Strong and Weak Spatial Segregation with Multilevel Discrimination Criteria

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This paper is influenced by the research of Thomas Schelling on *spatial segregation*; in his seminal work on the subject he used simple simulations to show that even highly tolerant individuals end up being spatially aggregated far beyond the local requirement of their tolerance level. In this paper we are not seeking to find the conditions, in terms of density of population and tolerance level, that lead to a global stable state where all the individuals are satisfied in view of their own neighborhoods. Here the context is: (*i*) a space full of agents where each individual is in continual contact with a maximum number of neighbors; and (*ii*) where both a principal and a secondary discrimination criterion compel people to leave their places. As, in general, the first hypothesis does not allow the population to converge within the meaning of Schelling, only incomplete segregation phenomena are observable. So the problematic will be to determine, according to the respective strength of the two discrimination criteria, the spatial repartition of the agents resulting from their moves; in such a general context, we will refer to segregation as being *strong* or *weak* or even *mixed*.

Keywords: spatial segregation; computational sociology; agent-based simulation; multilevel discrimination criteria

1. Introduction

The general context of this paper is formulated by Schelling [1, 2], in these words: "The [..] subject that occupied me in the seventies was the ways that individual behavioral choices could aggregate into social phenomena that were unintended or unexpected." In the specific context of spatial segregation, Schelling used simple simulations to show that even highly tolerant individuals end up being spatially aggregated far beyond the local requirement of their tolerance level. These simulations performed on a physical chessboard can be viewed as one of the first steps in what is now known as "computational sociology and agent-based modeling" [3, 4]. The result is a funding gap between the emergent global behavior and the local behaviors of each individual [5].

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The Schelling model of segregation has been extended to more than two types: to individual tolerance thresholds and to proximity networks other than a simple grid [5–11]. One particularity of the Schelling model is that it uses a grid network where some nodes are occupied by one agent while others are vacant; it is precisely the vacant places that: (*i*) allow movement of unsatisfied agents; (*ii*) in the long term, allow convergence toward configurations where all the agents are satisfied; and (*iii*) form the frontier between the aggregates.

In this paper we consider a tiny artificial world in the form of a grid composed of individuals where each one has their own behavior and can establish some links with others based on predetermined criteria. As there are no vacant places, the dynamics will rarely lead to a fixed point in the configuration space where all the agents are satisfied.

In the first part, we use one criterion only; it is in this context that we will make a first distinction between strong and weak segregation. This primary work will allow us to go deeper into the issue: in a "crowded world" (i.e., without vacant places), in what sense can one speak of convergence?

In a second step, we use both a primary and a secondary discrimination criterion. To show this idea with one example, the primary criterion would be the nationality and the second the native language; we assume: (*i*) two nationalities and two languages only; (*ii*) the same number of people in each country; and (*iii*) in each country, the same number of people practice each language. Saying that nationality is the primary criterion means that a person does not tolerate a person of the other nationality despite language. Saying that language is the secondary criterion means that a person does not tolerate people that practice the other language even if they have the same nationality. This second part will allow us to answer the questions: (*i*) to what extent does the existence of a secondary criterion change segregation due to the primary criterion?; and (*ii*) is the strong-weak dichotomy for segregation still fully relevant in the two-criteria context?

2. Micro-Motive versus Macro-Behavior

We assume that: (*i*) the world is a two-dimensional *grid* composed of cells (in all the simulations, the grid will be a square of 2500 cells); (*ii*) the world is populated by agents (let A be the population with $\sharp A = N$); and (*iii*) there is at most one agent per cell. To avoid edge effects, we impose that the world wraps horizontally and vertically, so the neighborhood of one agent is composed of the eight nearest cells surrounding it.

2.1 Agent Motive

For each agent $a_i \in A$, let $N_i(t)$ be the set of the neighboring agents at time t; the cardinal $\sharp N_i(t)$ is a measure of the "in/out" influence of the agent; obviously, $\sharp N_i \leq 8$.

2.1.1 Type of Agents

Apart from its location, an agent a_i is described by a *type* attribute t_i that remains constant over time. Let type $= \{t^0, \ldots, t^n\}$ be the set of type values; for instance, in the classical Schelling model there are only two type values: type $= \{t^0, t^1\}$. $\forall\, i \in \{0, \ldots, n\}$, let T^i be the set of agents with type value t^i; these subsets form a partition of A and $\sharp T^i = N / (n+1)$.

2.1.2 Dislike Function

For each type value $t \in$ type, the opposite types are defined by a *dislike* function δ from the set of types to its power set:

$$\begin{array}{rcl} \text{type} & \to & P(\text{type}) \\ t & \mapsto & \delta(t). \end{array}$$

For instance, in the classical Schelling model $\delta(t^0) = \{t^1\}$ and $\delta(t^1) = \{t^0\}$. A dislike function defines one discrimination criterion and we assume than there may be several criteria.

2.1.3 Agent Satisfaction

Regarding a discrimination criterion, the satisfaction of an agent at one time depends on its own type and on the type of its neighbors.

Let $a_i \in A$ with type t_i; we define the set of neighbors with an opposite type as $\Delta_i = \{a_k \in N(a_i) \mid t_k \in \delta(t_i)\}$. The *dislike ratio* (hereafter dr_i) defined by equation (1) measures the ratio of the number of neighbors of opposite types to the number of neighbors:

$$\mathrm{dr}_i = \begin{cases} \dfrac{\sharp \Delta_i}{\sharp N_i} & \text{if } (\sharp N_i(t) \neq 0) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then, for each agent a_i and for each dislike function δ, at one time we define the Boolean indicator *satisfied* as:

$$\text{satisfied}_i^\delta = (\mathrm{dr}_i \leq \tau) \tag{2}$$

where τ is a constant number in the range $[0..1]$ that denotes the threshold under which an agent is satisfied regarding the dislike function δ; we will refer to τ as the *tolerance* of the agents and we assume that its value is common to all agents.

As there may be several dislike functions δ_k, we will potentially need many tolerance values τ_k, one for each function. In such a case, one agent a_i is globally satisfied at one time if and only if it is satisfied for all the dislike functions; that is:

$$\text{satisfied}_i = \prod_{0<k<n} \text{satisfied}_i^{\delta_k} \tag{3}$$

where the product stands for the logical AND operator.

