

The Impact of Edge Correlations in Random Networks

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Random graphs are frequently used models of real-life random networks. The classical Erdős–Rényi random graph model is very well explored and has numerous nontrivial properties. In particular, a good number of important graph parameters that are hard to compute in the deterministic case often become much easier in random graphs. However, a fundamental restriction in the Erdős–Rényi random graph is that the edges are required to be probabilistically independent. This is a severe restriction, which does not hold in most real-life networks.

We consider more general random graphs in which the edges may be dependent. Specifically, two models are analyzed. The first one is called a p -robust random graph. It is defined by the requirement that each edge exist with probability at least p , no matter how we condition on the presence/absence of other edges. It is significantly more general than assuming independent edges existing with probability p , as exemplified via several special cases. The second model considers the case when the edges are positively correlated, which means that the edge probability is at least p for each edge, no matter how we condition on the presence of other edges (but absence is not considered). We prove some interesting, nontrivial properties about both models.

Keywords: random graph; dependent edges; monotone graph property; edge correlation; geometric random graph

1. Introduction

Practical scenarios often exhibit random networks. A characteristic example is the web graph, which is an abstraction of the World Wide Web. Further examples are various social networks, wireless ad-hoc networks and telephone call networks, as well as numerous others. Random graph models are frequently applied to describe and analyze such networks.

The historically first random graph model has been the Erdős–Rényi random graph $G_{n,p}$. This denotes a random graph on n nodes, such that each edge is added with probability p , and it is done independently for each edge. A multitude of deep results are proven about

such random graphs; see the expositions in [1–3]. In the following we list some examples. They are asymptotic results, and for simplicity and clarity we omit potential restrictions for the range of p , as well as ignore rounding issues (which means that an asymptotic formula may provide a noninteger value for a parameter that is defined as integer for finite graphs).

- The asymptotic size of a maximum independent set in $G_{n,p}$ is $(2n \ln d)/d$, where d is the average degree.
- The maximum clique size in $G_{n,p}$ is asymptotically $2 \log_{1/p} n$.
- The chromatic number of $G_{n,p}$ is asymptotically $n/(\log_b n)$, where $b = 1/(1-p)$.
- The longest cycle length in $G_{n,p}$ is asymptotically $n(1 - de^{-d})$, when the graph has a constant average degree d .
- The asymptotic size of a minimum dominating set in $G_{n,p}$ is $\log_b n$, where $b = 1/(1-p)$.
- The number of nodes of degree k in the $G_{n,p}$ is asymptotically $(d^k e^{-d})/k! n$, where d is the average degree.
- The diameter of $G_{n,p}$ is asymptotically $\log n/(\log(np))$, when $np \rightarrow \infty$. (In case the graph is not connected, then the diameter is defined as the largest diameter of its connected components.)

These results (and many others) make it possible that for random graphs we can find good and directly computable estimates of graph parameters that are hard to compute for deterministic graphs. Moreover, the parameters often show very strong concentration. For example, as listed above, the chromatic number of $G_{n,p}$ is asymptotically $n/(\log_b n)$, where $b = 1/(1-p)$. However, we can say more: the chromatic number of a random graph is so strongly concentrated that with probability approaching one, as $n \rightarrow \infty$, it falls on one of two consecutive integers (see [4]).

2. Random Graphs with Dependent Edges: p -Robust Random Graphs

In modeling real-life networks, the requirement that the edges be independent is often a too-strict restriction. As a result, quite a few attempts have been made to develop models with various types of dependencies among the edges; see, for example, a survey in [5]. Here we consider a quite general form of edge dependency, as defined below. Initial three-page abstracts of this model were presented by the

author in [6, 7]; here we add details and extend the investigation in a new direction.

Definition 1. (*p*-robust random graph). A random graph on n vertices is called *p*-robust if every edge in the graph is present with probability at least p , regardless of the status (present or not) of other edges. Such a random graph is denoted by $\tilde{G}_{n,p}$.

It is important to note that *p*-robustness does not imply independence. It allows that the probability of an edge may depend on other edges, possibly in a complicated way; it only requires that the probability never drop below p . Below we show some examples of *p*-robust random graphs.

