Self-Replicability of Composite Graph Reproduction System

D Venkata Lakshmi*

Jeganathan L[†]

School of Computing Science and Engineering
Vellore Institute of Technology
Vandalure-Kelambakkam Road, Chennai, India-600 127
*venkata.lakshmi2011@vit.ac.in
[†]jeganathan.l@vit.ac.in

The problem of self-replication initiated by John von Neumann has been investigated by many researchers from different perspectives. As a part of our earlier investigation into self-replication through graph reproduction, we introduced a comprehensive graph reproduction system and identified a few graph reproduction models that are self-replicable. This paper addresses the question of whether a combination of more than one non-self-replicable graph reproduction model is self-replicable or not. A few existing graph reproduction models were combined in two different ways and tested for self-replicability. Our study confirms that non-self-replicable graph reproduction models become self-replicable by combining themselves with other reproduction models.

1. Introduction

Self-replication, which means that a machine can make a copy of itself, was introduced by the eminent mathematician and physicist John von Neumann [1] in late 1940. Self-replication is a fundamental property of many natural and artificial forms of life that may be physical, chemical and biological systems, whose application in various fields has attracted computer scientists, mathematicians, physicists and chemists to study self-replicating structures or machines.

Self-replicating machines have diverse applications, ranging from the fabrication of nanomachines to space exploration. They could be useful in atomic-scale manufacturing, in robust electronic systems and in understanding the origin of life in a better way. Further, the concept of self-replication is used to model various complex biological and physical systems such as robotics, to analyze turbulence in fluids,

to study the formation of snowflakes, and to create an artificial form of life in a computer.

Based on the models used to study self-replication, the explorations carried out so far can be classified into four categories as described by Sipper in [2]: cellular automata–based self-replication, computer program–based self-replication, string-based self-replication and mechanical model–based self-replication. In [3], we take a different approach that does not come under any of the above four categories. We studied and investigated self-replication through the process of reproduction.

Richard Southwell and Chris Cannings generated different graph reproduction models [4] that produce different types of graphs from the given initial graph. These models are used to describe the growth of interactions between individuals within a population. In their models [5, 6], every vertex reproduces a new vertex that is connected to the existing vertices based on different constraints, thereby generating eight different models of graph-generating mechanisms. In their models, edges get eliminated because of the death of vertices. Their models do not consider the edges as a valid parameter for reproduction. Since graphs are characterized by both vertices and edges, we observe that any graph reproduction system (GRS) should consider the edges as well as the vertices of the graph.

From the given graph, the Southwell model focuses on the generation of different graphs through the reproduction process, and self-replication of the given graph is not its concern. With an idea to investigate the self-replication of graphs through GRS, we proposed a comprehensive model for a GRS [3] that is self-replicable. In contrast to the Southwell model, our model considers vertices as well as edges as valid parameters for the evolution of graphs. All the models developed by Southwell [4] are just specific instances of our model. Further, we analyzed some of the existing reproduction models through our system and identified the models that are self-replicable. We considered self-replication as an algorithmic process to investigate self-replicability of graphs. This paper addresses the question of whether a combination of more than one non-self-replicable graph reproduction model is self-replicable or not. Further, we investigate the composite of reproduction systems for self-replicability.

The paper is organized as follows. Section 2 proposes a generic model of GRS and discusses the graph self-replication system (GSS) [3]. Section 3 analyzes some of the existing reproduction models for self-replicability. Section 4 proposes two different composite graph reproduction systems (GRSs) and investigates the self-replicability of the composite model, a combination of more than one of Southwell's models. This section proves that certain composite models are self-replicable even though individual models may not be self-replicable.

Section 5 concludes with the description of future enhancements for our model.

2. Graph Reproduction System

We present a GRS that has the ability to produce copies of the graph given as an input to the system. Though biological organisms are the most familiar examples of a self-replication system, this paper explores the self-replication system with graphs as the basic unit. This artificial GSS is motivated by the desire to understand the fundamental information processing principles and algorithms involved in self-replication, independent of their physical realization. A better theoretical understanding of the GRS could be useful in a number of ways, from a computational as well as an engineering perspective.

As mentioned earlier, the main aim is to study the feasibility of any graph-generating system becoming a self-replication system. For this purpose, one should have a generating mechanism through which graphs get generated from the given graph through some well-defined process.

In that sense, we consider a GRS that generates graphs through the process of evolution/reproduction over discrete time step t. This paper assumes asexual reproduction with a single parent, in the sense that offspring are born with the parent's strategy (potential) and link up to the surroundings in a similar way to their parent. This will simulate the natural process of children inheriting the parent's genes and getting connected to the environment like their parents.

In our view, any asexual reproduction system in the social environment should not omit the following items.