2.2 Agent Behavior

Agent behavior is oriented on achieving and maintaining satisfaction (equation (2)): an unsatisfied agent is motivated to move toward another location, whereas a satisfied agent has no incentive to move.

2.2.1 Agent Mobility

Regarding the problem of finding a new place, two cases are distinguished depending on the density δ of agents on the grid: (*i*) if there are some vacant places (i.e., $\delta > 1$), an unsatisfied agent can choose a location randomly and moves into it if and only if this location is vacant; and (*ii*) else, if the grid is wholly occupied by agents (i.e., $\delta = 1$), and if possible, two unsatisfied agents swap places. In both cases, as the moves do not equate to immediate benefits, it is challenging to predict the overall emerging effect. The gap between micromotives and macro-behaviors underlined by Schelling [2] is due to overlapping neighborhoods: an agent that moves according to its own motivation affects not only the neighborhood it leaves and the one it arrives in, but also, in the long run, all the agents.

2.2.2 Agent Dislike Time

First of all, let us remember that in the model the time is discretized into periods. For each unsatisfied agent a_i, we define its *dislike time* at time t (hereinafter $\mathrm{dt}_i(t)$) as the duration of the current period during which it is unsatisfied. So, for a satisfied agent the dislike time is null, while it is greater than one for an unsatisfied agent.

2.3 Strong Segregation versus Weak Segregation

In this paper we will refer to segregation as being strong or weak. A *fixed point* in the configuration space is a configuration where all the agents are satisfied and then remain motionless forever. In such a case the *satisfaction rate*, that is, the proportion of satisfied agents, is equal to 1. We will say that there is *strong segregation* when the dynamics lead to a fixed point; if not, the dynamics can however, under certain conditions, lead to a configuration where a significant number of agents with the same type group together; in such cases, we will say that there is *weak segregation*.

Knowing that the dislike time of an agent is null if and only if it is satisfied, one can conclude that if for all the agents $\mathrm{dt}_i(t) = 0$, then a global fixed point is reached and so there is strong segregation. But if weak segregation occurs over time, the dislike time of one agent does not necessarily converge to a fixed value, so we have to look at the *dislike-time distribution* to highlight the global phenomenon. Whereas strong segregation is well-defined, the notion of weak segregation requires expansion; that is what we examine in the remainder of this paper.

3. Segregation with One Discrimination Criterion

Here, we assume that: (*i*) there is only one criterion that induces two types of agents denoted C and T (C stands for *circle* and T for *triangle*); (*ii*) the criterion is symmetrical, that is, $\mathrm{dislike}(C) = \{T\}$ and $\mathrm{dislike}(T) = \{C\}$; and (*iii*) there is the same number of agents in each type.

The global parameter τ is the tolerance of an agent *vis-à-vis* its neighbors with a different type. So, depending on equation (1), for tolerance we have to look at the eight threshold values $\tau^k = k / 8$ with $k \in [1;\, 8]$; each value in one of the intervals $[\tau^k,\, \tau^{k+1}[$ is equivalent to the lower boundary.

The model is based on symmetry because the population size as well as the rules of the dynamics is identical for both types; so one question will be whether the dynamic process preserves symmetry.

3.1 A Grid with Some Vacant Locations

A grid with some vacant nodes corresponds to the classical Schelling model where the density δ of agents is strictly below 1, so for each agent there are at most eight neighbors. In the following we will set $\delta = 0.9$.

3.1.1 Simulation and Results

Experimental simulations are performed with an implementation of the model in the NetLogo multi-agent programmable environment [12]. To ease the display: (*i*) according to its type, one agent will be represented either by a circle or a triangle; and (*ii*) a vacant place will be represented by a square.

For a very low value of τ (e.g., $\tau = 1/8$), no aggregation occurs; by contrast, Figures 1 and 2 show a strong segregation configuration reached after 1000 time steps with τ set to $2/8$ and $5/8$, respectively. The first case is of particular interest, as all the agents are satisfied and the frontier between the two types is a narrow strip composed of the vacant places [13].

Figure 3 shows the mean dislike ratio $\overline{\mathrm{dr}}$ according to the tolerance (results are averaged over 100 runs). As soon as $\tau \geq 2/8$, there is strong segregation and $\overline{\mathrm{dr}}$ increases with τ; moreover, in any case, it is less than what is required on the basis of the tolerance (black line).

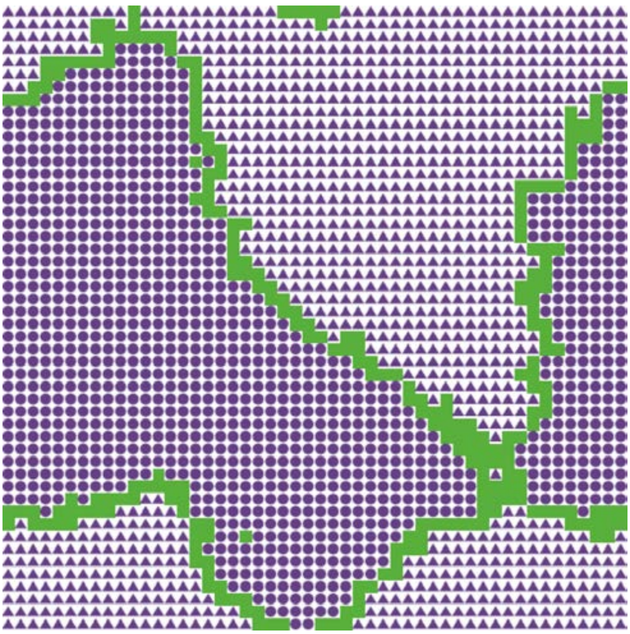


Figure 1. One criterion with vacant locations with $\tau = 2/8$. Strong segregation at $t = 1000$: $\overline{\mathrm{dr}} = 0.02$.

3.1.2 Discussion

As the mean dislike ratio is far below the tolerance, the global system is much more segregationist as it is locally required; as soon as the tolerance is greater than $2/8$, strong segregation occurs. All this corresponds well to the description and the results presented first by Schelling [1, 2]. These first results will serve as a baseline in the rest of the paper.

