

A Decidability Result for the Halting of Cellular Automata on the Pentagrid

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In this paper, we investigate the halting problem for deterministic cellular automata on the pentagrid. We prove that the problem is decidable when the cellular automaton starts its computation from a finite configuration and when it has two states, one of them being a quiescent state.

Keywords: tilings; hyperbolic plane; cellular automata; halting problem; decidability

1. Introduction

The paper belongs to the important field of cellular automata. The reader can find much information on the topic in [1]. However, the paper deals with very particular kinds of cellular automata: those that live in a hyperbolic space. For the sake of self-containment, let us recall generalities about cellular automata.

A cellular automaton is defined by two basic objects: the space of its cells and the finite automaton, a copy of which lies in each cell. The space of cells is assumed to be homogeneous enough in order to ensure that each cell has the same number of neighbors. This condition is naturally satisfied if the space of cells is associated to a tiling that is a tessellation based on a single regular tile. Then, each cell is associated to a tile that is called the *support* of the cell. Each cell has a state belonging to some finite set \mathcal{L} , called the set of states. As \mathcal{L} is finite, it can be seen as the alphabet used by the finite automaton that equips the cells. The cellular automaton evolves in a discrete time provided by a clock. At time t , each cell updates its state according to the current value of its state at time t and the values at the same time of the states of its neighbors. These current states constitute the neighborhood to which the finite automaton associates a new state, which will be the current state of the cell at time $t + 1$. The codification of this association is called a *rule* of the cellular automaton. There are finitely many rules constituting the *program* of the cellular automaton.

A *quiescent* state is a state ξ such that the cell remains in state ξ if all its neighbors are also in state ξ . The corresponding rule is called the *quiescent rule*. Usually, that state is called *white* or *blank*. In this paper, we will use the term *white* with another meaning. We keep the word *quiescent* for that state, which will also be denoted by W . A *configuration* at time t is the set of cells that are in a non-quiescent state together with the position of their supports in the tiling. Traditionally, the *initial* configuration of a cellular automaton is *finite*. This means that at time 0, the time that marks the beginning of the computation, the set of cells that are in the non-quiescent state is finite. Define the *distance* of a cell c to a cell d by the smallest number of cells needed to link c to d in a sequence where two consecutive cells are neighbors. Then, define the *disk* $D(c, n)$ of center c and radius n as the set of cells d whose distance from c is at most n . If we fix a cell c as the origin of the space, there is a smallest number N_0 such that the initial configuration is contained in $D(c, N_0)$. This means that all cells outside $D(c, N_0)$ are in the quiescent state. Call such an N_0 the *initial border number*. The reason for the index 0 will be clear later. Let $C(c, n)$ be the set of cells whose distance from c is exactly n . The definition of N_0 also entails that $C(c, N_0)$ contain at least one non-quiescent state. In this setting, the halting of a cellular automaton is reached by two identical consecutive configurations. Accordingly, there is a number k and a time t such that the configurations at time t and $t + 1$ are both contained in $D(c, k)$ and they are equal.

From now on, when we say cellular automaton, we need to understand “deterministic cellular automaton with a quiescent state.” The term *deterministic* means that a unique new state is associated to the current state of a cell and the current states of its neighbors.

From various papers of the author, we know the following on cellular automata in hyperbolic spaces: in the tessellations $\{5, 4\}$, $\{7, 3\}$ and $\{5, 3, 4\}$, namely the pentagrid, the heptagrid and the dodecagrid, respectively, it is possible to construct weakly universal cellular automata with two states only. In the case of the dodecagrid, the constructed automaton is rotation invariant; we restate the definition in Section 3. In the cases of the pentagrid and the heptagrid, the rules are not rotation invariant. Moreover, in the case of the pentagrid, we assume the Moore neighborhood; that is, we assume that the neighbors of the cell are the cells that share at least a vertex with it. It is known that with rotation-invariant rules and a von Neumann neighborhood, which means that the neighbors of a cell share a side with it, there is a strongly universal cellular automaton on the pentagrid with 10 states; see [2]. This means that the cellular automaton that is universal starts its computation from a finite configuration. If we

relax the rotation invariance, there is a weakly universal cellular automaton on the pentagrid with five states. And so, results concerning rotation invariance are also interesting.

Very little is known if we change something in the preceding assumptions.

The present paper is devoted to the proof of the following result:

Theorem 1. For deterministic cellular automata on the pentagrid whose initial configuration is finite and that have at most two states with one of them being quiescent, the halting problem is decidable.

The proof is split into two propositions dealing first with rotation invariance in Section 3, then when that condition is relaxed; see Section 4. In Section 5, we study what happens in an infinite motion of the cellular automaton when such a motion occurs. In Section 2, we present to the reader a minimal introduction of the pentagrid and of the implementation of cellular automata in that context. Section 6 brings in a few reflections on the topic.

We now turn to hyperbolic geometry and the tiling we consider in which the cellular automata later considered evolve.

2. The Pentagrid

This paper makes use of the model of the hyperbolic plane known as *Poincaré's disk*. Call *unit disk* a disk of the Euclidean plane fixed once and for all. Let D be the open unit disk. The model \mathcal{M} of the hyperbolic plane we consider is defined in D , which we call the *support* of \mathcal{M} . The points in \mathcal{M} are the points of the open disk. The lines in \mathcal{M} are the traces in D of circles that are orthogonal to ∂D , the border of D and the traces in D of straight lines that pass through the center of D .

Figure 1 represents a line ℓ and a point A out of ℓ . The figure also shows us four lines that pass through A . The line s cuts ℓ and is therefore called a *secant* with ℓ . The lines p and q touch ℓ on ∂D . The points P and Q where, respectively, p and q touch ℓ are called *points at infinity* of the hyperbolic plane but do not belong to that plane. The lines p and q are called *parallel* to ℓ . Last, but not least, the line m does not cut ℓ and it also does not touch it—not in D , nor on its border, nor outside D . The line m is called *non-secant* with ℓ . It is proved that two lines of the hyperbolic plane are non-secant if and only if they have a unique common perpendicular.

A theorem by Poincaré tells us that there are infinitely many tessellations in the hyperbolic plane whose basic tile is a triangle with angles π/p , π/q and π/r , provided that the positive numbers p , q

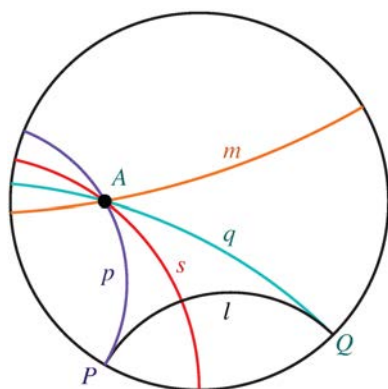


Figure 1. Poincaré's disk model of the hyperbolic plane. Here, the various relations between a line and a point out of the line, with other lines passing through the point, are shown.

and r satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

which simply means that the triangle with these angles lives in the hyperbolic plane. As a consequence, if we consider P the regular convex polygon with p sides and with interior angle $2\pi/q$, P tiles the plane by recursive reflections in its sides and in the sides of its images if and only if

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}.$$

When this is the case, the corresponding tessellation is denoted by $\{p, q\}$. Illustrated by Figure 2, the tessellation $\{5, 4\}$ is called a *pentagrid*.