- Offspring are born based on the parent's strategy (potential) and get connected to the environment like their parent.
- Individual organisms lose their reproductive potential over a period of time due to various reasons. That is, individual organisms may become infertile.
- Infertile organisms may gain fertility due to some medical treatment.
- Organisms lose their connectivity with other organisms over a period of time.
- Individual organisms die due to the aging process.
- As a social constraint, organisms might not produce more offspring, though they are capable. In other words, there may be a cap on the number of offspring that can be reproduced by an organism; for example, organisms are allowed to reproduce only once during their lifetime.

Considering all these items, we propose a graph-generating mechanism called a graph reproduction system (GRS), starting from an initial graph. As made clear, our main goal is to check the potential of this GRS to become a GSS, thereby possessing the ability to generate copies of the initial graph.

Definition 1. A graph reproduction system (GRS),

$$\rho = \left(RA, RI, CA, LCA, DA \right),$$

where RA is the reproduction axiom—the set of rules with which the vertices of a graph reproduce.

RI is the reproductive index—a positive integer, the maximum number of offspring that can be produced by an organism throughout its life.

CA is the connectivity axiom—the set of rules with which the offspring are connected to their parent and the other members of the society.

LCA is the loss of connectivity axiom—the set of rules by which the connectivity between different vertices of the graph is lost.

DA is the dying axiom—the set of rules by which the vertices die.

Definition 2. Language generated by a GRS: Let ρ be any GRS. Let G be any graph. $\rho(G)$ is the set of graphs reproduced by G; that is, $\rho(G) = U_{i=1}^{\infty} \rho^i(G)$, where $\rho^i(G)$ is the graph produced in the ith generation.

Note: The initial graph G is not included in $\rho(G)$, just to observe whether $\rho(G)$ produces G or not.

Since the vertices and edges are the only two parameters of a graph, the preceding 5-tuple GRS represents a comprehensive model of any GRS, in the sense that the system includes the birth and death of vertices along with the birth and death of edges. Any graph-generating reproducing mechanism will be a specific instance of our GRS. The generating reproducing system will differ only in the description of axioms.

Definition 3. A GRS ρ is said to be a graph self-replication system (GSS) of order k if there exists at least one graph G, and a $k \geq 1$ such that $\rho^k(G) \cap \{G\} \neq \Phi$. If for any k, $\rho^k(G)$ has disconnected components, the individual components of $\rho^k(G)$ will be considered as separate graphs for the computation of $\rho^k(G) \cap \{G\}$.

3. Representation of Southwell's Models through Our Model

Southwell and Cannings explored different graph reproducing models [4] that produce different types of graphs from the given initial graph.

In their models, every vertex reproduces a new vertex that is connected to the existing vertices based on different constraints, thereby generating eight different models of graph generating mechanism. In their models, edges get eliminated because of the death of vertices. These models concentrate on the evolution of different types of graphs, and self-replication of the given graph is not the authors' concern. The models proposed in [4] are the specific instances of our GRS. The eight models proposed in [4] are represented through our GRS as

$$\rho_i = \left(\text{RA}_i, \text{RI}_i, \text{CA}_i, \text{LCA}_i, \text{DA}_i\right),$$

where $i = 0, 1, 2, 3, 4, 5, 6, 7$.

In all the eight models, the axioms except the connectivity axiom remain the same. They differ only in the connectivity aspect, which means the models can be differentiated with respect to the connectivity axiom only.

The components in the given 5-tuple are described as follows:

- RA_i: All vertices produce an offspring vertex one at a time, $\forall\ i$.

- $\text{RI}_i = 1 \forall\ i$, which means all the vertices produce only one offspring vertex throughout their lifetime.

- $\text{LCA}_i = \Phi$ (empty set) $\forall\ i$.

- DA_i: Q, a positive integer, which means that every vertex of degree greater than Q will die.

The connectivity axioms are described as follows:

- CA_0: No offspring are connected among themselves and no offspring are connected to their parents.

- CA_1: Offspring are connected to their parent's neighbors.

- CA_2: Offspring are connected to their parents.

- CA_3: Offspring are connected to their parents and their parent's neighbors.

- CA_4: Offspring are connected to the offspring of their parent's neighbors.

- CA_5: Offspring are connected to their parent's neighbors and their parent's neighbor's offspring.

- CA_6: Offspring are connected to their parents and the offspring of their parent's neighbors.

- CA_7: Offspring are connected to their parents, their parent's neighbors and the offspring of their parent's neighbors.

The GRS ρ_i corresponds to the ith model ($i = 0, 1, 2, 3, 4, 5, 6, 7$) described in [4].

Apart from the respective descriptions of CA_i,

$$\rho_i = \left(RA_i,\ 1,\ CA_i,\ \Phi,\ Q\right)$$

conveys that RA_i is the same for all models, RI_i is 1, LCA_i is empty, and $DA_i = Q$.

Of the eight models, ρ_0, ρ_4, ρ_6 are found to be self-replication systems, and the models $\rho_1, \rho_2, \rho_3, \rho_5, \rho_7$ are not self-replicable [3].