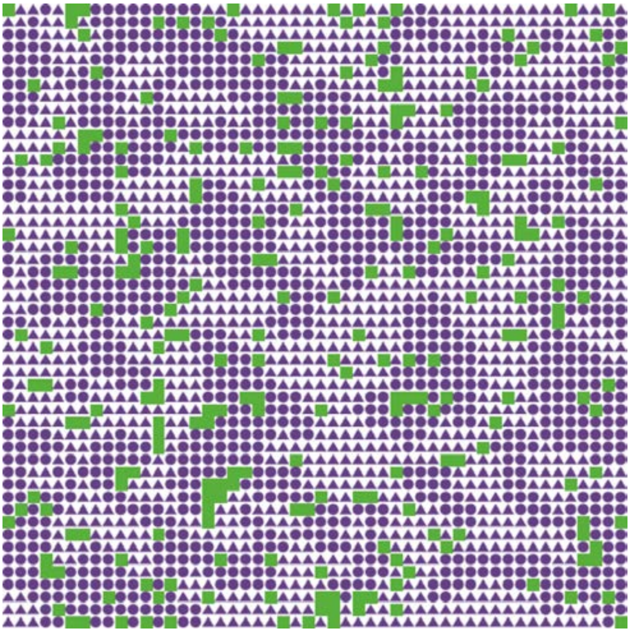


Figure 2. One criterion with vacant locations with $\tau = 5 / 8$. Strong segregation at $t = 1000$: $\overline{dr} \approx 0.23$.

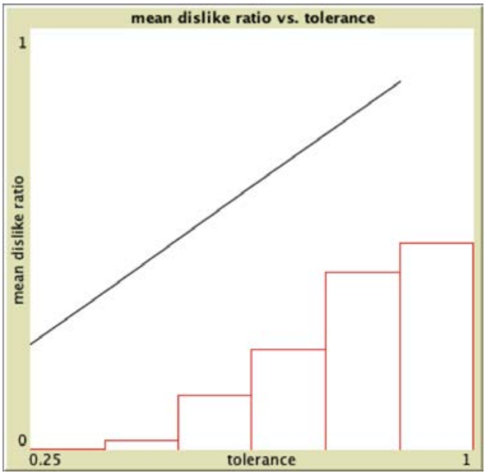


Figure 3. One criterion with vacant locations with mean dislike ratio versus tolerance at $t = 1000$.

3.2 A Crowded Grid without Vacant Locations

By the end of the paper, we assume that initially the density of agents on the grid is at its maximum ($\delta = 1$), so there is one and only one agent per cell. Moreover, as during the dynamics the unsatisfied agents swap places, the assumption is time invariant.

3.2.1 Simulation and Results

For very low values of the tolerance (i.e., $\tau = 1/8$), no aggregation occurs. By contrast, Figures 4 and 5 show a certain form of segregation reached after 1000 time steps with τ set to $2/8$ and $5/8$, respectively. The configuration obtained with $\tau = 2/8$ is of particular interest, as most agents are satisfied ($\overline{\mathrm{dr}} = 0.07$) and the few unsatisfied agents build the frontier between the two types. Let us note that the frontier is a narrow strip composed of both circles and triangles.

Figure 6 shows the mean dislike ratio according to the tolerance; in any case it is far below the tolerance value (black line) and $\overline{\mathrm{dr}}$ increases with τ (with the exception of the case $\tau = 2/8$).

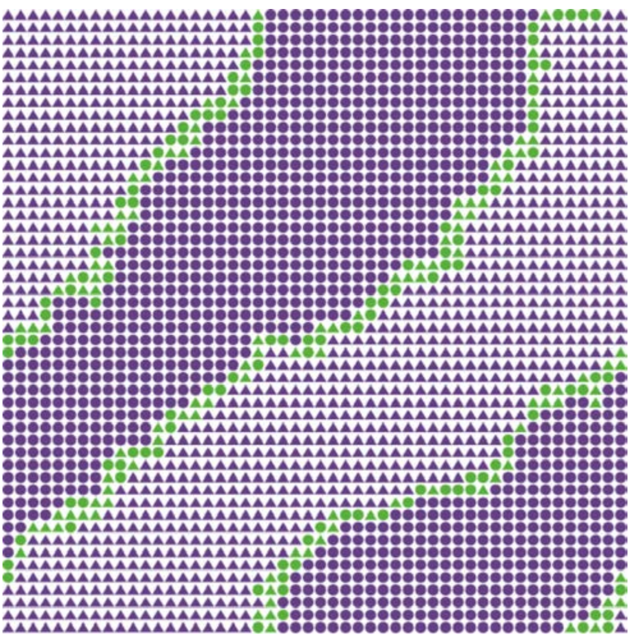


Figure 4. One criterion on a crowded grid with $\tau = 2/8$. Weak segregation at $t = 1000$: $\overline{\mathrm{dr}} \approx 0.07$.

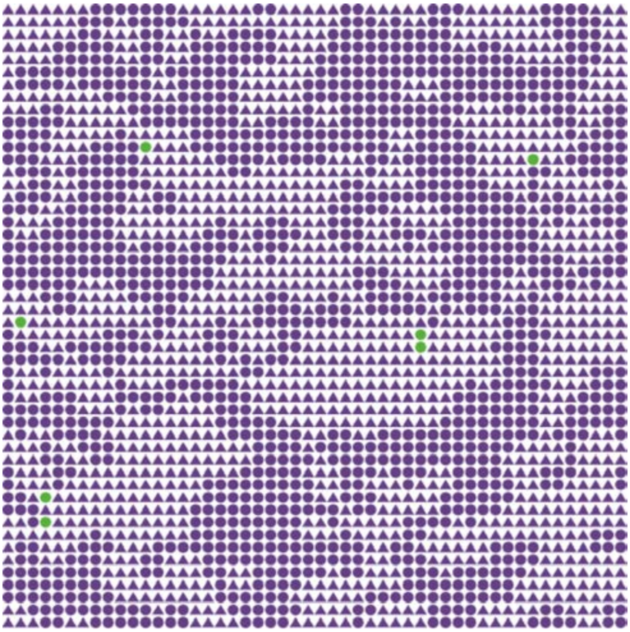


Figure 5. One criterion on a crowded grid with $\tau = 5/8$. Quasi strong segregation at $t = 1000$: $\overline{\mathrm{dr}} \approx 0.23$.