Example 1. First note that the classical Erdős–Rényi random graph $G_{n,p}$ is a special case of $\tilde{G}_{n,p}$, since our model also allows adding all edges independently with probability p .

Example 2. We can also allow, however, possibly messy dependencies. As an example, let $P(e)$ be the probability that a given edge e is contained in the graph, and let us condition on k the number of other edges in the whole graph. Let $P_e(k)$ denote the probability that there are k other edges in the graph. For any fixed k , let us choose

$$P(e | k) = 1 - \frac{k+1}{n^2}$$

as the probability that edge e exists, given that there are k other edges in the graph. Using that the total number of edges cannot be more than $n(n-1)/2$, we have that $k \leq n(n-1)/2 - 1$ always holds. Therefore,

$$P(e | k) \geq 1 - \frac{n(n-1)}{2n^2} = 1 - \frac{n-1}{2n} \geq \frac{1}{2},$$

for any k , implying

$$P(e) = \sum_{k=0}^{n(n-1)/2-1} P(e | k) P_e(k) \geq \frac{1}{2}.$$

Thus, with $p = 1/2$, this random graph is *p*-robust. At the same time, the edges are not independent, since the probability that e is present depends on how many other edges are present.

Example 3. For any given edge e , let $r(e)$ denote the number of edges that are adjacent to e (not including e itself). If e does not exist, then let $r(e) = 0$. Let us choose the conditional probability that edge e

exists, given that it has k adjacent edges as

$$P(e \mid r(e) = k) = \frac{1}{2} - \frac{1}{k+5}.$$

Note that the possible range of k is $0 \leq k \leq 2(n-2)$. Then we have

$$P(e \mid r(e) = k) \geq \frac{1}{2} - \frac{1}{5} = \frac{3}{10}.$$

The above inequality implies

$$\begin{aligned} P(e) &= \sum_{k=0}^{2(n-2)} P(e \mid r(e) = k)P(r(e) = k) \\ &\geq \frac{3}{10} \sum_{k=0}^{2(n-2)} P(r(e) = k) = \frac{3}{10}. \end{aligned}$$

Thus, with $p = 3/10$, this random graph is p -robust. At the same time, the edges are not independent, since the probability that e is present is influenced by the number of adjacent edges.

Example 4. Consider the same model presented in Example 3, but with the additional condition that every potential edge e have at least three edges adjacent to it, regardless of whether e is in the graph or not. What can be said about this conditional random graph? Repeating the derivation from Example 3, but using $k \geq 3$, provides that the new random graph will remain p -robust, but now with $p = 3/8$.

If we have a random graph like the ones in Examples 2–4 (and many possible others with dependent edges), then how can we estimate some parameter of the random graph, like the size of the maximum clique or the diameter? It may become very messy, due to the possibly complicated and partially unknown dependencies. We show that at least for a large family of properties, the so-called monotone graph properties, we can use the existing results about Erdős–Rényi random graphs as lower bounds.

Let \mathcal{Q} be a set of graphs. We use it to represent a graph property: a graph G has property \mathcal{Q} if and only if $G \in \mathcal{Q}$. Therefore, we identify the property with the set \mathcal{Q} . We are going to consider *monotone graph properties*, which are defined below.

Definition 2. (Monotone graph property) A graph property \mathcal{Q} is called monotone if it is closed with respect to adding new edges to the graph. That is, $G \in \mathcal{Q}$ and $G \subseteq G'$ together imply $G' \in \mathcal{Q}$.

It is important to note that many of the often-used graph properties are monotone. Let us present some examples: the graph has a Hamiltonian circuit, it contains k disjoint spanning trees, the graph

contains a clique of size k , its diameter is at most k , its chromatic number is at least k , it has a matching of size at least k , it has a dominating set of size at most k , as well as a large number of others. In fact, essentially almost all interesting graph properties have a monotone version. Our result is that for any monotone graph property and for any n, p , it always holds that $\tilde{G}_{n,p}$ is more likely to have the property than $G_{n,p}$ (or at least as likely).

Why is this useful? Because it allows the application of the rich treasury of results on Erdős–Rényi random graphs to the non-independent setting, as lower bounds on the probability of having a monotone property. Next we state and prove our general result.