In [3, 4], it is proved that the pentagrid is spanned by a tree. The left-hand side of Figure 3 shows us a quarter of the pentagrid spanned by the tree illustrated on the right-hand side of the same figure. That tree is called the *Fibonacci tree*. The reason for this name comes from the properties of the tree. The tree is a finitely branched tree generated by two rules:

$$B \longrightarrow BW \text{ and } W \longrightarrow BWW.$$

Indeed, we split the nodes into two kinds: black nodes and white nodes. Black nodes have two children, as suggested by the preceding rules: a black child, the left-hand-side child; and a white child, the right-hand-side one. White nodes have three children: a black child,

the leftmost one; and two white children, the others. The root of the tree may be a white node or a black one. From the preceding rules, it is not difficult to prove that there are exactly f_{2n+1} nodes lying on the n^{th} level of the tree, where f_k is the Fibonacci sequence where $f_0 = f_1 = 1$ when the root is a white node [3, 4]. When it is a black node, the number of nodes on the n^{th} level of the tree is f_{2n} . When the root of a Fibonacci tree is white/black, respectively, we say a white/black, respectively, Fibonacci tree.

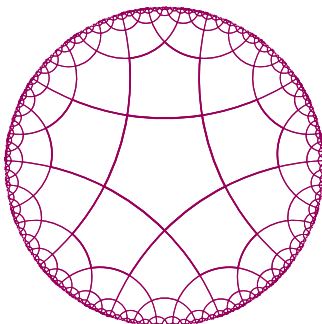


Figure 2. The pentagrid as it can be represented in Poincaré's disk model of the hyperbolic plane.

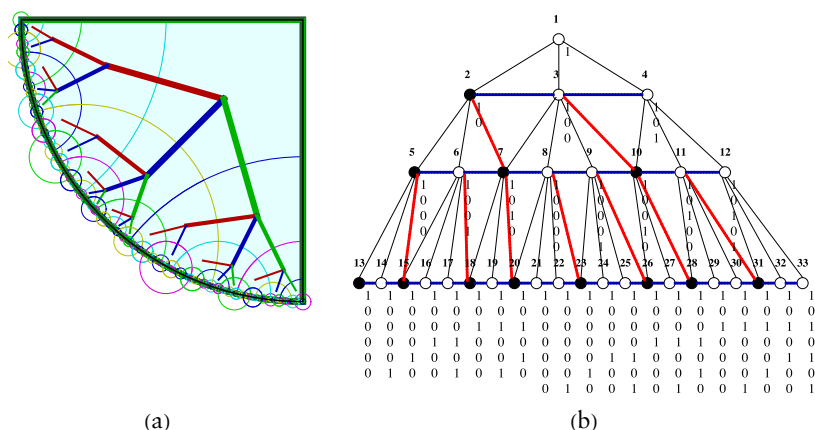


Figure 3. (a) A sector of the pentagrid generated by the Fibonacci tree illustrated in (b). (b) Under each node, vertically, we show the Fibonacci representation of the number attached to the node. We can check the preferred child property.

There is another, more striking property when the root is a white node. Number the nodes of the tree, starting from the root, which

receives 1, level after level and on each level from left to right. Then, represent these numbers in the Fibonacci sequence, choosing the representation whose number of digits is the largest. If $[n]$ is that Fibonacci representation of n , $[n00]$ is the Fibonacci representation of a child of n , which we call the *preferred child*. The preferred child of a black node is its black child. That of a white node is its middle child. The preferred child property can be checked in Figure 3(b).

Note that in Figure 2, one tile seems to play a different role than the others. It is the tile that contains the center of the support of D . As can be seen in Figure 2, not much can be seen from the tiling. We can see the central tile very well, as well as its neighbors, but going further from the central tile, we can see the tiles less and less. In fact, as the hyperbolic plane has no center, the pentagrid too has no tile playing a central role. We can view the support of our model as a *window* over the hyperbolic plane. We can imagine that we fly over that plane and that the window is a screen on the control board of our spacecraft. The center of that window is simply the point of the hyperbolic plane over which our spacecraft is flying. Indeed, we fly with instruments only, which we just defined. This window property of Poincaré's disk stresses that so little can be represented of this space in its Euclidean models. It is the reason why we choose the disk model.

We fix a tile τ_0 that we call from now on the *central tile*, and we will consider that the central tile is the tile in which the center of D lies in the figures. As illustrated by Figure 4(a), around the central tile, we can assemble five quarters like those defined in Figure 3(a) in order to construct the whole pentagrid. We call these quarters *sectors*. In each sector, the tiling is spanned by the white Fibonacci tree. It is not difficult to prove that the tiles that lie on the level k of a Fibonacci tree of a sector are at distance k from the central tile. We call *Fibonacci circle of level n* the set of tiles $C(\tau_0, n)$ denoted by \mathcal{F}_n . Similarly, we call *Fibonacci disks* the sets $D(\tau_0, n)$, which we denote by \mathcal{D}_n . Note that \mathcal{D}_n is the union of the Fibonacci circles \mathcal{F}_k with $0 \leq k \leq n$. In Figure 4, we illustrate the notion of Fibonacci circles and disks by marking in blue, green and gray the tiles that belong to \mathcal{F}_3 and by marking in pink those that belong to \mathcal{D}_2 .

We call the Fibonacci representation we attached to the number given to a node v of a white Fibonacci tree the *coordinate* of v , denoted by $[v]$. We identify the node with its number v . We locate the tiles of the pentagrid with 0 for the central tile and for the other tiles with two numbers: the number of the sector in which the tile lies and then the number of the node in the white Fibonacci tree that spans the sector, as is clear from Figure 4. We extend the coordinate of a tile: it is the number of its sector followed by its coordinate in the tree. We

will also say that the central tile is the support of the *central cell*. Again, the central cell is the cell on which we focus our attention at the given moment of our argumentation.