4. Composite Graph Reproduction System

As mentioned earlier, we observe that all the models of Southwell and Cannings differ only in the connectivity axiom, and all the other components of the reproduction models remain the same. We also observe that combining two or more of the Southwell models results in another Southwell model.

For example, consider the path graph with two vertices.
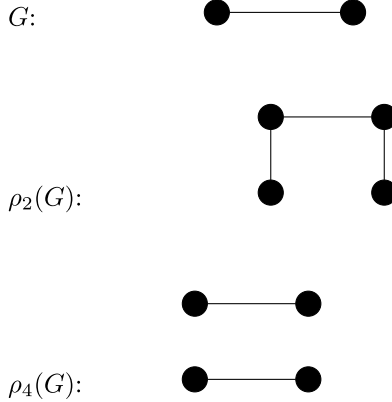
If we perform ρ_2 and ρ_4 (one after the other without any specific order), we get the following graph.
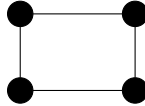
The preceding graph is obtained by applying ρ_6 to G.

That is, the effect of applying ρ_2 after ρ_4 on G (or ρ_4 after ρ_2 on G) is equivalent to the application of ρ_6 over G. We know ρ_2 is not self-replicable and ρ_4 is self-replicable. When we combine ρ_2 (a non-self-replicable system) with ρ_4 (a self-replicable system), we get ρ_6, which is a self-replicable system.

In a similar way, ρ_3 is obtained by combining ρ_1 and ρ_2; ρ_5 is obtained by combining ρ_1 and ρ_4; ρ_6 is obtained by combining ρ_2 and ρ_4; ρ_7 is obtained by combining ρ_1, ρ_2 and ρ_4. All the other models such as ρ_0, ρ_1, ρ_2, ρ_4 cannot be expressed as the combination of other ρ_i.

In other words, considering the path graph G, a combination of a non-self-replicable reproduction model with a self-replicable reproduction model gives rise to a self-replicable reproduction model. Thus, we get a new insight: a combination of two reproduction models may become self-replicable whether the individual models are self-replicable or not.

This motivated us to explore the combination of reproduction models (composite models) for self-replicability. In this section, we propose two different composite GRSs and investigate the self-replicability of the composite model obtained by the combination of various of Southwell and Cannings's models. We define two composite GRSs, wherein combination of GRSs happens in two different ways.

1. We combine two GRSs ρ_i and ρ_j such that ρ_i is applied first and then ρ_j. We call this a "system composite of ρ_i and ρ_j," written as $\rho_i \oplus \rho_j$.

2. We combine two GRSs ρ_i and ρ_j such that components of ρ_i and ρ_j are combined and applied as a single GRS. We call this a "component composite of ρ_i and ρ_j," written as $\rho_i \otimes \rho_j$.

▌ 4.1 System-Composite Graph Reproduction System

Definition 4. Let

$$\rho_i = \left(\text{RA}_i, 1, \text{CA}_i, \text{LCA}_i, \text{DA}_i\right)$$

and

$$\rho_j = \left(\text{RA}_j, 1, \text{CA}_j, \text{LCA}_j, \text{DA}_j\right)$$

be any two GRSs. We define the system composite of ρ_i and ρ_j as the GRS where ρ_i is applied first and then ρ_j (or ρ_j is applied first and then ρ_i).

The system composite of ρ_i and ρ_j, denoted as $\rho_i \oplus \rho_j$ (or $\rho_{i \oplus j}$), is defined as:

$$\rho_i \oplus \rho_j = \left(\text{RA}_i \vee \text{RA}_j, \text{RI}_i \vee \text{RI}_j, \text{CA}_i \vee \text{CA}_j, \text{LCA}_i \vee \text{LCA}_j, \text{DA}_i \vee \text{DA}_j\right),$$

where each component is interpreted as follows.

- $\text{RA}_i \vee \text{RA}_j$: RA_i or RA_j

- $RI_i \vee RI_j$: RI_i or RI_j

- $CA_i \vee CA_j$: CA_i or CA_j

- $LCA_i \vee LCA_j$: LCA_i or LCA_j

- $DA_i \vee DA_j$: DA_i or DA_j

Here we have used the notation "$\vee$", just to represent "OR" in logic theory.

Definition 5. Let $\rho_i \oplus \rho_j$ be any system-composite GRS. $(\rho_i \oplus \rho_j)(G)$ is the set of graphs reproduced by G, called a language generated by the GRS, that is, $(\rho_i \oplus \rho_j)(G) = \bigcup_{k=1}^{\infty} \rho_{i \oplus j}^k(G)$, where $\rho_{i \oplus j}^k$ is defined as the Cartesian product of $\{\rho_i^1, \rho_j^1\}$ with itself k times.