Theorem 1. For any monotone graph property Q the following inequality holds:

$$\Pr(G_{n,p} \in Q) \leq \Pr(\tilde{G}_{n,p} \in Q).$$

Proof. Our plan is to represent $\tilde{G}_{n,p}$ as the union of two random graphs, $G_{n,p}$ and G_2 . They are both on the same vertex set V . Here $G_{n,p}$ is the well-known Erdős–Rényi random graph; the other graph G_2 will be defined later. The union $G_{n,p} \cup G_2$ is meant such that if the same edge occurs in both graphs, then we merge them into a single edge. Our plan is to choose the edge probabilities in G_2 such that it produces $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$. Here the “ \sim ” sign denotes a relation between random graphs with the meaning that they have the same probability distribution; in other words, they are statistically indistinguishable. If we can accomplish this, then the claim will directly follow, since then a random graph distributed as $\tilde{G}_{n,p}$ can be obtained by adding edges to $G_{n,p}$, which cannot destroy a monotone property once $G_{n,p}$ has it. This will imply the claim.

Let us introduce some notations. The (potential) edges are denoted by e_1, \dots, e_m . For every i , let h_i be the indicator of the event that the edge e_i is included in $\tilde{G}_{n,p}$. Further, we use the abbreviation $h_i^m = (h_i, \dots, h_m)$. For any binary vector $a = (a_1, \dots, a_m) \in \{0, 1\}^m$, the event $\{h_1^m = a\}$ means that $\tilde{G}_{n,p}$ takes a realization in which edge e_i is included if and only $a_i = 1$. Similarly, if we want to start from an index i , rather than 1, then the notation $\{h_i^m = a_i^m\}$ abbreviates the event $\{h_i = a_i, \dots, h_m = a_m\}$. We also use the notation $a_i^m = (a_i, \dots, a_m)$.

Now we generate the random graphs $G_{n,p}$ and G_2 by the following procedure, which is a recursive procedure, starting from $i = m$ and processing downward to $i = 1$.

Step 1. Set $i = m$.

Step 2. If $i = m$, then choose $q_m = \Pr(h_m = 1)$. If $i < m$, then set $q_i = \Pr(h_i = 1 \mid h_{i+1}^m = a_{i+1}^m)$, where a_{i+1}^m indicates the formerly generated edges of $G_{n,p} \cup G_2$.

Step 3. Compute the value

$$p'_i = \frac{p(1 - q_i)}{1 - p}. \quad (1)$$

Step 4. With probability p , put the edge e_i into $G_{n,p}$, and with probability $q_i - p'_i$, put the edge e_i into G_2 .

Step 5. If $i > 1$, then decrement i by one and go to Step 2; else halt.

To analyze the procedure, first note that the value $q_i - p'_i$ in Step 4 can indeed serve as a probability. Why? First, we have $q_i - p'_i \leq 1$, as q_i is a probability and $p'_i \geq 0$. To show $q_i - p'_i \geq 0$, observe that

$$p'_i = \frac{p(1 - q_i)}{1 - p} \leq q_i,$$

since the inequality can be rearranged into

$$p(1 - q_i) \leq q_i(1 - p),$$

which simplifies to $p \leq q_i$. The latter is indeed true, due to

$$q_i = \Pr(h_i = 1 \mid h_{i+1}^m = a_{i+1}^m) \geq p,$$

which follows from the p -robust property.

Next we show that the algorithm generates the random graphs $G_{n,p}$ and G_2 in a way that they satisfy $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$. We prove it by induction, starting from $i = m$ and progressing downward to $i = 1$. For any i , let $G_{n,p}^i$, G_2^i denote the already generated parts of $G_{n,p}$, G_2 , respectively, after executing Step 4 $m - i + 1$ times, so they can only contain edges with index $\geq i$. Further, let $\tilde{G}_{n,p}^i$ be the subgraph of $\tilde{G}_{n,p}$ in which we only keep the edges with index $\geq i$, that is, $\tilde{G}_{n,p}^i = \tilde{G}_{n,p} - \{e_{i-1}, \dots, e_1\}$. The inductive proof will show that $G_{n,p}^i \cup G_2^i \sim \tilde{G}_{n,p}^i$ holds for every i . At the end of the induction, having reached $i = 1$, we are going to get $G_{n,p}^1 \cup G_2^1 \sim \tilde{G}_{n,p}^1$, which is the same as $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$.

