, but for the configurations of $\mathcal{S}_2$ the Lyapunov exponent and the spreading rate are the same.

Changing the order of maximization and taking limits in the definition of the spreading rate, the *right* and *left strict spreading rates*, $s\gamma^+(x)$ and $s\gamma^-(x)$ respectively, of x are defined by the following, when the limit values exist:

$$s\gamma^+(x) \equiv \max_{j \in \mathbf{Z}} \left\{ \lim_{t \to \infty} \frac{DFR\left(g^t(x), g^t\left(n_j(x)\right) - j\right.}{t} \right\},$$

$$s\gamma^-(x) \equiv \min_{j \in \mathbf{Z}} \left\{ \lim_{t \to \infty} \frac{DFL\left(g^t(x), g^t\left(n_j(x)\right) - j\right.}{t} \right\}.$$

The expression in the upper curly brackets is the right extending rate of the rightmost different site due to the change of the state x_j, and $s\gamma^+(x)$ is the maximal rate over $j \in \mathbf{Z}$.

Theorem 2 holds evidently.

Theorem 2. For ECA g, at $x \in \mathcal{A}$ and $t \in \mathbf{N}$, if the limiting values exist, then we have

$$s\gamma^+(x) \leq \gamma^+(x) \leq \lim_{t \to \infty} \frac{\Lambda_t^+(x)}{t}, \quad \lim_{t \to \infty} \frac{\Lambda_t^-(x)}{t} \leq \gamma^-(x) \leq s\gamma^-(x).$$

For rule 40 the spreading rate and the Lyapunov exponent of each configuration in $\mathcal{S}_2$ are coincident with each other, but the strict spreading rate is not necessarily equal to them. These examinations are shown next in Section 3.

3. Dynamical Properties of Rule 40

Following [4], we have Lemma 2.

Lemma 2. (1) For every configuration $x \in \mathcal{S}_2$, $g_{40}(x) = \sigma_L(x)$, then $\mathcal{S}_2$ is a subshift.

(2) For

$$x = (\ldots, 0, \mathbf{1}_{m_{-2}}, 0, \mathbf{1}_{m_{-1}}, \overset{j}{\smile}{0}, \overset{j+1}{\smile}{0}, \ldots),$$
$$m_i = 1 \text{ or } 2,\ i = -1, -2, \ldots,$$

we have

$$g_{40}^t(x) = (\ldots, 0, \mathbf{1}_{m_{-2}}, 0, \mathbf{1}_{m_{-1}}, \overset{j-t}{\smile}{0}, 0, \ldots, \overset{j}{\smile}{0}, \overset{j+1}{\smile}{0}, \ldots).$$

(3) For

$$x = (\ldots, 0, \overset{j}{\smile}{0}, 0, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, \ldots),$$
$$m_i = 1 \text{ or } 2,\ i = 1, 2, \ldots,$$

we have

$$g_{40}(x) = (\ldots, 0, \overset{j}{\smile}{0}, 0, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, \ldots).$$

(4) For

$$x = (\ldots, 0, \overset{j}{\smile}{0}, 0, 1, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, \ldots),$$
$$m_i = 1 \text{ or } 2,\ i = 1, 2, \ldots,$$

we have

$$g_{40}(x) = (\ldots, 0, \overset{j}{\smile}{0}, 0, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, \ldots),$$

$$g_{40}^2(x) = (\ldots, 0, \overset{j}{\smile}{0}, 0, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, \ldots).$$

Lemma 2 part (2) tells us that the block 00 extends to the left by one site at every time step, whatever the neighborhood around the block is. Lemma 2 parts (3) and (4) show us that the block 01 on the right-hand side of the block 00 disappears in one step, the block 011 disappears in two steps, and the block 00 extends to the right by one site to be the block 000 (see Figure 1).

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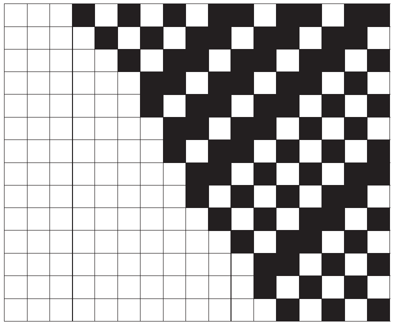

Figure 1. An example time-development of a configuration of $\mathcal{S}_2$, where white and black boxes mean the 0 and 1 state cells, respectively.