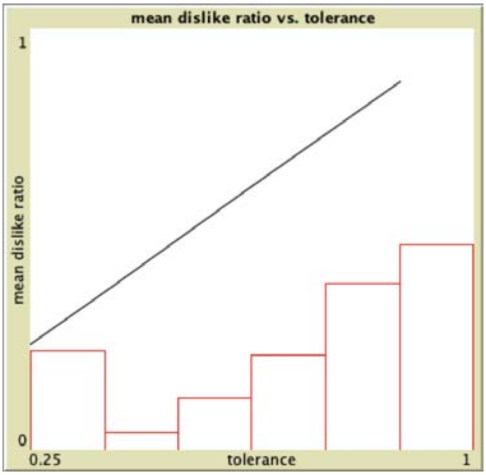


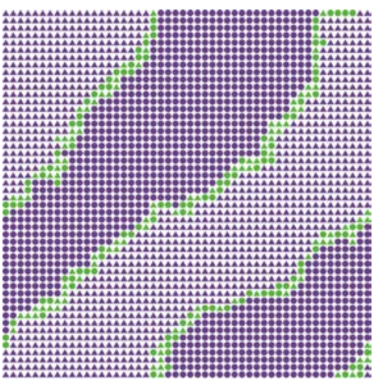
Figure 6. One criterion on a crowded grid with mean dislike ratio versus tolerance at $t = 1000$.

3.2.2 Discussion

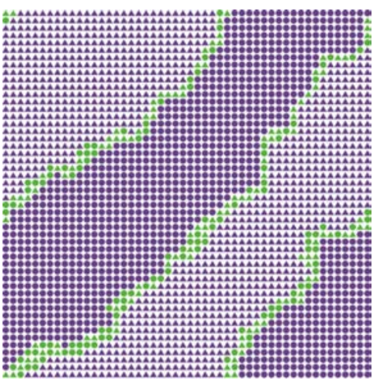
For tolerance above $4/8$, after 5000 runs unsatisfied agents remain very rare, and thus there is a quasi strong segregation (Figure 5). Results are qualitatively comparable to what would be observed with vacant places, because such places are not essential to ensure the convergence.

It is for $\tau = 2/8$ that a new phenomenon can be observed. First, let us remember that for $\delta = 0.9$ and $\tau = 2/8$ the system converges toward a fixed point due to the vacant places located on the frontier between the two subpopulations (Figure 1). Here, for $\delta = 1$ and $\tau = 2/8$, the system reaches a certain form of segregation but without reaching a fixed point. The major part of the population is certainly satisfied ($\approx 90\%$), but a few unsatisfied agents remain located at the frontier between the two types; it is in this sense that we are talking about weak segregation.

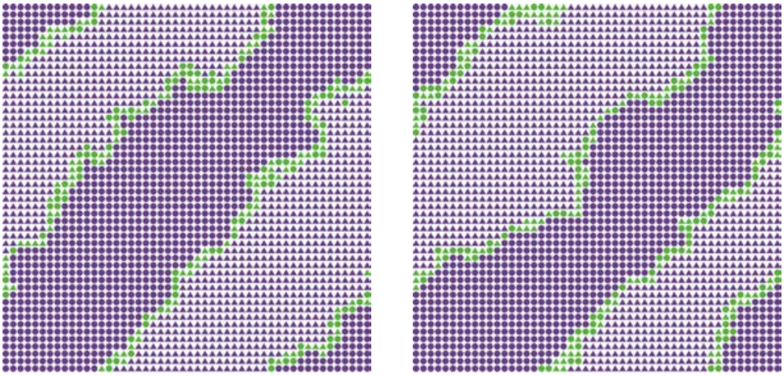
Figures 7(a) to 7(d) show the configuration space at time 1000, 5000, 15 000 and 20 000, respectively, for a same significant run. We can observe that the two big zones of satisfaction shift with time; such a global move is due to the local relocations of unsatisfied agents. Even though at the global level segregation occurs, at each time, agents close to the frontier can possibly swap their locations. You can observe this on Figure 8(a), which shows the time evolution of the dislike time for one particular agent over 20 000 time steps; let us remember that $\mathrm{dt}_i = 0$ if and only if the agent a_i is satisfied. Figure 8(b) shows that the dislike-time distribution of unsatisfied agents roughly follows a decreasing exponential distribution; that means the waiting process for becoming satisfied is memoryless. Whatever the tolerance is, the long-term issue is qualitatively similar for the two types, so the dynamics do not break symmetry.



(a) $t = 1000$

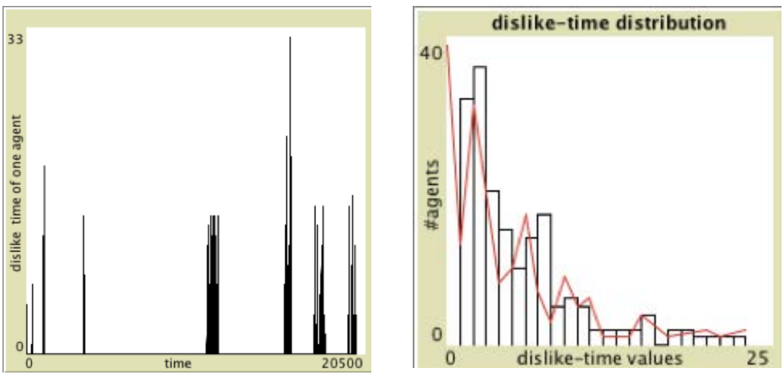


(b) $t = 5000$



(c) $t = 15\,000$ (d) $t = 20\,000$

Figure 7. One criterion on a crowded grid with $\tau = 2/8$ (one significant run). Weak segregation: $\overline{\text{dr}} \approx 0.07$ and 87% of satisfied agents.



(a) (b)

Figure 8. One differentiating criterion without vacant locations and $\tau = 2/8$. (a) Time evolution of the dislike time of one single agent. (b) Dislike-time distribution of unsatisfied agents ($t = 20\,000$).