That is, $\rho_{i \oplus j}^k = \{\rho_i^1, \rho_j^1\} \times \{\rho_i^1, \rho_j^1\} \times \ldots k$ times:

$$\rho_{i \oplus j}^1(G) = \{\rho_i^1, \rho_j^1\}(G) = \{\rho_i^1(G), \rho_j^1(G)\}$$

$\rho_i^1(G)$ means that ρ_i is applied to G only once and the offspring get connected as per CA_i. We put the superscript 1 in ρ_i^1, to indicate that ρ_i is applied to G only once.

In other words, the language generated by the system-composite GRS $\rho_{i \oplus j}(G)$ is obtained as follows.

1. In the first iteration, we apply ρ_i and ρ_j over G only once and obtain $\rho_i^1(G)$ and $\rho_j^1(G)$.

2. In the second generation, we apply ρ_i and ρ_j once, to all the graphs obtained in the first iteration. That is, we obtain $\rho_i^1(\rho_i^1(G))$, $\rho_j^1(\rho_i^1(G))$, $\rho_i^1(\rho_j^1(G))$, $\rho_j^1(\rho_j^1(G))$.

3. In the third iteration, we apply ρ_i and ρ_j once, to all the graphs obtained in the second iteration. Similarly we obtain the other iterations.

Thus,

$$\rho_{i \oplus j}^1(G) = \{\rho_i^1(G), \rho_j^1(G)\}$$
$$\rho_{i \oplus j}^2(G) = \{\rho_i^1(\rho_i^1(G)), \rho_j^1(\rho_i^1(G)), \rho_i^1(\rho_j^1(G)), \rho_j^1(\rho_j^1(G))\}$$
$$\rho_{i \oplus j}^3(G) = \{\rho_i^1(\rho_i^1(\rho_i^1(G))), \rho_i^1(\rho_i^1(\rho_j^1(G))), \rho_i^1(\rho_j^1(\rho_i^1(G))), \rho_i^1(\rho_j^1(\rho_j^1(G))),$$
$$\rho_j^1(\rho_i^1(\rho_i^1(G))), \rho_j^1(\rho_i^1(\rho_j^1(G))), \rho_j^1(\rho_j^1(\rho_i^1(G))), \rho_j^1(\rho_j^1(\rho_j^1(G)))\}$$

Here is the tree diagram for the language generation of $\rho_{i \oplus j}$:
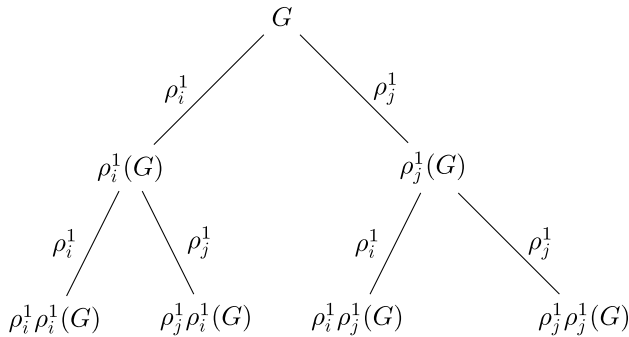
We illustrate the system-composite GRS with the reproduction models of Southwell.

4.1.1 Illustration for the System-Composite Graph Reproduction System $\rho_2 \oplus \rho_4$

Let G be a path graph with two vertices.

In generation 1, we obtain $\{\rho_2^1(G), \rho_4^1(G)\}$. $\rho_2^1(G)$ means that ρ_2 is applied to G only once, the offspring are born, and offspring get connected to the other vertices based on CA_2. Similarly $\rho_4^1(G)$ is obtained by using CA_4.

In generation 2, we obtain

$$\{\rho_2^1(\rho_2^1(G)), \rho_4^1(\rho_2^1(G)), \rho_2^1(\rho_4^1(G)), \rho_4^1(\rho_4^1(G))\}.$$

These can be clearly observed from the following figure:
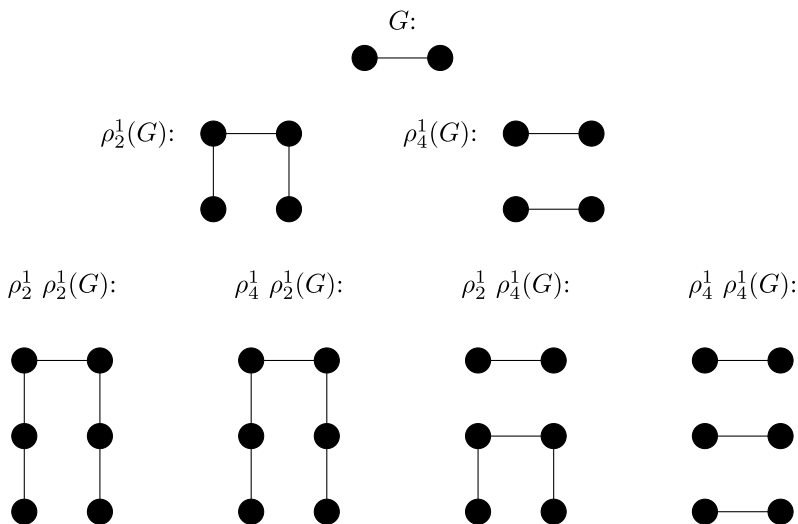
Note: $\rho_i \oplus \rho_i$ is just ρ_i itself.