3.1 Dynamical Properties of $\boldsymbol{x} \in \mathcal{S}_2$ and $n_j(\boldsymbol{x})$

For $\boldsymbol{x} = (\ldots, 0, \mathbf{1}_{m_0}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots)$, $\forall\, i \in \mathbf{Z}$, $1 \leq m_i \leq 2$, and $j \in \mathbf{Z}$, we may classify $n_j(\boldsymbol{x})$ into the following cases.

(1) When $x_j = 1$,

$$\text{(1-i)} \qquad \boldsymbol{x} = (\ldots, 1, 0, \overset{\overset{j}{\smile}}{1}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 1, 0, \overset{\overset{j}{\smile}}{0}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$\text{(1-ii)} \qquad \boldsymbol{x} = (\ldots, 1, 0, \overset{\overset{j}{\smile}}{1}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 1, 0, \overset{\overset{j}{\smile}}{0}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$\text{(1-iii)} \qquad \boldsymbol{x} = (\ldots, 1, 0, 1, \overset{\overset{j}{\smile}}{1}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 1, 0, 1, \overset{\overset{j}{\smile}}{0}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

For case (1-ii), following the local rule g_{40}, we have the next-step configurations as

$$g_{40}(\boldsymbol{x}) = (\ldots, 1, 0, 1, \overset{\overset{j}{\smile}}{1}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$g_{40}(n_j(\boldsymbol{x})) = (\ldots, 1, 0, 0, \overset{\overset{j}{\smile}}{0}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

(2) When $x_j = 0$,

$$\text{(2-i)} \qquad \boldsymbol{x} = (\ldots, 0, 1, 1, \overset{\underset{\smile}{j}}{0}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 0, 1, 1, \overset{\underset{\smile}{j}}{1}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$\text{(2-ii)} \qquad \boldsymbol{x} = (\ldots, 1, 0, 1, \overset{\underset{\smile}{j}}{0}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 1, 0, 1, \overset{\underset{\smile}{j}}{1}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$\text{(2-iii)} \qquad \boldsymbol{x} = (\ldots, 0, 1, 1, \overset{\underset{\smile}{j}}{0}, 1, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 0, 1, 1, \overset{\underset{\smile}{j}}{1}, 1, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$\text{(2-iv)} \qquad \boldsymbol{x} = (\ldots, 1, 0, 1, \overset{\underset{\smile}{j}}{0}, 1, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$n_j(\boldsymbol{x}) = (\ldots, 1, 0, 1, \overset{\underset{\smile}{j}}{1}, 1, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

For case (2-i),

$$g_{40}(\boldsymbol{x}) = (\ldots, 1, 0, \overset{\underset{\smile}{j}}{1}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$g_{40}(n_j(\boldsymbol{x})) = (\ldots, 1, 0, \overset{\underset{\smile}{j}}{0}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

For case (2-ii),

$$g_{40}(\boldsymbol{x}) = (\ldots, 1, 0, \overset{\underset{\smile}{j}}{1}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$g_{40}(n_j(\boldsymbol{x})) = (\ldots, 1, 1, \overset{\underset{\smile}{j}}{0}, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

For case (2-iii),

$$g_{40}(\boldsymbol{x}) = (\ldots, 1, 0, \overset{\underset{\smile}{j}}{1}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$g_{40}(n_j(\boldsymbol{x})) = (\ldots, 1, 0, \overset{\underset{\smile}{j}}{0}, 0, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

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For case (2-iv),

$$\boldsymbol{g}_{40}(\boldsymbol{x}) = (\ldots, 1, 0, \overset{\overset{j}{\smile}}{1}, 1, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$\boldsymbol{g}_{40}(n_j(\boldsymbol{x})) = (\ldots, 1, 1, \overset{\overset{j}{\smile}}{0}, 0, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

For visual depictions of one-step developments for cases (1) and (2), see Figures 2 through 5.

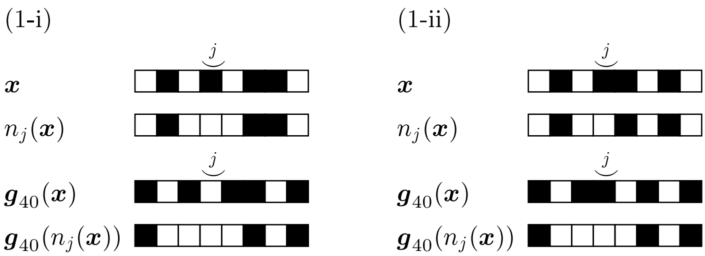

Figure 2. Time-development for cases (1-i) and (1-ii).