4. Segregation with Multilevel Discrimination Criteria

In this section we assume that there is not just one criterion but two: a *primary* one represented by the shape (circle or triangle) and a *secondary* one represented by the size (small or large). Let us note that in the previous section the sole criterion corresponded to the primary one. So from now on there are: (*i*) four type values: large circle, small circle, large triangle and small triangle; (*ii*) two dislike functions δ_1 and δ_2; and (*iii*) two tolerance thresholds τ_1 and τ_2.

4.1 Where Are the Dislikes?

For convenience we will denote $\{C, T, c, t\}$ the set of types. The two dislike functions are defined as follows (Table 1): (*i*) a circle, however large or small it is, does not tolerate a triangle according to tolerance τ_1 and vice versa; (*ii*) a large circle C does not tolerate a small circle c according to tolerance τ_2 and vice versa; and (*iii*) a large triangle T does not tolerate a small triangle t according to tolerance τ_2 and vice versa.

Type	δ_1(type)	δ_2(type)
C	$\{T, t\}$	$\{c\}$
T	$\{C, c\}$	$\{t\}$
c	$\{T, t\}$	$\{C\}$
t	$\{C, c\}$	$\{T\}$

Table 1. Dislike functions.

In the following one can raise the questions of knowing if there is weak or strong segregation for both criteria and possibly even a mixed situation, with weak segregation for one criterion and strong segregation for the other.

4.2 Simulation and Results

The tolerances τ_1 and τ_2 are two global parameters. For each value of τ_1, we will examine all the possible values for τ_2; as there are eight significant values for tolerance ($\tau^k = k / 8$ with $k \in [1; 8]$), we have to examine 64 cases, each one corresponding to a couple $(\tau_1^k, \tau_2^{k'})$. Let us note that $\tau_2 = 1$ corresponds to the previous situation with one criterion only (see the point to the far right in the 16 graphs in Figures 9 and 10).

The main objective here is to distinguish weak segregation from strong segregation, so for each of the 64 cases we will run one execution for a period of 2000 time steps and then collect the satisfaction rate for both circles (left graphs in Figures 9 and 10) and triangles (right graphs in Figures 9 and 10). All these results are summarized in Table 2.

Each one of these 64 experiments results from one particular run, and so, due to the sensitivity of initial conditions and the randomness of the dynamics, they produce qualitatively different issues. However, the satisfaction rate for the two primary subpopulations computed after 2000 time steps is relevant enough to be able to draw conclusions about the dynamics.

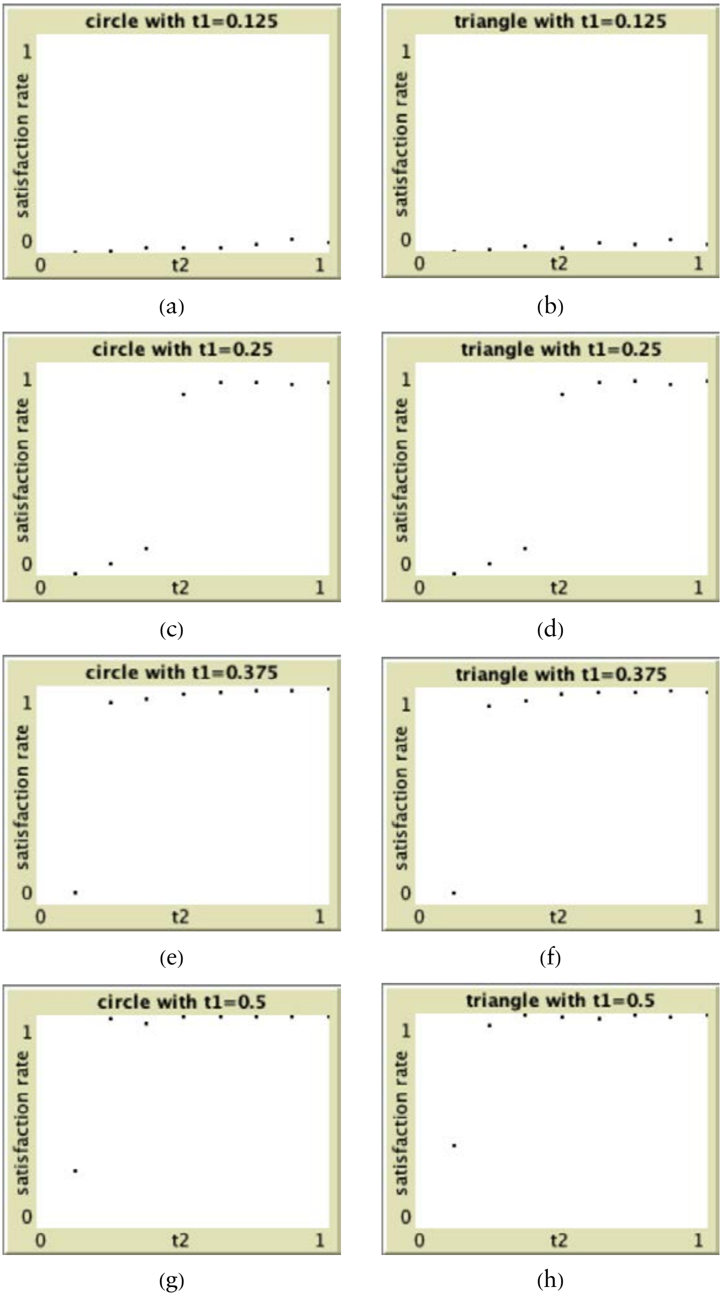


Figure 9. Two differentiating criteria: proportion of satisfied circles (left) and triangles (right) at $t = 2000$ ($\tau_1 \leq 4/8$).