Let us first look at the base case $i = m$. Then we have

$$\Pr(e_m \in G_{n,p}) = \Pr(e_m \in G_{n,p}^m) = p$$

by Step 4. Then in Step 4, edge e_m is put into G_2 with probability $q_m - p'_m$, yielding $\Pr(e_m \in G_2^m) = q_m - p'_m$. Now observe that equation (1) is chosen such that p'_i is precisely the solution of the equation

$$p + q_i - p'_i - (q_i - p'_i)p = q_i \quad (2)$$

for p'_i . For $i = m$ the equation becomes

$$p + q_m - p'_m - (q_m - p'_m)p = q_m, \quad (3)$$

and

$$p'_m = \frac{p(1 - q_m)}{1 - p}$$

is the solution of this equation. Since by Step 4 we have $\Pr(e_m \in G_{n,p}^m) = p$ and $\Pr(e_m \in G_2^m) = q_m - p'_m$, therefore, we get that the left-hand side of equation (3) is precisely the probability of the event $\{e_m \in G_{n,p}^m \cup G_2^m\}$. By equation (3), this probability is equal to q_m , which is set to $q_m = \Pr(h_m = 1) = \Pr(e_m \in \tilde{G}_{n,p}^m)$ in Step 2. This means that $G_{n,p}^m \cup G_2^m \sim \tilde{G}_{n,p}^m$, as desired.

For the induction step, assume that the claim is true for $i + 1$; that is, $G_{n,p}^{i+1} \cup G_2^{i+1} \sim \tilde{G}_{n,p}^{i+1}$ holds. We show that it carries over to i . In Step 4, edge e_i is added to $G_{n,p}^{i+1}$ with probability p . It is also added to G_2^{i+1} with probability $q_i - p'_i$. Therefore, just like in the base case, we get

$$p + q_i - p'_i - (q_i - p'_i)p = \Pr(e_i \in G_{n,p}^i \cup G_2^i).$$

We already know that p'_i satisfies equation (2), so e_i is added to $\tilde{G}_{n,p}^{i+1}$ with probability $q_i = \Pr(h_i = 1 \mid h_{i+1}^m = a_{i+1}^m)$, given the already generated part, represented by a_{i+1}^m . By the inductive assumption, h_{i+1}^m is distributed as $\tilde{G}_{n,p}^{i+1}$, which is the truncated version of $\tilde{G}_{n,p}$, keeping only the $\geq i + 1$ indexed edges. Hence, for h_{i+1}^m , we can write by the chain rule of conditional probabilities:

$$\Pr(h_{i+1}^m = a_{i+1}^m) = \Pr(h_m = a_m) \prod_{j=i+1}^{m-1} \Pr(h_j = a_j \mid h_{j+1}^m = a_{j+1}^m).$$

After processing e_i (i.e., adding it with probability q_i), we get

$$\begin{aligned}
 \Pr(h_i^m = a_i^m) &= \Pr(h_i = a_i \mid h_{i+1}^m = a_{i+1}^m) \Pr(h_{i+1}^m = a_{i+1}^m) \\
 &= \Pr(h_i = a_i \mid h_{i+1}^m = a_{i+1}^m) \times \\
 &\quad \frac{\Pr(h_m = a_m) \prod_{j=i+1}^{m-1} \Pr(h_j = a_j \mid h_{j+1}^m = a_{j+1}^m)}{\Pr(h_{i+1}^m = a_{i+1}^m)} \\
 &= \Pr(h_m = a_m) \prod_{j=i}^{m-1} \Pr(h_j = a_j \mid h_{j+1}^m = a_{j+1}^m).
 \end{aligned}$$

Now we observe that by the chain rule this is indeed the distribution of $\tilde{G}_{n,p}^i$, completing the induction.