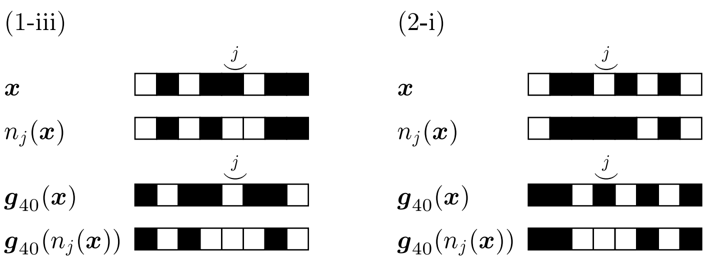

Figure 3. Time-development for cases (1-iii) and (2-i).

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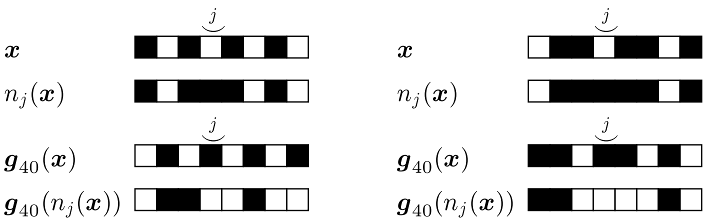

Figure 4. Time-development for cases (2-ii) and (2-iii).

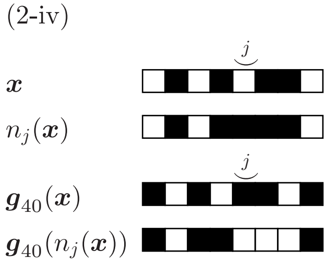

Figure 5. Time-development for case (2-iv).

By Lemma 2 parts (1) and (2), we easily have the following. For cases (2-i) and (2-iii):

$$DFL\big(\boldsymbol{g}^t(\boldsymbol{x}),\ \boldsymbol{g}^t\big(n_j(\boldsymbol{x})\big) = j - t + 1.$$

For cases (1), (2-ii), and (2-iv):

$$DFL\big(\boldsymbol{g}^t(\boldsymbol{x}),\ \boldsymbol{g}^t\big(n_j(\boldsymbol{x})\big) = j - t.$$

Thus, we have Theorem 3.

Theorem 3. The left Lyapunov exponent, the left spreading, and the left strict spreading rates are -1 for every configuration of $\mathcal{S}_2$.

From cases (1) and (2), for every $\boldsymbol{x} \in \mathcal{S}_2$ and every $j \in \mathbf{Z}$,

$$\boldsymbol{x} = (\ldots,\ \overset{\overset{j}{\smile}}{1},\ 0,\ \mathbf{1}_{m_1},\ 0,\ \mathbf{1}_{m_2},\ 0,\ \ldots), \tag{1}$$

$$n_j(\boldsymbol{x}) = (\ldots,\ \overset{\overset{j}{\smile}}{0},\ 0,\ \mathbf{1}_{m_1},\ 0,\ \mathbf{1}_{m_2},\ 0,\ \ldots), \tag{2}$$

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or

$$g_{40}(x) = (\ldots, \underset{\smile}{\overset{j \text{ or } j+1}{}}\!\!\!1\;, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots),$$

$$g_{40}(n_j(x)) = (\ldots, \underset{\smile}{\overset{j \text{ or } j+1}{}}\!\!\!0\;, 0, \mathbf{1}_{m_1}, 0, \mathbf{1}_{m_2}, 0, \ldots).$$

$DFR\left(x, n_j(x)\right)$ and $DFR\left(g(x), g\left(n_j(x)\right)\right.$ differ by at most one, and this difference tends to zero by taking the limit in the definition of the right spreading rates and the right Lyapunov exponents, then it is sufficient to examine them for cases (1) and (2).

A configuration $x \in \mathcal{S}_2$ is generally written as

$$x = \left((01)_{p_i}, (011)_{q_i}\right)_{i=-\infty}^{\infty}, \;\; p_i \geq 1, \;\; q_i \geq 1, \;\; i \in \mathbf{Z}. \tag{3}$$

Then, noticing Lemma 2 parts (1), (3), and (4), we have Lemma 3.