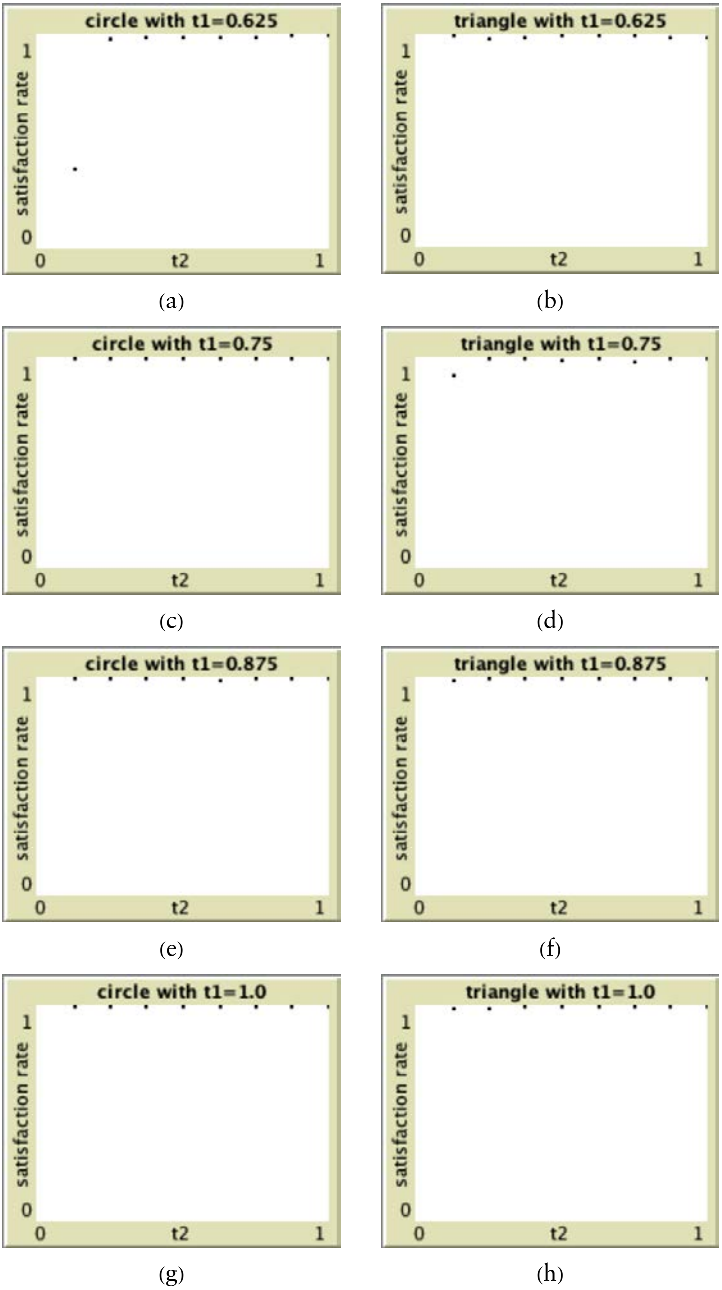


Figure 10. Two differentiating criteria: proportion of satisfied circles (left) and triangles (right) at $t = 2000$ ($\tau_1 > 4/8$).

τ_1	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
1/8	–	–	–	–	–	–	–	–
2/8	–	–	–	**W**	W	W	W	W
3/8	–	W	W	S	S	S	S	S
4/8	–	S	S	S	S	S	S	S
τ_2								
5/8	**M**	S	S	S	S	S	S	S
6/8	S	S	S	S	S	S	S	S
7/8	S	S	S	S	S	S	S	S
1	S	S	S	S	S	S	S	S

Table 2. Weak, mixed and strong segregation.

4.2.1 Rather Intolerant Agents for the Main Criterion

Figure 9 examines the cases of rather intolerant agents regarding the primary criterion (i.e., $\tau_1 \leq 4/8$).

- If $\tau_1 = 1/8$, whatever τ_2 is, no segregation occurs (Figures 9(a) and 9(b)).
- If $\tau_1 = 2/8$, there is a sudden transition from no segregation to weak segregation at $\tau_2 = 4/8$ (Figures 9(c) and 9(d)).
- If $\tau_1 = 3/8$, there is no segregation for $\tau_2 = 1/8$, weak segregation for $\tau_2 = 2/8$ and $3/8$ and quasi strong segregation (Figures 9(e) and 9(f)).
- If $\tau_1 = 4/8$, there is quasi strong segregation except for $\tau_2 = 1/8$ (Figures 9(g) and 9(h)).

4.2.2 Rather Tolerant Agents for the Main Criterion

Figure 10 examines the cases of rather tolerant agents regarding the primary criterion (i.e., $\tau_1 \geq 5/8$).

- If $\tau_1 = 5/8$:
- For the circles, if $\tau_2 = 1/8$ there is weak segregation and above there is strong segregation (Figure 10(a)).
- For the triangles, whatever τ_2 is, there is strong segregation (Figure 10(b)).
- If $\tau_1 \geq 6/8$, whatever τ_2 is, strong segregation occurs (Figures 10(c) to 10(h)).

4.3 Discussion

In order to help us build our understanding of the different forms of segregation, we will examine in detail two specific cases (bold letters in Table 2).

4.3.1 Weak Segregation

At low tolerance for the main criterion (i.e., $\tau 1 = 2/8$), there is a sudden transition phenomenon regarding the tolerance τ_2 for the value $4/8$ [6, 14]; at that *transition threshold* the system abruptly switches from quasi no segregation to weak segregation (Figures 9(c) and 9(d)). Let us note our approach differs from the paper [6] from Gauvin et al. because, in this work, the main results have been summarized as a phase diagram in the (density, tolerance) plane.