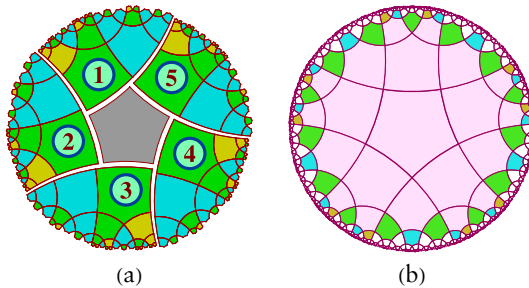


Figure 4. (a) How sectors are assembled around the central cell in order to get the pentagrid. (b) The Fibonacci circle of level 3. Together with the tiles of the Fibonacci circle, the tiles in pink, that is, the central cell and the tiles of levels 0 and 1, constitute the Fibonacci disk of level 2.

In our proof of Theorem 1, we need to consider a node v on \mathcal{F}_n together with its closest nodes, also on \mathcal{F}_n : one on the left-hand side of v —denote it by $v \ominus 1$ —and one on the right-hand side of v —denote it by $v \oplus 1$. Most often, $v \ominus 1 = v - 1$ and $v \oplus 1 = v + 1$. Now, if v is on the rightmost branch of the white Fibonacci tree, we have then that $v = f_{2n+2} - 1$ and $v \oplus 1 = f_{2n}$, but on the white tree of the next sector. Similarly, if $v = f_{2n}$, we have that $v \ominus 1 = f_{2n+2} - 1$ on the white tree of the previous sector. Of course, $v \oplus (k + 1) = (v \oplus k) + 1$ and similarly, $v \ominus (k -) = (v \ominus k) - 1$ for small values of k .

These considerations allow us to implement cellular automata in the pentagrid as performed in [3, 5]. As mentioned in the introduction, to each tile we associate a cell of the cellular automaton. We will also identify the cell by the number of its support. If η is the state of the cell attached to the tile v , we say that v is also an η -cell. As we have black nodes and white nodes for the *status* of the nodes in the Fibonacci tree, we will say W-cell or quiescent cell for a white cell in order to avoid any confusion with the status of its support.

In order to note the rules of a deterministic cellular automaton in the pentagrid, we introduce a numbering of the sides of each tile. The numbering starts from 1 and it is increased by 1 for each side while turning counterclockwise around the tile. For the central cell, side 1 is fixed once and for all, and for the other tiles, side 1 is the side of the tile shared with its parent, the central cell being the parent of the root of the tree. Neighbor i of a cell v shares with v the side i of v . The precision is required because the side shared by two tiles does not receive the same number in both tiles.

If η_0 is the current state of the cell, if η_0^1 is its new state and if η_i is the state of its neighbor i at the current time, then the rule giving η_0^1 from η_0 and the η_i is written as a word in $\{\mathcal{L}\}^*$, where \mathcal{L} is the set of states of the cellular automaton: $\underline{\eta_0}\eta_1\dots\eta_5\underline{\eta_0^1}$. The underscore is put under η_0 and η_0^1 in order to facilitate the reading. In a rule $\underline{\eta_0}\eta_1\dots\eta_5\underline{\eta_0^1}$, we say that $\underline{\eta_0}\eta_1\dots\eta_5$ is the *context* of the rule and we say that the word $\eta_1\dots\eta_5$ is the *state neighborhood* of the cell. In our situation, where our cellular automata have only two states, we denote the quiescent state by W and the non-quiescent one by B. Accordingly, the state neighborhood is a word of $\{W, B\}^5$. A finite sequence of nodes $v, v+1, \dots, v+k$ on \mathcal{F}_n gives rise to a word in $\{B, W\}^{k+1}$, where its i^{th} letter is the state of $v+i-1$. We call that word a *state pattern* on \mathcal{F}_n .

3. Rotation-Invariant Cellular Automata on the Pentagrid with Two States

By definition, the rules of a cellular automaton A on the pentagrid are said to be *invariant by rotation*, in short, *rotation invariant* and A is said to be a *rotation-invariant cellular automaton*, if for each rule present in the program of A , namely, $\underline{\eta_0}\eta_1\dots\eta_5\underline{\eta_0^1}$, the rules $\underline{\eta_0}\eta_{\pi(1)}\dots\eta_{\pi(5)}\underline{\eta_0^1}$ are also present, where π runs over the circular permutations on $[1..5]$. When the cellular automaton is rotation invariant, we usually indicate the rule where, after the current state, we have the state of neighbor 1.

The goal of this section is to prove:

Proposition 1. For any deterministic cellular automaton on the pentagrid, if its initial configuration is finite, if it has two states with one of them being quiescent, and if its rules are rotation invariant, then its halting problem is decidable.

Our proof is based on the following considerations. If the halting of the computation of a cellular automaton halts, it means that the computation remains in some \mathcal{D}_N forever. Note that the computation may remain within some \mathcal{D}_N and not halt. But in that case, after a certain time, the computation becomes periodic. And this can be detected: it is enough to find two identical configurations during the computation; this generalizes the situation of the halting. What is not that easy to detect is the case when the configuration extends to infin-

ity in the sense that for each circle \mathcal{F}_k , there is a time when that circle contains a non-quiescent cell.

Let us take a closer look at such a case. Let N_0 be the initial border number. We know that there is at least one tile v of \mathcal{F}_{N_0} that is a B-cell, at time 0. Call \mathcal{F}_{N_0} the *front* at time 0. The front at time t is \mathcal{F}_{N_t} , where N_t is the smallest k such that \mathcal{F}_k contains all configurations at time τ , with $\tau \leq t$, and such that all cells outside \mathcal{D}_k are quiescent. This is the reason why the initial border number is denoted by N_0 : \mathcal{F}_{N_0} is the front at time 0.

Our proof of Proposition 1 lies on the analysis of how a B-cell on the front at time t can propagate to the front at time $t+1$. Note that a similar remark is at the basis of Codd's proof that cellular automata in the Euclidean plane with two states and a von Neumann neighborhood have a decidable halting problem when the computation starts from a finite configuration; see [6]. If we can prove that N_t is a nondecreasing function of t that tends to infinity, we then prove that the computation of the cellular automaton does not halt. The main property that will allow us to detect such a situation is that a cell on \mathcal{F}_{n+1} has at most two neighbors on \mathcal{F}_n and the others on \mathcal{F}_{n+2} . So that if v is a node of the front that is a B-cell, the state neighborhood of its children is either BW^4 or B^2W^3 . That situation occurs if and only if the node $v \ominus 1$ of the front is also a B-cell. We say that a B-cell is *isolated* on \mathcal{F}_n if v being its support, $v \ominus 1$ and $v \oplus 1$ are both W-cells. These considerations significantly reduce the number of rules to consider and, consequently, the number of cases to scrutinize. More precisely, we have the following lemma.