Lemma 1. The system-composite operation is not associative but commutative.

Proof. Since, $\rho_{i \oplus j}^k = \{\rho_i^1, \rho_j^1\} \times \{\rho_i^1, \rho_j^1\} \times \ldots k$ times, we conclude that the system-composite operation is commutative.

The system-composite operation is not associative. That is, for i, j, k $(\rho_i \oplus \rho_j) \oplus \rho_k \neq \rho_i \oplus (\rho_j \oplus \rho_k)$, as a Cartesian product is not associative. $\square$

Now we apply the system-composite operation on Southwell's model to test whether the $\oplus$ operation induces self-replicability.

4.1.2 Self-Replicability of the System-Composite Graph Reproduction System

Definition 6. A system-composite GRS $\rho_i \oplus \rho_j$ is self-replicable if there exists at least one graph G and a $k \geq 1$ such that $(\rho_i \oplus \rho_j)^k(G) \cap \{G\} \neq \phi$.

If for any k, $(\rho_i \oplus \rho_j)^k(G)$ has disconnected components, the individual components of $(\rho_i \oplus \rho_j)^k(G)$ will be considered as separate graphs for the computation of $(\rho_i \oplus \rho_j)^k(G) \cap \{G\}$.

Theorem 1. $\rho_i \oplus \rho_j$, for $i \neq j$, is a graph self-replication system (GSS), whenever either ρ_i or ρ_j is a GSS.

Proof. Hypothesis: Without loss of generality, assume ρ_i is a GSS. Claim: $\rho_i \oplus \rho_j$, for $i \neq j$, is a GSS.

Consider any graph G. Graphs that get generated in the first iteration of $\rho_i \oplus \rho_j$ are

$$[\rho_i \oplus \rho_j]^1(G) = \{\rho_i^1(G), \rho_j^1(G)\}.$$

Graphs that get generated in the first iteration of ρ_i are

$$\rho_i^1(G) = \{\rho_i^1(G)\}.$$

Graphs that get generated in the second iteration of $\rho_i \oplus \rho_j$ are

$$[\rho_i \oplus \rho_j]^2(G) = \{\rho_i^1(\rho_i^1(G)), \rho_j^1(\rho_i^1(G)), \rho_i^1(\rho_j^1(G)), \rho_j^1(\rho_j^1(G))\}.$$

Graphs that get generated in the second iteration of ρ_i are

$$\rho_i^2(G) = \{\rho_i^1(\rho_i^1(G)), \rho_j^1(\rho_i^1(G)), \rho_i^1(\rho_j^1(G)), \rho_j^1(\rho_j^1(G))\}.$$

Thus, we observe that all the graphs that get generated in the kth iteration of $\rho_i(G)$ are present in the kth iteration of $(\rho_i \oplus \rho_j)(G)$.

Thus, $\rho_i^k(G) \subseteq (\rho_i \oplus \rho_j)^k(G)$ for all k.

$$\Rightarrow \rho_i(G) \subseteq (\rho_i \oplus \rho_j)(G).$$

By hypothesis, ρ_i is self-replicable. Hence $\rho_i \oplus \rho_j$ is self-replicable to complete the proof. $\square$

Theorem 2. Let ρ_i and ρ_j be any two non-self-replication systems. Then $\rho_i \oplus \rho_j$ for $i \neq j$ is a GSS except when $i = 1, j = 5$.

Proof. We know that ρ_i for $i = 1, 2, 3, 5, 7$ are not self-replicable. Let $i, j \in \{1, 2, 3, 5, 7\}$.

Part I : Without loss of generality, we prove that $\rho_i \oplus \rho_j$ is self-replicable for any two specific values of i and j except when $i = 1, j = 5$.

Let $i = 2, j = 5$. Consider the path graph G with two vertices. In generation 1, we obtain $\rho_2(G), \rho_5(G)$.