Lemma 3. For

$$x = (\ldots, \overset{j}{\smile}\!\!\!\!1, (01)_{p_1}, (011)_{q_1}, (01)_{p_2}, (011)_{q_2}, \ldots),$$

$$n_j(x) = (\ldots, \overset{j}{\smile}\!\!\!\!0, (01)_{p_1}, (011)_{q_1}, (01)_{p_2}, (011)_{q_2}, \ldots),$$

we have the following.

(i) For $\sum_{i=1}^{k} (p_i + 2\, q_i) \leq t < \sum_{i=1}^{k} (p_i + 2\, q_i) + p_{k+1}$,

$$DFR\left(g_{40}^t(x), g_{40}^t\left(n_j(x)\right)\right) - j = \sum_{i=1}^{k} (p_i + q_i) + t - \sum_{i=1}^{k} (p_i + 2\, q_i).$$

(ii) For $\sum_{i=1}^{k} (p_i + 2\, q_i) + p_{k+1} \leq t < \sum_{i=1}^{k+1} (p_i + 2\, q_i)$,

$$DFR\left(g_{40}^t(x), g_{40}^t\left(n_j(x)\right)\right) - j =$$

$$\sum_{i=1}^{k} (p_i + q_i) + p_{k+1} + \left[\frac{t - \sum_{i=1}^{k} (p_i + 2\, q_i) - p_{k+1}}{2}\right]$$

where [ ] is the Gaussian symbol.

Figure 6 shows us the time-development of the configuration $n_j(x)$ in Lemma 3 and how the rightmost different site between $g_{40}^t(x)$ and $g_{40}^t\left(n_j(x)\right)$ extends to the right as $t \to \infty$, which is marked by thick lines in the figure.

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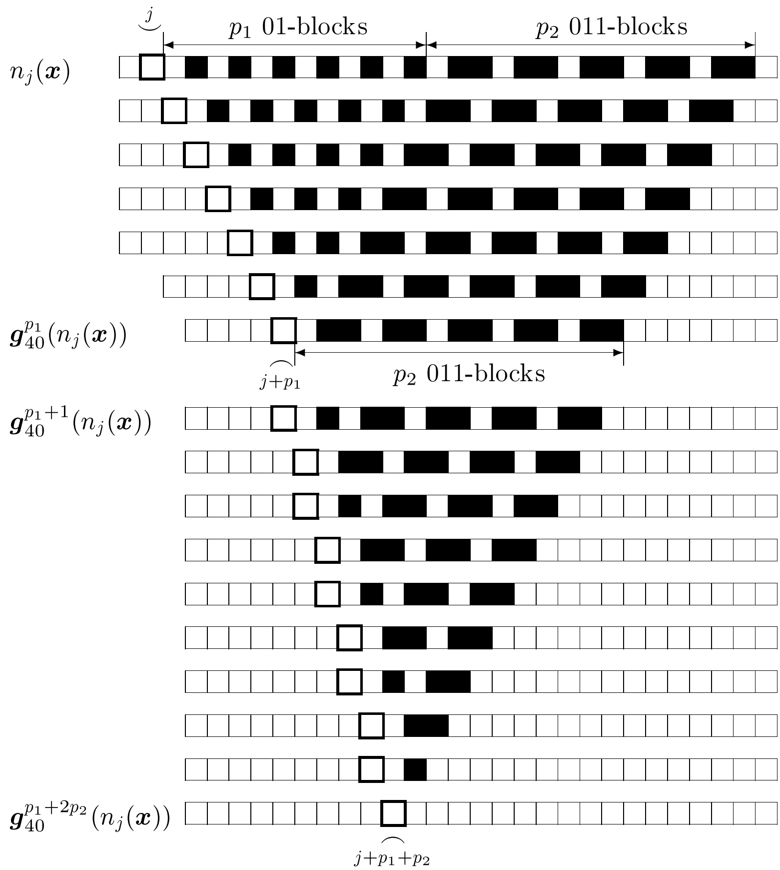

Figure 6. p_1 01-blocks disappear in p_1 time steps and the rightmost different site j moves over p_1 sites to the right. p_2 011-blocks disappear in $2\,p_2$ time steps and the rightmost different site $j+p_2$ moves over p_2 sites to the right. The rightmost different sites between $g_{40}^t(x)$ and $g_{40}^t(n_j(x))$ are marked by thick lines.

4. Right Lyapunov Exponents, Right Spreading, and Right Strict Spreading Rates of Configurations in $\mathcal{S}_2$

Theorem 4. For rule 40, the spreading rate and the Lyapunov exponent of each configuration in $\mathcal{S}_2$ are equivalent.