Lemma 1. Let A be a deterministic cellular automaton on the pentagrid with two states, one being quiescent, and whose rules are rotation invariant. If the rule $\underline{WBW^4W}$ occurs in the program of A , the front at time $t+k$ is the same as the front at time $t+1$ for $k \geq 2$; that is, $N_{t+k} = N_{t+1}$ for the same values of k . If it is not the case, that is, if the rule $\underline{WBW^4B}$ occurs in the program of A , then if the front at time t contains a B-cell, the front at time $t+1$ also contains a B-cell; that is, we have $N_{t+1} = N_t + 1$.

Proof. Let v be the tile of \mathcal{F}_{N_t} which is a B-cell. Assume that v is an isolated B-cell of the front at time t . Let σ be a child of v . Whether σ is a black node or a white one, σ is a W-cell as well as its children. Accordingly, its state neighborhood is BW^4 , so that the rule $\underline{WBW^4W}$ applies. Consequently, σ remains a W-cell at time t .

If $v \oplus 1$ is also a B-cell at time t , let σ be the black child of $v \oplus 1$. Then, the state neighborhood of v is B^2W^3 . If the program of A

contains the rule $\underline{WB^2W^3W}$, then σ remains a W-cell at time $t + 1$, as well as the other children of ν . If the program contains the rule $\underline{WB^2W^3B}$, then σ becomes a B-cell at time $t + 1$ but the cells $\sigma \ominus 1$ and $\sigma \oplus 1$ are white nodes, so that whatever the state of their parent, they remain W-cells at time $t + 1$, as either the quiescent rule or the rule $\underline{WBW^4W}$ applies to them. Accordingly, in that case, the cell σ is an isolated B-cell of the front at time $t + 1$. Now, what we proved in the previous paragraph shows us that the children of σ remain W-cells at the time $t + 1$, so that the front at time $t + 2$ is the same as at time $t + 1$ and it remains the same afterward. This proves the part of the lemma concerning the rule $\underline{WBW^4W}$.

Assume that the rule $\underline{WBW^4B}$ occurs in the program of A . From our previous study on the children of ν , at least one of them is a white node, which means that its neighborhood is BW^4 . Accordingly, if ν is a B-cell, that white child becomes a B-cell at the next time, so that $\mathcal{F}_{N_{t+1}} = \mathcal{F}_{N_t+1}$. \square

We are now in position to prove Proposition 1. If the initial configuration is empty, that is, if all tiles are W-cells at time 0, there is nothing to prove: the configuration remains empty forever. Accordingly, if the initial configuration is not empty, N is definite, so that \mathcal{F}_N contains at least one B-cell. From Lemma 1, if the rule $\underline{WBW^4B}$ occurs in the program of A , the front moves forward by one step at each time, so that the computation of the cellular automaton does not halt. If that rule does not occur, then necessarily, the rule $\underline{WBW^4W}$ is present in the program of A . From Lemma 1, we know that at most, we have $\mathcal{F}_{t_1} = \mathcal{F}_1$, but that necessarily, $\mathcal{F}_{t_k} = \mathcal{F}_1$ for $k \geq 1$. \square

4. When the Rules Are Not Rotation Invariant

Here again, we deal with a deterministic cellular automaton with a quiescent state that starts its computation from a finite configuration. But in this section, we relax the assumption of rotation invariance. The convention we fixed in Section 2 for the numbering of the sides of a tile have their full meaning in this section. And so, a rule $\eta_0\eta_1\dots\eta_5\eta_0^1$ may be different from a rule $\eta_0\eta_{\pi(1)}\dots\eta_{\pi(5)}\eta_0^1$ where π is a permutation over $[1..5]$. Note that this time, the order of the letters in the state neighborhood associated to the rule is meaningful.

Consider a cell $\nu \in \mathcal{F}_{n+1}$. In all cases, its neighbor 1 is its parent, which, by construction, belongs to \mathcal{F}_n . If ν is a black node, as already

noticed in previous sections, v has exactly two neighbors that belong to \mathcal{F}_n : neighbor 1, as it is the parent, and also neighbor 2. Consider N_0 the initial border number. From what we just noticed, a rule can make a state B move from \mathcal{F}_{N_0} to \mathcal{F}_{N_0+1} if its state neighborhood starts with BW, WB or B²: the last two cases may happen if the considered cell of \mathcal{F}_{N_0+1} is a black node. As an example, the state neighborhood of the tile v of \mathcal{F}_{N_0+1} cannot be WWBWW: if a rule whose state neighborhood is WWBWW is applied to a cell of \mathcal{F}_n , its neighbor that is a B-cell belongs to \mathcal{F}_{n+1} .

Lemma 2. Let A be a deterministic cellular automaton on the pentagrid with two states, where one of them is a quiescent state (Table 1). If the program of A contains the rule $\underline{\text{WBW}}^4\underline{\text{W}}$ and the rule $\underline{\text{WWBW}}^3\underline{\text{W}}$, then $\mathcal{F}_{N_1+k} = \mathcal{F}_{N_1}$ for all positive integers k with $k \geq 2$.

Proof. The proof comes from the fact that the state neighborhood of a child of a node v that is an isolated B-cell of the front is BW⁴. If v is a B-cell and if $v \ominus 1$ is a W-cell, then the state neighborhood of the leftmost child of v is WBW³. Now, the children of an isolated B-cell of the front at time t remain quiescent at time $t+1$. If the program of A contains the rule $\underline{\text{WB}}^2\underline{\text{W}}^3\underline{\text{B}}$, then if the state pattern BB is present on the front at time t , say at the nodes $v \ominus 1$ and v , then the just-mentioned rule applies to the leftmost child σ of v , but the white children of $v \ominus 1$ and those of v remain W-cells at time $N_t + 2$. Accordingly, σ is an isolated B-node of the level $N_t + 1$ so that, from the previous study, all cells on \mathcal{F}_{N_t+2} remain quiescent, so that $\mathcal{F}_{N_t+k} = \mathcal{F}_{N_t+2}$ for all positive integers k . Clearly, the same conclusion holds if the program of A contains the rule $\underline{\text{WB}}^2\underline{\text{W}}^3\underline{\text{W}}$. \square

BW	$\underline{\text{WBW}}^4\underline{\text{B}}$	$\overline{\text{BW}}$	$\underline{\text{WBW}}^4\underline{\text{W}}$
WB	$\underline{\text{WWBW}}^3\underline{\text{B}}$	$\overline{\text{WB}}$	$\underline{\text{WWBW}}^3\underline{\text{W}}$
BB	$\underline{\text{WB}}^2\underline{\text{W}}^3\underline{\text{B}}$	$\overline{\text{BB}}$	$\underline{\text{WB}}^2\underline{\text{W}}^3\underline{\text{W}}$

Table 1. Rules of a deterministic cellular automaton on the pentagrid with two states that apply to the children of a node of the front, W being the quiescent state.