G:
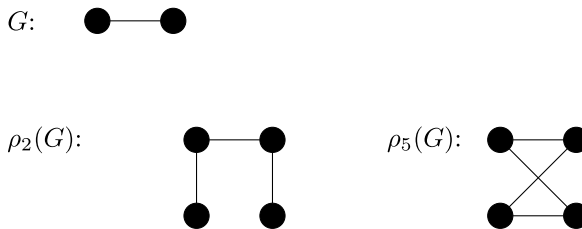
$\rho_2(G)$: $\rho_5(G)$:

Since all vertices in $\rho_2(G)$, $\rho_5(G)$ are of degree 2, by the dying axiom no vertices will die for any $Q > 1$. Now in generation 2 we obtain

$$\{\rho_2(\rho_2(G)), \rho_5(\rho_2(G)), \rho_2(\rho_5(G)), \rho_5(\rho_5(G)\}.$$

Consider $\rho_2\left(\rho_5(G)\right)$, which is obtained by applying ρ_2 to $\rho_5(G)$ obtained in generation 1.

$\rho_2(\rho_5(G))$:
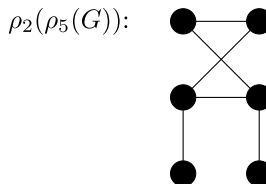
In $\rho_2\left(\rho_5(G)\right)$, there are two vertices of degree 3. By the dying axiom, for $Q \geq 2$ those two vertices of degree 3 will die and the edges incident to those vertices will be removed. So we obtain a graph as shown following, that is, a path graph with two vertices and two isolated vertices.
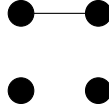
Hence as $G \in [\rho_{2\oplus5}]^2(G)$, $\rho_2 \oplus \rho_5$ is a GSS though ρ_2, ρ_5 are not self-replicative.

Analogously, we can prove $\rho_i \oplus \rho_j$, $i \neq j$ are GSSs for any i, $j \in \{1, 2, 3, 5, 7\}$ except when $i = 1, j = 5$.

Part II: We prove that $\rho_1 \oplus \rho_5$ is not a GSS. Let G be any graph. Graphs that get generated in the first iteration are

$$[\rho_1 \oplus \rho_5]^1(G) = \{\rho_1(G), \rho_5(G)\}.$$

Since ρ_1, ρ_5 are not GSSs, G will not be in the language generated by $\rho_1(G), \rho_5(G)$.

Graphs that get generated in the second iteration are

$$[\rho_1 \oplus \rho_5]^2(G) = \{\rho_1(\rho_1(G)), \rho_5(\rho_1(G)), \rho_1(\rho_5(G)), \rho_5(\rho_5(G))\}.$$

Since ρ_1, ρ_5 are not GSSs, G will not be in the language generated by $\rho_1(\rho_1(G)), \rho_5(\rho_5(G))$.

Now, we check whether G belongs to the language generated by $\rho_5(\rho_1(G)), \rho_1(\rho_5(G))$.

Consider $\rho_5(\rho_1(G))$.

Assume, without loss of generality, G has a vertex v of degree $k \geq 1$. In $\rho_1(G)$, this vertex v produces an offspring vertex, say v^1, and using the CA_1 axiom, v^1 is connected to neighbors of v, as offspring vertices are connected to parent neighbors in the CA_1 axiom.

So, applying ρ_1 to G, the degree of vertex v will become $2k$, and the degree of vertex v^1 will become k.

In $\rho_5(G)$, using the CA_5 axiom, offspring vertices are connected to the parent's neighbors and the parent's neighbor's offspring.

Now, applying ρ_5 to $\rho_1(G)$, the degree of vertex v will become $3k$, the degree of v^1 will become $2k$, and the degree of v^{11} (offspring vertex of v^1) will also become $2k$. So, we have three vertices v, v^1, v^{11} with degrees $3k$, $2k$, $2k$, respectively. For applying the dying axiom, we have to choose a value for Q.

Case 1: $Q = k$. The vertices of degree $Q > k$ will die. So, the vertex v will die, since the degree of v is greater than k, and edges incident to v will be removed. Now, the graph G will have isolated vertices, as v^1, v^{11} are connected to v.

Note: If we choose Q as any value less than k, G will still have only isolated vertices.

Case 2: $Q = 2k$. The vertices of degree $Q > 2k$ will die. So the vertex v will die, since the degree of v is $> 2k$, and edges incident to v will be removed. Now the graph G will have isolated vertices, as v^1, v^{11} are connected to v.

Note: If we choose Q as any value between k and $2k$, G will still have only isolated vertices.

Case 3: $Q \geq 3k$: The vertices of degree $Q > 3k$ will die. So the vertex v of degree $3k$ will remain, and in this generation this graph does not contain any vertex of degree k.

Now we observe the following: there are only two possibilities for a vertex v of degree k in a graph G.

1. The degree of the vertex becomes zero if the value of $Q > k$, when the dying axiom is applied. This vertex will have degree 0 in all the subsequent generations due to the CA axioms of ρ_1 and ρ_5.

2. If the value of Q is chosen such that the vertex v will not die, then the degree of vertex v increases to $6k$, $9k$, ...

That is, a vertex v (of the graph G) with degree k in generation 0 will not have degree k in any of the subsequent generations.

The phenomenon can be observed for one vertex v of G. Extending this phenomenon to all vertices of G, we conclude that the original graph G cannot be obtained in any of the subsequent generations.