We therefore study the case $(\tau_1; \tau_2) = (2/8; 4/8)$. Figures 11 and 12 show the configuration space for one significant run first at $t = 2000$ and then at $t = 20\,000$, respectively. The dynamics lead to the emergence of a shape (hereinafter *satisfaction shape*) where satisfied agents constitute an overwhelming majority and are grouped according to their type. Although segregation occurs for both criteria, the system does not converge to a fixed point, as only around 85% of the agents are satisfied. Let us note that the satisfaction-rate is quasi identical for circles and triangles. Thus, at each time step, around 15% of agents continuously swap places, which results in revisiting local satisfaction: some satisfied agents become unsatisfied and vice versa. Indeed, as can be observed on the configuration space, this

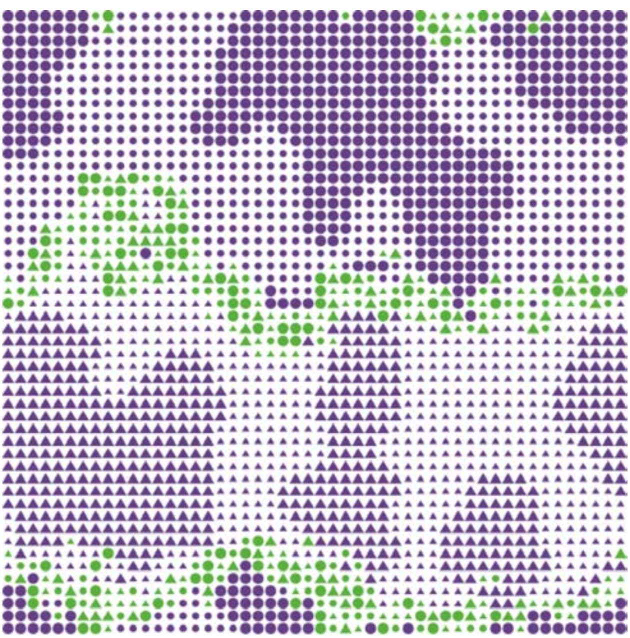


Figure 11. Weak segregation at $t = 2000$ $(\tau_1; \tau_2) = (2/8; 4/8)$: satisfaction rate ≈ 0.85.

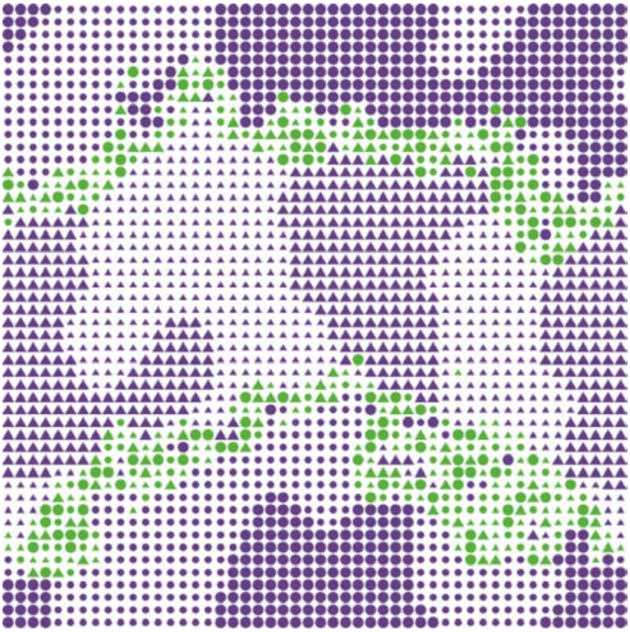


Figure 12. Weak segregation at $t = 20\,000$ $(\tau_1; \tau_2) = (2/8; 4/8)$: satisfaction rate ≈ 0.85.

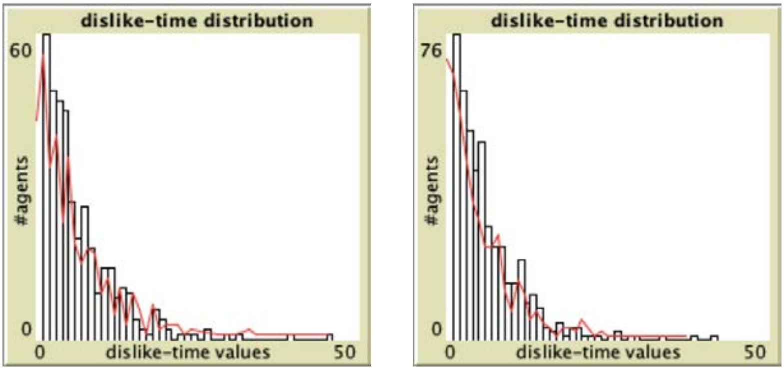
leads to a dynamic move of the global satisfaction shape, with a large strip composed of unsatisfied agents as a frontier between satisfied agents.

Figures 13(a) and 13(b) show the dislike-time distribution for the unsatisfied agents at two specific time steps; that is, the heights of the bars represent the numbers of agents with each dislike-time value: after a first transitional period (not represented here), whatever the time is, the distribution follows an exponential decrease.

4.3.2 Mixed Segregation

In all the previous cases, the issue is similar for both types except for one particular case. The notable exception is for $(\tau_1; \tau_2) = (5/8; 1/8)$, where the dynamics lead to a *breaking of symmetry*. With this setting, on the one hand, circles and triangles are tolerant with respect to each other, and on the other hand, large and small shapes are intolerant relative to each other.

In the long term, the state of the circle and the triangle populations differs (Figures 10(a) and 10(b)). For this very specific case, there is strong segregation for one value of the main criterion and weak segregation for the other; it is in this sense that we use the term *mixed segregation*. Figures 14 and 15 show the configuration space for one



(a) $t = 2000$ (b) $t = 20\,000$

Figure 13. Weak segregation: $(\tau_1; \tau_2) = (2/8; 4/8)$ (one significant run): satisfaction rate ≈ 0.85.

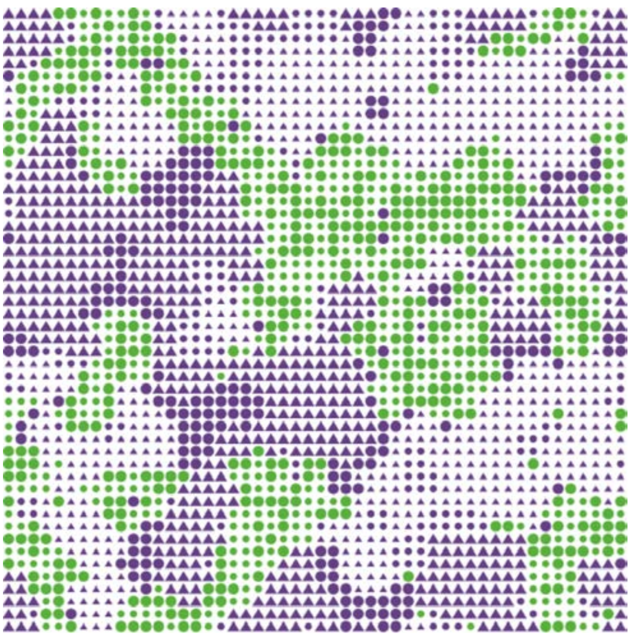


Figure 14. Mixed segregation at $t = 2000$: $(\tau_1; \tau_2) = (5/8; 1/8)$: satisfaction rate circle ≈ 0.33; satisfaction rate triangle $= 1$.

significant run first at $t = 2000$ and then at $t = 20\,000$, respectively; in both circumstances, the satisfaction rate is 1 for triangles, while it is only about 0.33 for circles. Let us note the situation could be the reverse because the issue depends both on the initial configuration and on the random nature of the process; in any case there would be strong segregation for one main type and weak segregation for the other. As the location of each triangle is the same at time 2000 and 20 000, all the triangles are permanently satisfied and so motionless.