Thus, eventually, a realization $a = a_1^m \in \{0, 1\}^m$ of $\tilde{G}_{n,p}$ is generated with probability

$$\Pr(h_1^m = a) = \Pr(h_m = a_m) \prod_{j=1}^{m-1} \Pr(h_j = a_j \mid h_{j+1}^m = a_{j+1}^m),$$

which indeed assigns $\tilde{G}_{n,p}$ its correct probability. As a result, we get $G_{n,p} \cup G_2 \sim \tilde{G}_{n,p}$. From this we conclude $\tilde{G}_{n,p}$ arises by adding edges to $G_{n,p}$, and such additions cannot destroy a monotone property. From this the statement of the theorem follows, completing the proof. \square

3. An Example

As a sample application of the result, consider the random graph described in Example 3 of Section 2. For handy access, let us repeat the example here:

Example 3. For a given edge e , let $r(e)$ denote the number of edges that are adjacent with e (not including e itself). If e does not exist, then let $r(e) = 0$. Let the conditional probability that edge e exists, given that it has k adjacent edges, be

$$P(e \mid r(e) = k) = \frac{1}{2} - \frac{1}{k+5}.$$

Note that the possible range of k is $0 \leq k \leq 2(n-2)$. Then we have

$$P(e \mid r(e) = k) \geq \frac{1}{2} - \frac{1}{5} = \frac{3}{10}.$$

This implies

$$P(e) = \sum_{k=0}^{2(n-2)} P(e \mid r(e) = k)P(r(e) = k) \geq \frac{3}{10} \sum_{k=0}^{2(n-2)} P(r(e) = k) = \frac{3}{10}.$$

Thus, with $p = 3/10$, this random graph is p -robust. At the same time, the edges are not independent, since the probability that e is present is influenced by the number of adjacent edges.

We can ask several natural questions about this edge-dependent random graph model, regarding the asymptotic value of various parameters:

- What is the chromatic number?
- How large is the maximum clique?
- What is the diameter?
- How large is a minimum dominating set?

Answering these questions via direct analysis of the model would be a daunting task. On the other hand, using the known results about the Erdős-Rényi random graph $G_{n,p}$ (listed in the Introduction), and complementing them with our Theorem 1, we can quickly obtain concise bounds on the parameters:

- The chromatic number is asymptotically at least $n / (\log_b n)$, where the parameter b is given by $b = 1 / (1 - p) = 10 / 7$.
- This random graph asymptotically has a maximum clique of size at least $2 \log_{10/3} n$.
- The minimum dominating set is asymptotically at most $\log_b n$, where $b = 1 / (1 - p) = 10 / 7$.
- The diameter of the graph is asymptotically at most $\log n / (\log(3n / 10))$.

Observe that obtaining such bounds from the direct analysis of the model would be extremely hard.

4. Positive Edge Correlation

So far we have analyzed the edge-dependency structure that we called p -robust random graph; see Definition 1. This property can also be stated such that the probability of an edge cannot decrease if we condition on the arbitrary presence/absence of other edges.

Now we look into a less demanding condition: we require that the probability of an edge cannot decrease if we condition on the arbitrary presence of other edges, but ignoring absence. It can be formulated this way:

Definition 3. (Positive edge correlation) A random graph has the positive edge correlation property if for any (potential) distinct edges e, e_1, \dots, e_k it holds that

$$\Pr(e \mid e_1, \dots, e_k) \geq \Pr(e).$$

In other words, the presence of a given edge cannot be made less likely by the presence of other edges.

It is natural to ask: which random graphs have this property? While currently we do not have a full answer, we conjecture that all *geometric random graphs* have this property. A random graph is called *geometric* if it arises in the following way: pick random points independently from some distribution over a metric space; they represent the nodes. The edges are obtained by connecting any two points if their distance is at most a given parameter $r > 0$. Then we can state our conjecture as follows:

Conjecture 1. All geometric random graphs have the positive edge correlation property.

The requirement of positive edge correlation is weaker than the property of p -robustness, so we cannot expect that a strong result like Theorem 1 would still hold. Nevertheless, we can still prove an interesting property. In order to state it, let us introduce some notations.

Definition 4. (Erdős–Rényi core) Assume G is a random graph with n nodes, in which any given edge exists with the same probability p , but the edges may not be independent. Then the Erdős–Rényi random graph $G_{n,p}$ is called the Erdős–Rényi core of G .