Hence $G \notin [\rho_1 \oplus \rho_5]^k(G)$, for $k \geq 1$. That is, G does not belong to the language generated by $\rho_1 \oplus \rho_5$. Thus, $\rho_1 \oplus \rho_5$ is not a GSS. $\square$

We summarize the self-replicability of the system composite of reproduction models of Southwell in the following table.

$\oplus$	ρ_7	ρ_6	ρ_5	ρ_4	ρ_3	ρ_2	ρ_1
ρ_0	✓	✓	✓	✓	✓	✓	✓
ρ_1	✓	✓	✕	✓	✓	✓	
ρ_2	✓	✓	✓	✓	✓		
ρ_3	✓	✓	✓	✓			
ρ_4	✓	✓	✓				
ρ_5	✓	✓					
ρ_6	✓						

▌ 4.2 Component-Composite Graph Reproduction System

Definition 7. Let

$$\rho_i = (\text{RA}_i, \text{RI}_i, \text{CA}_i, \text{LCA}_i, \text{DA}_i)$$

and

$$\rho_j = \left(\mathrm{RA}_j, \mathrm{RI}_j, \mathrm{CA}_j, \mathrm{LCA}_j, \mathrm{DA}_j\right)$$

be any two GRSs.

We define the component-composite GRS of ρ_i and ρ_j as the GRS whose components are the combination of the respective components of ρ_i and ρ_j. The component composite of ρ_i and ρ_j, denoted as $\rho_i \otimes \rho_j$ (also written as $\rho_{i \otimes j}$), is defined as:

$$\rho_i \otimes \rho_j = \left(\mathrm{RA}_i \wedge \mathrm{RA}_j, \min\left(\mathrm{RI}_i, \mathrm{RI}_j\right),\right.$$
$$\left.\mathrm{CA}_i \wedge \mathrm{CA}_j, \mathrm{LCA}_i \wedge \mathrm{LCA}_j, \max\left(\mathrm{DA}_i, \mathrm{DA}_j\right)\right),$$

where:

- $\mathrm{RA}_i \wedge \mathrm{RA}_j$: RA_i and RA_j
- $\min\left(\mathrm{RI}_i, \mathrm{RI}_j\right)$: indicates the reproduction index that is acceptable to both ρ_i and ρ_j
- $\mathrm{CA}_i \wedge \mathrm{CA}_j$: CA_i and CA_j
- $\mathrm{LCA}_i \wedge \mathrm{LCA}_j$: LCA_i and LCA_j
- $\max\left(\mathrm{DA}_i, \mathrm{DA}_j\right)$: indicates the dying axiom that is acceptable to both ρ_i and ρ_j

Here we have used the notation "$\wedge$", just to represent "And" in logic theory.

In the component composite of ρ_i and ρ_j, respective components of ρ_i and ρ_j get combined. The connectivity axiom of $\rho_i \otimes \rho_j$ will be a combination of both CA_i and CA_j. If CA_i: Offspring are connected to the parent's neighbors and CA_j: Offspring are connected to the parents, $\mathrm{CA}_i \wedge \mathrm{CA}_j$ means that the offspring are connected to the parents and the parent's neighbors. If for example, RA_i is 2 (say), RA_j is 1 (say), then the reproduction index of $\rho_i \otimes \rho_j$ is taken as the $\min\left(\mathrm{RI}_i, \mathrm{RI}_j\right)$ just to have the reproduction index that is acceptable to both ρ_i and ρ_j. Similarly, the dying axiom of $\rho_i \otimes \rho_j$ is taken as the $\max\left(\mathrm{DA}_i, \mathrm{DA}_j\right)$. All the vertices with degree greater than or equal to DA_i will die in ρ_i and all the vertices with degree greater than or equal to DA_j will die in ρ_j. $\max\left(\mathrm{DA}_i, \mathrm{DA}_j\right)$ is the dying axiom index acceptable to both ρ_i and ρ_j.

Language generated by the GRS $\rho_i \otimes \rho_j$

$$\left(\rho_i \otimes \rho_j\right)(G) = \bigcup_{k=1}^{\infty} \left(\rho_i \otimes \rho_j\right)^k(G),$$

where $\left(\rho_i \otimes \rho_j\right)^k$ is the graph produced in the kth generation.

Note: $\rho_i \otimes \rho_i = \rho_i$ itself.

In the component-composite GRS, the composition (or combination) occurs at the component level whereas in the system-composite GRS, the composition (or combination) occurs at the system level. Only for this reason, we apply the system ρ_i first and then apply ρ_j (or ρ_j first and then ρ_i) in the computation of $\rho_i \oplus \rho_j$.