Proof. Noticing

$$\Gamma_t^+(x) = \max_{j \in \mathbf{Z}} \left\{ DFR\left(g^t(x),\, g^t\left(n_j(x)\right) - j\right\},\right.$$

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$$\Lambda_t^+(x) = \max_{j \in \mathbf{Z}} \max_{y \in W_{j+1}^+(x)} \left\{ DFR\left(g^t(x), g^t(y)\right) - j \right\},$$

we prove the inequality $\Lambda_t^+(x) \leq \Gamma_t^+(x)$.

For $y \in W_{j+1}^+(x)$, letting $i_0 \equiv \max\{i \mid x_i \neq y_i, i \in \mathbf{Z}\}$, we have $i_0 \leq j$ and then we may write

$$x = \left(\ldots, x_{i_0}, x_{i_0+1}, \ldots, x_j, \ldots\right),$$
$$y = \left(\ldots, y_{i_0}, x_{i_0+1}, \ldots, x_j, \ldots\right), \; x_{i_0} \neq y_{i_0}.$$

Noticing the dynamical properties of rule 40, we have

$$DFR\left(g_{40}^t(x), g_{40}^t(y)\right) \leq DFR\left(g_{40}^t(x), g_{40}^t\left(n_{i_0}(x)\right)\right).$$

Since $i_0 \leq j$,

$$DFR\left(g_{40}^t(x), g_{40}^t(y)\right) - j \leq DFR\left(g_{40}^t(x), g_{40}^t\left(n_{i_0}(x)\right)\right) - i_0 \leq \Gamma_t^+(x)$$

holds for every $y \in W_{j+1}^+(x)$, and then the intended inequality holds. □

Theorem 5. For every rational number v/u between $1/2$ and 1, where the fraction is irreducible and v and u are positive integers, there exists a configuration in $\mathcal{S}_2$ whose right Lyapunov exponent is v/u.

Proof. When $p_i = p$ and $q_i = q$ with $(i = 1, 2, \ldots)$ in equation (3), we have from Lemma 3 parts (i) and (ii),

$$\lim_{t \to \infty} \frac{\Gamma_t^+(x)}{t} = \frac{p+q}{p+2q}.$$

Then, setting $p = 2v - u$ and $q = u - v$,

$$\lim_{t \to \infty} \frac{\Gamma_t^+(x)}{t} = \frac{v}{u}. \;\square$$

Example 1. This configuration x shows us that the right spreading and the strict spreading rates are not necessarily equivalent:

$$x = \big(\ldots, 0, 1, 0, 1, 0, 1, (01)_p,$$
$$(011)_q, (01)_p, (011)_q, (01)_p, (011)_q, \ldots\big),$$

where $p = 2v - u$, $q = u - v$, and $1/2 \leq v/u \leq 1$ is irreducible and u and v are positive integers. For this x

$$\forall\, j \in \mathbf{Z}, \; \lim_{t \to \infty} \frac{DFR\left(g_{40}^t(x), g_{40}^t\left(n_j(x)\right)\right) - j}{t} = \frac{v}{u},$$

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which means that the strict spreading rate of the configuration is v/u. Otherwise, noticing Lemma 3 and Figure 6,

$$\forall\, t \geq 0,\ \ \exists\, j << -t,\ \ DFR\big(g^t(x),\, g^t\big(n_j(x)\big)\big) - j = t,$$

and then

$$\forall\, t \geq 0,\ \ \sup_{j \in \mathbf{Z}} \frac{DFR\big(g^t(x),\, g^t\big(n_j(x)\big)\big) - j}{t} = 1.$$

Thus, the spreading rate and the Lyapunov exponent of this configuration are 1.

Next, we prove a property of the strict spreading rate of a configuration in $\mathcal{S}_2$. The proof requires Lemma 4.