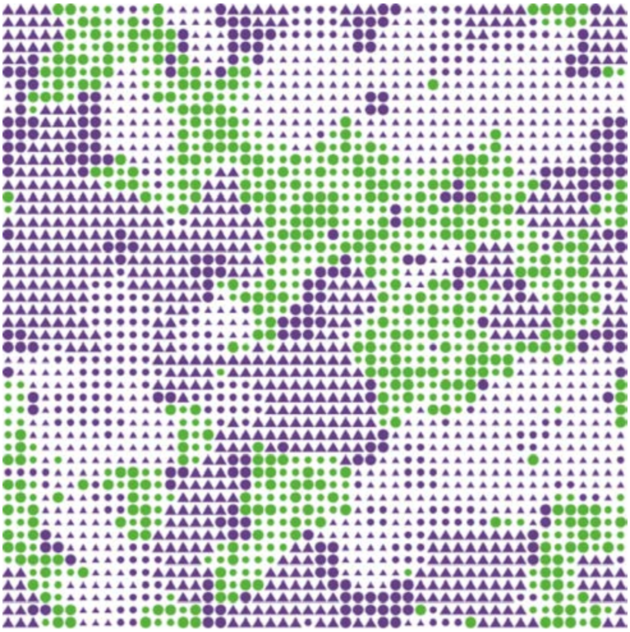
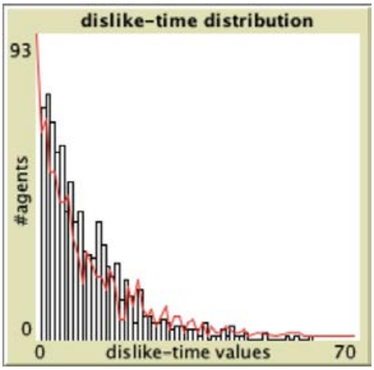


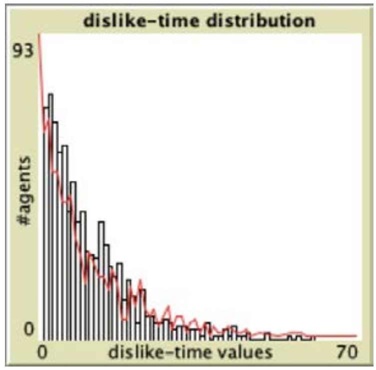
Figure 15. Mixed segregation at $t = 20\,000$: $(\tau_1; \tau_2) = (5/8; 1/8)$: satisfaction rate circle ≈ 0.33; satisfaction rate triangle $= 1$.

Since the tolerance is low on the second criterion, large (resp. small) triangles use circles (satisfied as well as unsatisfied) as a frontier to "protect" against small (resp. large) triangles; unlike the previous cases, the frontier is made up of large compact blocks of circles scattered over the space.

Figures 16(a) and 16(b) show the dislike-time distribution for the unsatisfied agents at the two specific time steps: whatever the time is, the distribution follows an exponential decrease again. One important point is that the dislike time is null for all the triangles, which means they always remain satisfied and so motionless; in other terms, satisfaction has become a frozen state for the triangle population.



(a) $t = 2000$



(b) $t = 20\,000$

Figure 16. Mixed segregation: $(\tau_1; \tau_2) = (5/8; 1/8)$ (one significant run): satisfaction rate circle ≈ 0.33; satisfaction rate triangle $= 1$.

5. Conclusion

This paper extends the classical Schelling model of spatial segregation to a "crowded world" without vacant locations. Whereas in the classical model the dynamics lead to *strong segregation*—that is, the formation of static aggregates of satisfied agents with the same characteristic—in this paper we are able to observe the emergence of shifting aggregates marginally separated by unsatisfied agents on the move.

In the first part we have used only one discrimination criterion with two type values. We have highlighted an unknown situation in the classical model with vacant locations. Indeed, assuming that agents are rather intolerant, the system weakly converges in the sense that after a transitional phase, a very large proportion of agents is satisfied and the unsatisfied agents are located at the frontier, comparable with a narrow strip of land between two communities; this is what we have called weak segregation. Satisfied agents with the same type are aggregated in big zones that shift with time; such a global move is due to the local relocations of unsatisfied agents who swap places. Let us note that, whatever the tolerance is, the long-term issue is qualitatively similar for the two types, so the dynamics maintain symmetry.

In the second part we have used two discrimination criteria with two primary types and two secondary types. Considering all the cases for the two tolerance values, we have highlighted two special cases:

- In the first case, agents are rather intolerant for the primary criterion and, by varying the tolerance for the secondary criterion, we observed a sudden *transitional phase*; before the threshold there is no segregation, while after it there is weak segregation.

- The second case corresponds to a particular setting where agents are tolerant for the primary criterion and intolerant for the secondary criterion. As in the long term the future of the two primary populations differs, we observed a *symmetry break*: there is strong segregation for one main type and weak segregation for the other; this is what we have called mixed segregation.

In the future we project to further the present work in the following ways: (*i*) study more than two criteria; (*ii*) refine the concept of frontier in relation with the notion of multilevel discrimination criteria; (*iii*) study the possibility for two opposite groups to communicate across their common frontier; (*iv*) deepen the relation between weak segregation and phase transition; and (*v*) deepen the relation between mixed segregation and symmetry breaking.

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