4.2.1 Illustration for the Component-Composite Graph Reproduction System $\rho_2 \otimes \rho_4$

In generation 1, we obtain $\{(\rho_2 \otimes \rho_4)^1(G)\}$. The connectivity axiom of $\rho_2 \otimes \rho_4$ will be a combination of both CA_2 and CA_4, which is CA_6. ρ_2 is applied to G once, the offspring are born, and the offspring get connected to the other vertices based on CA_2. Now apply ρ_4 to $\rho_2(G)$ component-wise by using CA_4. Then we get a graph obtained by applying CA_6. This can be clearly observed from the following figure.
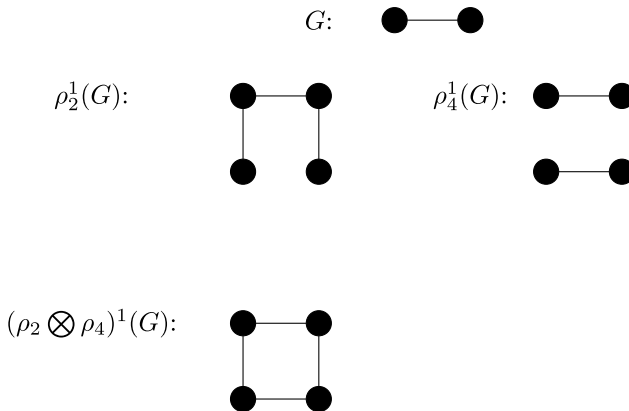
Now we apply the component-composite operation on the Southwell model to test whether the $\otimes$ operation induces self-replicability.

4.2.2 Self-Replicability of Component-Composite Graph Reproduction System

As observed already, all Southwell models differ only in the connectivity axiom, and all the other components of the reproduction models remain the same.

As mentioned already,

$$CA_1 \wedge CA_2 = CA_3, \ CA_2 \wedge CA_4 = CA_6, \ CA_1 \wedge CA_5 = CA_6.$$

$$\rho_2 \otimes \rho_4 = (RA, 1, CA_2 CA_4, \Phi, Q) = (RA, 1, CA_6, \Phi, Q).$$

That is, $\rho_2 \rho_4 = \rho_6$.

Based on this, we construct the composition table for the operation "$\wedge$" among the CA_i as follows.

$\wedge$	CA_7	CA_6	CA_5	CA_4	CA_3	CA_2	CA_1
CA_0	CA_7	CA_6	CA_5	CA_4	CA_3	CA_2	CA_1
CA_1	CA_7	CA_7	CA_5	CA_5	CA_3	CA_3	
CA_2	CA_7	CA_6	CA_7	CA_6	CA_3		
CA_3	CA_7	CA_7	CA_7	CA_7			
CA_4	CA_7	CA_6	CA_5				
CA_5	CA_7	CA_7					
CA_6	CA_7						

Theorem 3. Here is the composition table for the operation $\otimes$ among the eight Southwell GRSs' ρ_i, $i = 0, 1, 2, 3, 4, 5, 6, 7$.

$\otimes$	ρ_7	ρ_6	ρ_5	ρ_4	ρ_3	ρ_2	ρ_1
ρ_0	ρ_7	ρ_6	ρ_5	ρ_4	ρ_3	ρ_2	ρ_1
ρ_1	ρ_7	ρ_7	ρ_5	ρ_5	ρ_3	ρ_3	
ρ_2	ρ_7	ρ_6	ρ_7	ρ_6	ρ_3		
ρ_3	ρ_7	ρ_7	ρ_7	ρ_7			
ρ_4	ρ_7	ρ_6	ρ_5				
ρ_5	ρ_7	ρ_7					
ρ_6	ρ_7						

Proof. The proof of the theorem is obvious. $\square$

From the composition table given in Theorem 3, we observe that

- $\rho_i \otimes \rho_j$, for $i = 0$, $j = 2, 3, 4, 6$ is self-replicable.

- $\rho_i \otimes \rho_j$, for $i = 1$, $j = 2, 3$ is self-replicable.

- $\rho_i \otimes \rho_j$, for $i = 2$, $j = 3, 4$ is self-replicable.

- $\rho_i \otimes \rho_j$, for $i = 4$, $j = 6$ is self-replicable.

The other combinations are not self-replicable.

5. Conclusion

In this paper, we have investigated the self-replicability of the composite graph reproduction system (GRS) viz., system-composite GRS and the component-composite GRS. We introduced two types of composite graph reproduction systems (GRSs). A study may be initiated to explore the possibility of other types of composite GRSs that may be self-replicable.

Our study has revealed that the composite GRS may be self-replicative even though the individual GRSs are not self-replicative. Though this paper focused only on Southwell's GRSs [4–6], this study can be extended to any type of GRS. An intensive generalized study of the composite GRS may lead to a characterization of a GRS for self-replicability.

We can attempt an algorithm to identify whether the given GRS is a composite GRS, obtained by the combination of more than one GRS, and to compute the individual GRS if it is a composite GRS.

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