Let $v \oplus 1, \dots, v \oplus k$ be a sequence of consecutive nodes on the circle \mathcal{F}_n . Then, the word $\eta_1 \dots \eta_k$ with $\eta_i \in \{\text{B}, \text{W}\}$, $i \in \{1 \dots k\}$ is called a *state pattern*.

Let us now prove Theorem 1. From Lemma 2, the computation remains within \mathcal{D}_{N_1+2} if the program of A contains both rules $\overline{\text{BW}}$ and

\overline{WB} . Accordingly, we may assume that it contains either the rule BW or the rule WB.

Consider the case when the rule BW belongs to the program of A . If a node v of the front at time t is a B-cell, from the proof of Lemma 2 we know that there is also a B-cell on the front at time $t + 1$ and that we have $N_{t+1} = N_t + 1$, as any white child of v is a B-cell on \mathcal{F}_{N_t+1} at time $N_t + 1$.

Consider the case when the rules \overline{BW} and WB belong to the program of A . If a node v of the front at time t is a B-cell, we have to look at the case when the state pattern BW is present on the front or not. As by definition the front contains at least one B-cell, if the state pattern BW is not present, this means that all tiles of the front are B-cells. In that case, all black nodes of \mathcal{F}_{N_t+1} have the state neighbor B^2W^3 . Accordingly, as the white nodes of \mathcal{F}_{N_t+1} remain quiescent at time $t + 1$, the evolution depends on the rule whose context is \overline{WB}^2W^3 .

If the rule is \overline{BB} , then the black nodes of \mathcal{F}_{N_t+1} remain quiescent at time $t + 1$, which entails that all nodes of \mathcal{F}_{N_t+1} remain quiescent at time $t + 1$. Now, if at time $t + 1$ at least one node of \mathcal{F}_{N_t} at time t becomes a W-cell at time $t + 1$ and at least one remains a B-cell, then the pattern BW occurs, say on the nodes $v \ominus 1$ and v . Then, if σ is the black node of v , its state neighborhood at time $t + 1$ is WBW^3 , so that the rule WB applies and σ becomes a B-cell at time $t + 2$. From the rule \overline{BW} , we know that the white children of the B-cells on \mathcal{F}_{N_t} remain W-cells, and so the black nodes on \mathcal{F}_{N_t+1} are isolated B-cells on the level N_t+1 . Accordingly, the rule WB applies to their black children on \mathcal{F}_{N_t+2} at time $t + 2$. The argument applies again to those nodes that are also isolated B-cells on the new front. So that $\mathcal{F}_{N_{t+k}} = \mathcal{F}_{N_t+k}$ for all positive integers k .

We remain with the situation when all the nodes of the front at time t are B-cells and all of them become W-cells at time $t + 1$. We can repeat the preceding analysis to time $t + 2$. If at that time all nodes are again B-cells, say that this situation is an alternation of B and W. If such a situation is repeated long enough, as in the case where the front does not go beyond \mathcal{D}_{N_t+1} , the computation remains forever within that disk and so the computation is periodic. We know that such an evolution can be detected: it is enough to observe two identical configurations. If this is not the case, we find a situation where the front contains the state pattern BW, so that the rule WB applies endlessly, as already seen.

If the program of A contains the rule BB , as it also contains the rule $\overline{\text{BW}}$, the state pattern BW occurs on \mathcal{F}_{N_t+1} at time $t + 1$ as soon as the pattern BB occurs on \mathcal{F}_{N_t} at time t . If the pattern BB does not occur, clearly, the pattern BW occurs on \mathcal{F}_{N_t} at time t , so that we have the same conclusion as we had with the rule $\overline{\text{BB}}$ when the pattern BW occurs on the front: a non-halting computation that is detected by the occurrence of that pattern.

We can summarize the discussion as shown in Table 2.

Rules	Front	Evolution
BW	a B-cell	$N_{t+1} = N_t + 1$ from some t_0
$\overline{\text{BW}}, \overline{\text{WB}}$	any	within $\mathcal{D}_{N_{t_0}}$ from some t_0
$\overline{\text{BW}}, \text{WB}$	a BW	$N_{t+1} = N_t + 1$ from some t_0
$\overline{\text{BW}}, \text{WB}$	never BW	within $\mathcal{D}_{N_{t_0}}$ from some t_0

Table 2. Table of the evolutions of the computation of A depending on its rules of Table 1 and on the patterns that can be seen on the front.

Accordingly, as we have analyzed all possible cases and as each one can be detected, we conclude that the proof of Theorem 1 is completed. \square

5. Propagation of the Front

Although we solved the question about the halting problem for such cellular automata, it can be interesting to examine their behavior in the case when the computation does not halt with an unbounded occurrence of non-quiescent cells. We will focus on the front. Up to now, we have seen that a motion to infinity exactly means that the front is increasing starting from some time t_0 . This happens in different settings, as shown by Table 2. It could be interesting to have more information about such a motion. However, as the situation may be intricate in some cases, as can be seen in the proof of Theorem 1 when the rules are $\overline{\text{BW}}$, WB and $\overline{\text{BB}}$, we will restrict our attention to what happens on the front. We will see that with only two states, the study of this restricted aspect is not that trivial.

From Table 2, we know that we basically have to consider two cases: the case when the program of A contains the rule BW and the

case when it contains the rule \overline{BW} together with the rule WB in the case that the pattern BW appears at some time on the front.

Consider that latter case. From the proof of Theorem 1, we know that if the pattern BW appears on the front at time t , it will also appear on the front at time $t + 1$. But the proof has given us more exact information. If v is the node of the B-cell of some BW pattern of the front at time t , the application of the rule WB to the black child σ of $v \oplus 1$ produces a BW pattern on the nodes σ and $\sigma \oplus 1$, as $\sigma \oplus 1$ remains a quiescent cell due to the fact that $v \oplus 1$ is a W-cell and that $\sigma \oplus 1$ is a white node. And so, σ and $\sigma \oplus 1$ define a BW pattern on the front at time $t+1$. Notice that the B-cell of the pattern BW on $\mathcal{F}_{N_{t+1}}$ is isolated. The same arguments can be repeated concerning the black child of $\sigma \oplus 1$. Consequently, a pattern BW where the B-cell is isolated on the front at time t generates a sequence of such patterns on each front at time $t + k$, with k being a positive integer, the B-cell of such a pattern being isolated and being the black child of the W-cell of the same pattern on the previous front. We can call this sequence a *line of patterns* BW. Accordingly, if there are k patterns BW on the front at time t , each of them generates a line of patterns BW on the successive fronts after time t .

From now on, consider the case when the program of A contains the rule BW.