Lemma 4. For

$$x = (\ldots,\, \overset{j}{\breve{1}},\, (01)_{p_1},\, (011)_{q_1},\, (01)_{p_2},\, (011)_{q_2},\, \ldots),$$

$$n_j(x) = (\ldots,\, \overset{j}{\breve{0}},\, (01)_{p_1},\, (011)_{q_1},\, (01)_{p_2},\, (011)_{q_2},\, \ldots),$$

if

$$\lim_{i \to \infty} \frac{p_i + q_i}{p_i + 2\, q_i} = \alpha \text{ and } \lim_{k \to \infty} \frac{p_{k+1}}{\sum_{i=1}^{k} (p_i + 2\, q_i)} = 0$$

then

$$\lim_{t \to \infty} \frac{DFR\big(g_{40}^t(x),\, g_{40}^t\big(n_j(x)\big)\big) - j}{t} = \alpha.$$

Proof. We only prove Lemma 4 for Lemma 3, case (i). Let

$$\sum_{i=1}^{k} (p_i + 2\, q_i) \leq t < \sum_{i=1}^{k} (p_i + 2\, q_i) + p_{k+1}.$$

From Lemma 3, case (i) we have

$$\frac{DFR\big(g_{40}^t(x),\, g_{40}^t\big(n_j(x)\big)\big) - j}{t} =$$

$$\frac{\sum_{i=1}^{k} (p_i + q_i)}{t} + 1 - \frac{\sum_{i=1}^{k} (p_i + 2\, q_i)}{t}.$$

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Noticing the given condition for t,

$$\frac{\sum_{i=1}^{k}(p_i + 2\,q_i)}{\sum_{i=1}^{k}(p_i + 2\,q_i) + p_{k+1}} < \frac{\sum_{i=1}^{k}(p_i + 2\,q_i)}{t} \leq \frac{\sum_{i=1}^{k}(p_i + 2\,q_i)}{\sum_{i=1}^{k}(p_i + 2\,q_i)} = 1.$$

The leftmost term tends to 1 as $j \to \infty$ by the condition of this lemma. Then,

$$\lim_{j\to\infty} \frac{\sum_{i=1}^{k}(p_i + 2\,q_i)}{t} = 1.$$

We also have

$$\frac{\sum_{i=1}^{k}(p_i + q_i)}{\sum_{i=1}^{k}(p_i + 2\,q_i) + p_{k+1}} < \frac{\sum_{i=1}^{k}(p_i + q_i)}{t} \leq \frac{\sum_{i=1}^{k}(p_i + q_i)}{\sum_{i=1}^{k}(p_i + 2\,q_i)}.$$

From the conditions of this lemma,

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k}(p_i + q_i)}{\sum_{i=1}^{k}(p_i + 2\,q_i) + p_{k+1}} = \alpha, \quad \lim_{k\to\infty} \frac{\sum_{i=1}^{k}(p_i + q_i)}{\sum_{i=1}^{k}(p_i + 2\,q_i)} = \alpha$$

holds, and then

$$\lim_{t\to\infty} \frac{\sum_{i=1}^{k}(p_i + q_i)}{t} = \alpha.$$

Thus, we finally have

$$\lim_{t\to\infty} \frac{DFR\left(\mathbf{g}_{40}^{t}(\mathbf{x}),\, \mathbf{g}_{40}^{t}\left(n_j(\mathbf{x})\right)\right) - j}{t} = \alpha. \tag{4}$$

The proof is similar for Lemma 3, case (ii). □

Theorem 6. For every real number α between $1/2$ and 1, there exists a configuration whose strict spreading rate is α.

Proof. We construct $\{p_i\}_{i\geq 1}$ and $\{q_i\}_{i\geq 1}$ satisfying the conditions of Lemma 4.

Let $\{v_i / u_i\}_{i\geq 1}$ be a series of rational numbers that converges to α, and each term satisfies $1/2 \leq v_i / u_i \leq 1$. Then, $2\,v_i - u_i > 0$ and $u_i - v_i > 0$.

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For a sequence of positive numbers $\{\epsilon_i\}_{i\geq 1}$ that converges to 0, we take a series of integers $\{i_j\}_{j\geq 1}$ as

$$\frac{2\,v_{k+1} - u_{k+1}}{\sum_{j=1}^{k} i_j\,u_j} < \epsilon_k,\ k = 1, 2, \ldots.$$

Using the series $\{i_j\}_{j\geq 1}$, we define $\{p_i\}_{i\geq 1}$ and $\{q_i\}_{i\geq 1}$ as

$$\begin{aligned} p_{i_1+\cdots+i_{j-1}+1} &= \cdots = p_{i_1+\cdots+i_j} = 2\,v_j - u_j, \\ q_{i_1+\cdots+i_{j-1}+1} &= \cdots = q_{i_1+\cdots+i_j} = u_j - v_j,\ \ j = 1, 2, \ldots \end{aligned}$$

where we promise $i_0 = 0$.