If a B-cell occurs on the front at time t on the node v , the white children of v become B-cells at time $t + 1$, as seen in the proof of Theorem 1. Accordingly, not only does the front at time $t + 1$ contain a B-cell, it also contains the pattern BB. If $v \oplus 1$ is a W-cell, its black child is a B-cell if and only if the program of A contains the rule WB. In that case, the front at time $t + 1$ contains the pattern BBB. The occurrence of the pattern BB on the front raises the question of which rule, BB or \overline{BB} , belongs to the program of A .

The easiest case to analyze is the case when together with the rule BW, we also have the rules WB and BB.

In Figure 5, the result of applying the rules BW, WB and BB, respectively, yields the cells in blue, purple and green, respectively. Clearly, the white neighbor of the purple neighbor of the central cell is its parent; see times 1, 2, 3 and 4. It is assumed that the state B is permanent: once a cell gets to that state, it remains unchanged. The figure also assumes that we start from a single B-cell on the front at time 0. That cell is placed as the central cell of Figures 5 through 7 in order to focus attention on the evolution of the computation from that cell.

In Figure 6, as opposed to Figure 5, it is assumed that a B-cell at time t becomes quiescent and remains in that state later on. Note that this representation allows us to better see the propagation of the front

in the case of the motions ruled by the occurrence of the rule BW in the program of A . As we assume that the rules WB and BB also belong to the program of A , we can easily see that the children of a B-cell in node v at time t are B-cells at time $t + 1$, whatever the states at time t of the nodes $v \ominus 1$ and $v \oplus 1$. Accordingly, on the front at time $t + k$, the B-cells occupy at least the whole level k of the Fibonacci tree rooted at v , whether v is a black node or a white one.

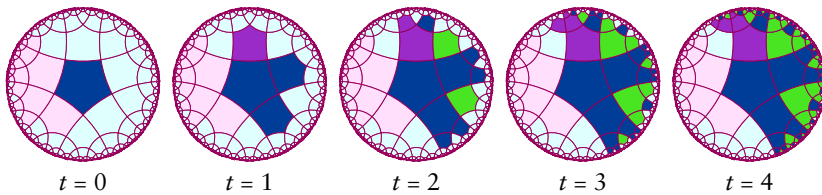


Figure 5. The program contains the rules BW, WB and BB. From left to right, times 0, 1, 2, 3 and 4. It is assumed that once a node is a B-cell, it remains in this situation. The light pink cells represent the circles that are behind the front at time t .

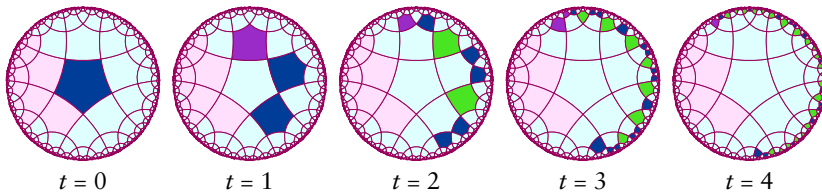


Figure 6. The program contains the rules BW, WB and BB. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a B-cell, it becomes a quiescent cell at the next time.

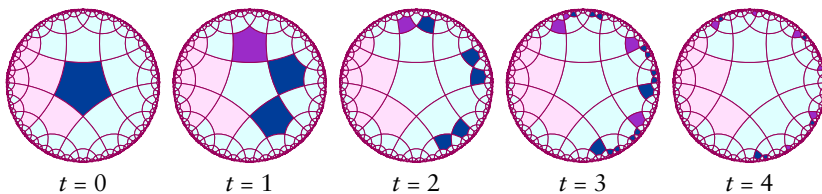


Figure 7. The program contains the rules BW, WB and \overline{BB} . From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a B-cell, it becomes a quiescent cell at the next time.

Still assume that we have the rules BW and WB, but that we have the rule \overline{BB} . Figure 7 illustrates the propagation of the front in that case, starting with a single B-cell on the front at the initial time. The

graphic at time 3 in that figure indicates that the pattern BBB appears on the front of that time and the graphic at time 4 seems to indicate the same property and that no new pattern appears. Let us prove this property.

Lemma 3. Let A be a deterministic cellular automaton on the pentagrid with two states, one of them being quiescent. Assume that the program of A contains the rule BW. The states of the cells attached to the white children of a white node v at time $t + 1$ are the state of the cell of v at time t .

Proof. Indeed, let v be a cell of the front at time t supported by a white node. Its white children belong to \mathcal{F}_{N_t+1} and, as white nodes, they have one neighbor on \mathcal{F}_{N_t} and four of them on \mathcal{F}_{N_t+2} . At time t , those four neighbors are W-cells by definition of the front at time t , so that the quiescent rule applies if v is a W-cell and the rule BW applies if v is a B-cell. In both cases, we get the conclusion of the lemma. \square

Lemma 3 shows that among the children of a white node on the front, two of them always have the same state at the next time. We can now state:

Lemma 4. Let A be a deterministic cellular automaton on the pentagrid with two states, one of them being quiescent. Assume that the program of A contains the rules BW, WB and BB. Then, the front at time t with $t \geq 3$ does not contain either the pattern WBW or the pattern BBBB.

Proof. Assume that the pattern WBW occurs at time $t + 2$. Let v , $v \oplus 1$ and $v \oplus 2$ be the nodes supporting that pattern. From the rules and from Lemma 3, the nodes v , $v \oplus 1$ and $v \oplus 2$ cannot have the same parent, which should be a white node. Accordingly, the parents of v , $v \oplus 1$ and $v \oplus 2$ are different, say φ and $\varphi \oplus 1$. Assume that $\varphi \oplus 1$ is the parent of $v \oplus 1$ and $v \oplus 2$, so that $v \oplus 1$ is a black node and $v \oplus 2$ is a white one. Also, v must be a white child of φ . By Lemma 3, both φ and $\varphi \oplus 1$ should be W-cells, so that by the quiescent rule, $v \oplus 1$ should be a W-cell, a contradiction with our assumption.

And so, we have that φ is the parent of v and $v \oplus 1$ and that $\varphi \oplus 1$ is the parent of $v \oplus 2$.