For $i_{j-1} < k \leq i_j$, $p_k = 2\,v_j - u_j$ and $q_k = u_j - v_j$. Then we have

$$\frac{p_k + q_k}{p_k + 2\,q_k} = \frac{v_j}{u_j} \to \alpha,$$

as $k \to \infty$ and then $j \to \infty$.

For $i_{j-1} < k < i_j$,

$$\frac{p_{k+1}}{\sum_{i=1}^{k} (p_i + 2\,q_i)} =$$

$$\frac{2\,v_j - u_j}{\sum_{l=1}^{j-1} i_{j-1}\,u_{j-1} + \left(k - i_{j-1}\right) 2\,v_j} < \frac{2\,v_j - u_j}{\sum_{l=1}^{j-1} i_{j-1}\,u_{j-1}} < \epsilon_j$$

and for $k = i_j$

$$\frac{p_{k+1}}{\sum_{i=1}^{k} (p_i + 2\,q_i)} = \frac{2\,v_{j+1} - u_{j+1}}{\sum_{l=1}^{j} i_j\,u_j} < \epsilon_{j+1}.$$

Hence, noticing $\epsilon_i \to 0$, the constructed $\{p_i\}_{i\geq 1}$ and $\{q_i\}_{i\geq 1}$ satisfy the conditions of Lemma 4 and then define a configuration whose strict spreading rate is α. □

5. Bernoulli Measure of $\mathcal{S}_2$ is 0

In this section we prove that the Bernoulli measure of $\mathcal{S}_2$ is 0. This theorem explains why the configuration in $\mathcal{S}_2$ could not be observed, making the left-shift dynamical property of rule 40 undetectable by computer simulations for randomly given initial configurations.

Let (q, p) be a probability distribution on $\{0, 1\}$ and $\mathbf{P}$ be the product probability measure $(q, p)^{\mathbf{Z}}$ on $(\mathcal{A}, \mathcal{F})$, where $\mathcal{F}$ is the σ field generated by the set of all cylinder subsets of $\mathcal{A}$.

We prove $\mathbf{P}(\mathcal{S}_2) = 0$, using the following notations:

$$\mathcal{A}^- = \{(\ldots, x_{-2}, x_{-1}) \mid x_i \in \{0, 1\},\ i = -1, -2, \ldots\},$$

$$[a_0, a_1, \ldots, a_n] = \{(x_0, \ldots, x_n, x_{n+1}, \ldots) \mid x_i = a_i\, (0 \le i \le n),\\ x_i = 0 \text{ or } 1\, (i \ge n+1)\},$$

$$A = \{\boldsymbol{a},\ \boldsymbol{b}\},\ A^n = \{\boldsymbol{a},\ \boldsymbol{b}\}^n,\ \boldsymbol{a} = (0, 1),\ \boldsymbol{b} = (0, 1, 1).$$

It is easily seen that

$$\forall\, \boldsymbol{\alpha} \in A^n,\ [\boldsymbol{\alpha}, \boldsymbol{b}] \subset [\boldsymbol{\alpha}, \boldsymbol{a}],$$

then

$$\boldsymbol{\alpha},\ \boldsymbol{\beta} \in A^n,\ \boldsymbol{\alpha} \neq \boldsymbol{\beta} \Longrightarrow [\boldsymbol{\alpha}, \boldsymbol{a}] \bigcap [\boldsymbol{\beta}, \boldsymbol{a}] = \phi,$$

$$\bigcup_{\boldsymbol{\alpha} \in A^{n+1}} [\boldsymbol{\alpha}] = \bigcup_{\boldsymbol{\alpha} \in A^n} [\boldsymbol{\alpha}, \boldsymbol{a}],$$

and noticing the independence property of the product measure $\mathbf{P}$, we have

$$\mathbf{P}\left(\mathcal{A}^- \times \bigcup_{\boldsymbol{\alpha} \in A^{n+1}} [\boldsymbol{\alpha}]\right) = \sum_{\boldsymbol{\alpha} \in A^n} \mathbf{P}(\mathcal{A}^- \times [\boldsymbol{\alpha}, \boldsymbol{a}]) =$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(q\, p^2\right)^k (q\, p)^{n-k}\, (q\, p) = q\, p\left(q\, p + q\, p^2\right)^n =$$

$$p^{n+1}\, q^{n+1}\, (1+p)^n = p^{n+1}\, (1-p)\left(1-p^2\right)^n.$$

The following inclusion is easily verified for every $n \ge 1$:

$$\mathcal{S}_2 \subseteq \left(\mathcal{A}^- \times \bigcup_{\boldsymbol{\alpha} \in A^{n+1}} [\boldsymbol{\alpha}]\right) \cup$$

$$\left(\mathcal{A}^- \times \{(\overset{0}{\breve{1}})\} \times \bigcup_{\boldsymbol{\alpha} \in A^{n+1}} [\boldsymbol{\alpha}]\right) \cup \left(\mathcal{A}^- \times \{(\overset{0}{\breve{1}}, 1)\} \times \bigcup_{\boldsymbol{\alpha} \in A^{n+1}} [\boldsymbol{\alpha}]\right).$$