Then, φ , which we consider on the front at time $t + 1$, cannot be a white node, as its white children would bear different states at time $t + 2$. So φ is a black node and v is its black child and v is its white one. Accordingly, φ is a B-cell at time $t + 1$. As $v \oplus 2$ is a W-cell at time $t + 2$, $\varphi \oplus 1$ must also be a B-cell at time $t + 1$. Let us look at what happens at time $t + 3$. Let σ be the white child of v , which is a black node. Then, the rule BW applies to σ , $\sigma \oplus 1$, $\sigma \oplus 2$ and $\sigma \oplus 3$, producing the pattern WBBB. Now, the rule WB applies to $\sigma \oplus 4$ as

that node is a black one, and by Lemma 3, $\sigma \oplus 5$ is quiescent, so that starting from σ , the children of ν , $\nu \oplus 1$ and $\nu \oplus 2$ produce the state pattern WBBBBW at time $t + 3$. Applying the rules in a similar way, at time $t + 4$, starting from the rightmost child of σ , we obtain the pattern WBBWBBWBBWBBW, where the rightmost W is the first white child of $\sigma \oplus 5$.

Now, consider the case of a pattern WBBBBW on the front at time $t + 1$, and let ν be the node that gives the leftmost W at that time. By Lemma 3, ν , $\nu \oplus 1$ and $\nu \oplus 2$ cannot be the children of a white node φ . A similar contradiction would occur if we assume that ν and $\nu \oplus 1$ are children of a black node. We conclude that ν is the rightmost child of a node φ . If we assume that $\nu \oplus 2$ and $\nu \oplus 3$ are the children of $\varphi \oplus 1$, which should accordingly be a black node supporting a B-cell, we have a contradiction between the state of $\varphi \oplus 2$ at time t , which would be a white node, and that of $\nu \oplus 3$ and $\nu \oplus 4$ at time $t + 1$. Accordingly, $\varphi \oplus 1$ must be a white node and $\nu \oplus 4$ and $\nu \oplus 5$ are children of $\varphi \oplus 2$, so that we find the situation associated with the pattern WBW.

We have seen that the pattern produced by the children of the nodes supporting WBBBBW does not contain either WBW or BBBB. It contains four occurrences of BB, two of them being separated by a single W.

Now, let us look at the pattern WBBW, which we assume to be on the front at time $t + 1$. Let ν be the node that supports the left-hand-side W of the pattern. The nodes ν , $\nu \oplus 1$ and $\nu \oplus 2$ can be the children of a white node φ , which is necessarily a B-cell at time t . As ν is a W-cell at time $t + 1$, φ must be a B-cell at time t . This indicates which kinds of nodes ν , $\nu \oplus 1$ and $\nu \oplus 2$ are, and clearly, $\nu \oplus 3$ is a black node. Applying the rules to the children of these nodes, we get that, from σ is the rightmost child of ν until the leftmost white child of $\nu \oplus 3$, the nodes $\nu \oplus i$ produce the pattern WBBBWBBW at time $t + 2$.

However, $\nu \oplus 1$, $\nu \oplus 2$ and $\nu \oplus 3$ cannot be the children of a white node, as $\nu \oplus 2$ and $\nu \oplus 3$ have different states. Another disposition for the parents of the node we have seen is that the parent of ν , say φ , is a black node, so that $\varphi \oplus 1$ is a white one. Necessarily, φ is a B-cell at time t and $\varphi \oplus 1$ is a white one. Looking at the children of the nodes ν , $\nu \oplus 1$, $\nu \oplus 2$ and $\nu \oplus 3$, starting from the rightmost child σ of ν until the leftmost white child of $\nu \oplus 3$, we find this time the pattern WBBBWBBW.

Let us now consider the pattern WBBBW on the front at time $t + 2$. Again, let ν be the node that supports the leftmost W-cell of this pattern. We can see that the nodes ν , $\nu \oplus 1$ and $\nu \oplus 2$ can be the children of a node φ , which must be a B-cell at time t , while the node $\varphi \oplus 1$ must be a W-cell at the same time. It is not difficult to see that under

that assumption on φ and $\varphi \oplus 1$ with respect to the nodes $\nu \oplus i$, if σ is the rightmost child of ν , we get the pattern WBBWBBWBBW on the front at time $t + 2$ until the leftmost white child of $\nu \oplus 4$.

If ν and $\nu \oplus 1$ would be the children of a black node φ , the other nodes $\nu \oplus i$ would be the children of the necessarily white node $\varphi \oplus 1$. The nodes $\nu \oplus 3$ and $\nu \oplus 4$ have different colors at time $t + 1$, a contradiction with Lemma 3. And so, another configuration that is possible this time, is that the nodes $\nu \oplus i$ we consider have three parents: φ is the parent of ν only, $\varphi \oplus 1$ is a white node or a black one, respectively, it does not matter; and $\varphi \oplus 2$ is the parent of $\nu \oplus 4$ or of $\nu \oplus 3$ and $\nu \oplus 4$, respectively. In both cases, $\nu \oplus 1$ is a black node; that is the important point. It can be checked that in both cases, if σ is the rightmost child of ν , the pattern on the front at time $t + 2$ starting from σ and ending on the first white child of $\nu \oplus 4$ is WBBWBBWBBW.

With this analysis, the proof of Lemma 4 is completed. \square

We have analyzed the situation when the program of A contains the rules BW and WB. With programs containing the rule BW, the case of the rule \overline{WB} remains to be considered. Figure 8 illustrates the propagation of the front whatever the rule, BB or \overline{BB} .

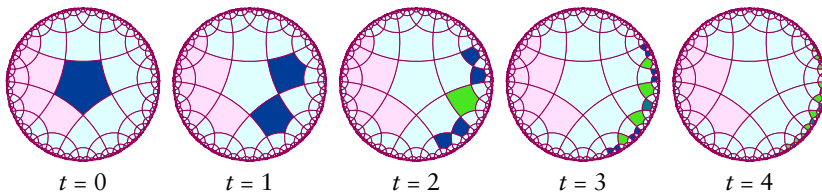


Figure 8. The program contains the rules BW and \overline{WB} . In green and dark blue are the B-cells produced when using the rule BB too. When the program contains the rule \overline{BB} , the B-cells are restricted to the dark blue cells. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a B-cell, it becomes a quiescent cell at the next time.

In fact, the figure illustrates both cases: as mentioned in the caption of the figure, a different coloration is applied to the cells produced directly by the application of the rule BB or to further applications of all rules in the tree rooted at the node where a first application of the rule BB was performed. When the rules BW, \overline{WB} and BB are applied starting from an isolated B-cell supported by a node ν on the front at time t , the evolution of the computation concerns the Fibonacci tree rooted at ν and on the front at time $t + k$; the trace of that computation is the whole level k of that tree. Say that a node ν is *hereditary*

white if there is a sequence of k white nodes v_i , with $i \in \{1 \dots k\}$ such that v_{i+1} is a white child of v_i with $i \in \{1 \dots k-1\}$ and $v = v_k$. When the rule \overline{BB} is used in place of the rule BB , the trace is restricted to hereditary white nodes only.

We can summarize our analysis by appending Table 3 to Table 2. The table assumes that we start from a B-cell supported by a white node of the front at time t . In order to better analyze the patterns, we remind the reader that the number of nodes on the level k of a white, black Fibonacci tree, respectively, is f_{2k+1} , f_{2k} , respectively.

Rules	Patterns at $t+k$
BW, WB, BB	$WB^{f_{2k+1}+f_{2k-2}}W$
BW, WB, \overline{BB}	WBBW, WBBBW in a range wider than $f_{2k+1} + f_{2k-2}$ nodes
BW, \overline{WB} , BB	$WB^{f_{2k+1}}W$
BW, \overline{WB} , \overline{BB}	B on hereditary white nodes in a range wider than f_{2k+1} nodes

Table 3. Patterns on the front at time $t + k$ when the program of A contains the rule BW starting from a B-cell in a white node of the front at time t .

We still need to append information regarding the case when the program of A is rotation invariant. The first remark is that in such a situation, there is no difference between the rules BW and WB as well as between the rules \overline{BW} and \overline{WB} . As we assume the rule BW, there is no consideration of a rule \overline{WB} . This also means that in a situation where we applied the rule WB when rotation invariance is relaxed, in the case of rotation invariance, we apply the rule BW. However, note that the rules BW and \overline{WB} are contradictory under rotation invariance, as in that case, \overline{WB} is the same rule as \overline{BW} , which, by definition, is opposite to BW. And so, we are concerned with the first two rows of Table 3. However, there is a special phenomenon that occurs here and may not occur in the situation where the rotation invariance does not take place. It is illustrated by Figures 9 and 10.

In Figure 9, we assume that besides the rule BW, the rule BB also belongs to the program of A . In this case too, the rules BW and WB are the same up to a circular permutation on the neighbors.

Comparing Figure 6 with Figure 9 on one hand and Figure 7 with Figure 10 on the other hand, we can see in both cases that the B-cells are at the same places during the propagation.

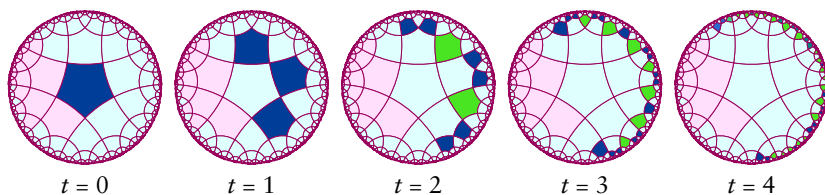


Figure 9. The program contains the rotation-invariant rules BW and BB. In green are the B-cells produced when using the rule BB too. From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a B-cell, it becomes a quiescent cell at the next time.

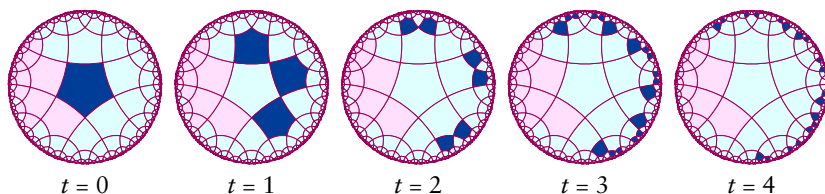


Figure 10. The program contains the rotation-invariant rules BW and \overline{BB} . From left to right, times 0, 1, 2, 3 and 4. It is assumed that when a node is a B-cell, it becomes a quiescent cell at the next time.

6. Conclusion

Of course, the first question is what can be said for three states? That issue is more difficult. We already have seen a rather difficult situation in the proof of Theorem 1 when it could happen that the front enters a situation when all cells are in the state B at some time t and then all of them are in the state W at time $t + 1$, repeated starting from time $t + 2$. As then the front remains at the same place during a certain time, the discussion was how long such a situation might last. A worse situation occurs with three states for something that we could ignore with two states: the point is what happens behind the front? In fact, in case a node changes its state from W to B behind the front, the worst thing that might happen is that another line of B-cells might propagate but, in that case, another line already occurred, so that we are in the situation of a constant advance of the front. Accordingly, it does not change the issue for the halting of the computation.

Things are different with three states. Let the three states be W, B and R, where W is the quiescent state that is associated to the quiescent rule possessed by the program of our cellular automaton. As a third state enters play, we may have the following rules:

BWB: $\underline{WBW^4B}$, BWR: $\underline{WBW^4R}$, RWB: $\underline{WRW^4B}$.

RWR: $\underline{WRW^4R}$.

RWW: $\underline{WRW^4W}$.

Clearly, if we have the rule BWB or the rule RWR, we have a constant progression of the front once a non-quiescent cell occurs on the front. A similar conclusion occurs if we have both rules BWR and RWB: they call each other in some sense, again, once a non-quiescent cell occurs on the front. What happens if, instead of both rules BWR and RWB we have, for instance, both rules BWR and RWW? In that case, assume that the rule BWR applies to the node φ of the front at time t . Let ν be the first white child of φ and let σ be the first white child of ν . Then, at time $t + 1$, ν becomes an R-cell and, due to RWW, at time $t + 2$, σ becomes (remains) a quiescent cell. Now, it may happen that at time $t + 2$, ν becomes a B-cell. In that case, σ becomes an R-cell at time $t + 3$. However, even if the transformation of ν from an R-cell to a B-cell happens at time $t + 2$, we are not guaranteed that the same transformation will happen for σ at time $t + 4$. The reason is that in those cases, the transformation depends on what happened behind the front. Note that in our discussions with a single non-quiescent state, it was enough to look at the rules that apply to a quiescent cell and not to look at those that apply to a cell in a non-quiescent state, although in the figures, in order to obtain nice images, we made implicit assumptions on rules applied to a B-cell or to a W-cell behind the front whose neighborhood may be different from BW^4 , WBW^3 or B^2W^3 . If we ignore the complex discussion involving a huge number of rules, we might expect an argument on how long we have to wait for a new transformation of ν from R to B. Even if we have an argument on the number of possible configurations within \mathcal{D}_{N_t} , to repeat the same argument to σ requires us to consider the number of possible configurations within \mathcal{D}_{N_t+1} , which is much bigger. Accordingly, this leads to no conclusion, so that the case with three states is open, even with rotation invariance.

Other questions may be considered. We know that strong universality is possible for deterministic cellular automata on the pentagrid with a quiescent state with 10 states; see [2]. That cellular automaton is rotation invariant. What can be performed if we relax rotation invariance? The answer is not straightforward, as the cellular automaton of [2] is based on a cellular automaton on the line that is strongly universal with 11 states, and six states of that automaton could be absorbed by the cellular automaton of the pentagrid that implements the cellular automaton on the line. And so, for that direction too, a new approach is needed.

Accordingly, as the gap between two states and 10 states seems to be not a small one, there is still a huge amount of work ahead.

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