Hence,

$$\mathbf{P}(\mathcal{S}_2) \le p^{n+1}(1-p)\left(1-p^2\right)^n + p\, p^{n+1}(1-p)\left(1-p^2\right)^n +\\ p^2\, p^{n+1}(1-p)\left(1-p^2\right)^n \to 0\ (n \to \infty),$$

and we finally have Theorem 7.

Theorem 7. $\mathbf{P}(\mathcal{S}_2) = 0$.

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When a configuration is randomly given, the state of each cell is determined to be 0 or 1 with probability $1/2$, which is the case of $p = q = 1/2$ for the stated product probability measure.

Theorem 7 tells us that the configurations in $\mathcal{S}_2$ are not observed in a computer simulation when an initial configuration is randomly given.

6. Concluding Remarks

In this paper, elementary cellular automaton (ECA) rule 40 in Wolfram's class I is examined and configurations of $\mathcal{S}_2$ are shown to have diverse Lyapunov exponent values. Spreading rates of a configuration are calculated by a practical construction given any exponents and rates. Furthermore, the Lyapunov exponents are shown to be the spreading rates for rule 40 and may not generally reflect the complexity of space-time patterns generated by other ECAs.

The Lyapunov exponents and the spreading rates for rule 40 are essentially determined by the relative frequencies of the special blocks 010 and 011 in configurations of $\mathcal{S}_2$.

We typically think that the space-time patterns generated by rule 40 die out in a finite number of time steps and that the rule does not have a left-shift dynamical subsystem $(\mathcal{S}_2, \sigma_L)$. Since the Bernoulli measure of $\mathcal{S}_2$ is 0 as shown in Section 5, it is natural that this left-shift dynamical subsystem has not been observed. Our work, however, suggests that even ECAs in Wolfram's classes I or II can have interesting space-time patterns under intense examination.

It is possibly true that one rule can show different kinds of space-time patterns depending on the patterns of initial configurations, and in this sense, generally, one rule may have some different rules embedded in it; for example, rule 40 is left-shift on $\mathcal{S}_2$ and has the unique attractor $\{\mathbf{0}\}$ on $\mathcal{A}\backslash\mathcal{S}_2$. However, we need more evidence to support this assertion for future work.

Furthermore, we still have the problem of calculating the Lyapunov exponents of F. Bagnoli and R. Rechtman [6], which are defined by using Boolean derivatives. It will be an interesting problem to determine what kind of values can be used as the Lyapunov exponents.

7. References

[1] A. Ilachinski, *Cellular Automata: A Discrete Universe*, River Edge, NJ: World Scientific, 2001.

[2] M. A. Shereshevsky, "Lyapunov Exponents for One-Dimensional Cellular Automata," *Journal of Nonlinear Science*, **2**(1), 1992 pp. 1–8.

https://doi.org/10.25088/ComplexSystems.19.4.323

[3] S. Wolfram, *A New Kind of Science*, Champaign, IL: Wolfram Media, Inc., 2002.

[4] F. Ohi, "Chaotic Properties of the Elementary Cellular Automaton Rule 40 in Wolfram's Class I," *Complex Systems*, **17**(3), 2007 pp. 295–308.

[5] G. Cattaneo, E. Formenti, and L. Margara, "Topological Definitions of Deterministic Chaos," in *Cellular Automata: A Parallel Model* (M. Delorme and J. Mazoyer, eds.), Boston: Kluwer Academic Publishers, 1999 pp. 213–259.

[6] F. Bagnoli and R. Rechtman, "Lyapunov Exponents and Synchronization of Cellular Automata," in *Complex Systems* (E. Goles and S. Martínez, eds.), Boston: Kluwer Academic Publishers, 2001 pp. 69–103.

https://doi.org/10.25088/ComplexSystems.19.4.323