In other words, the Erdős–Rényi core has the same number of nodes and the same edge probability as G , but the edges are independent, in contrast to arbitrary edge dependencies in G .

Definition 5. (Q -subgraph count) For any graph G and for any property Q of graphs, the Q -subgraph count in G means how many subgraphs of G have property Q . We denote this quantity by $\mathcal{N}(G, Q)$.

Now we are ready to prove our result:

Theorem 2. For an arbitrary random graph G with n vertices, assume that any given edge exists with the same probability p , but the edges may not be independent. Let $G_{n,p}$ be the Erdős–Rényi core of G . Assume that G has the positive edge correlation property (Definition 3). Then for any graph property Q , it holds that the expected

value of the \mathcal{Q} -subgraph count in G is at least as large as the expected value of the \mathcal{Q} -subgraph count in $G_{n,p}$. In formula:

$$E(\mathcal{N}(G, \mathcal{Q})) \geq E(\mathcal{N}(G_{n,p}, \mathcal{Q})). \quad (4)$$

Proof. Let G_0 be any fixed graph that may be a subgraph of G with positive probability. If G_0 has property \mathcal{Q} , this fact is denoted by $G_0 \in \mathcal{Q}$. If it also has at most n nodes, then we use the notation $G_0 \in \mathcal{Q}_n$. Further, the event that G_0 is a subgraph of the (random) graph G is denoted by $G_0 \subseteq G$. Then we can express the expected \mathcal{Q} -subgraph count in G as

$$E(\mathcal{N}(G, \mathcal{Q})) = \sum_{G_0 \in \mathcal{Q}_n} \Pr(G_0 \subseteq G).$$

For a fixed G_0 , let e_1, \dots, e_k denote its edges. Using the notation $\Pr(e_1, \dots, e_k \in G) = \Pr(e_1, \dots, e_k)$, we can rewrite the expression of $E(\mathcal{N}(G, \mathcal{Q}))$ as

$$E(\mathcal{N}(G, \mathcal{Q})) = \sum_{G_0 \in \mathcal{Q}_n, E(G_0) = \{e_1, \dots, e_k\}} \Pr(e_1, \dots, e_k). \quad (5)$$

Next we show that the positive correlation property implies

$$\Pr(e_1, \dots, e_k) \geq \Pr(e_1) \dots \Pr(e_k).$$

We use induction with respect to k . For $k = 1$ the statement is trivial. Assume it holds for any $k - 1$ edges, implying

$$\Pr(e_2, \dots, e_k) \geq \Pr(e_2) \dots \Pr(e_k). \quad (6)$$

Then we carry it over to k , as follows. First we write

$$\Pr(e_1, \dots, e_k) = \Pr(e_1 | e_2, \dots, e_k) \Pr(e_2, \dots, e_k).$$

Then we observe that the conditional probability $\Pr(e_1 | e_2, \dots, e_k)$ can be bounded from below by $\Pr(e_1)$, due to the positive correlation property. Further, the probability $\Pr(e_2, \dots, e_k)$ can be bounded from below by equation (6), so we indeed obtain

$$\Pr(e_1, \dots, e_k) \geq \Pr(e_1) \dots \Pr(e_k).$$

Using it in equation (5) yields

$$E(\mathcal{N}(G, \mathcal{Q})) \geq \sum_{G_0 \in \mathcal{Q}_n, E(G_0) = \{e_1, \dots, e_k\}} \Pr(e_1) \dots \Pr(e_k). \quad (7)$$

Now observe that

$$\Pr(G_0 \subseteq G_{n,p}) = \Pr(e_1) \dots \Pr(e_k).$$

This yields

$$\begin{aligned} E(\mathcal{N}(G_{n,p}, Q)) &= \sum_{G_0 \in Q_n} \Pr(G_0 \subseteq G_{n,p}) = \\ &= \sum_{G_0 \in Q_n, E(G_0) = \{e_1, \dots, e_k\}} \Pr(e_1) \dots \Pr(e_k). \end{aligned} \quad (8)$$

Comparing equations (7) and (8), we obtain precisely equation (4) as desired. \square

5. Interpretation of the Results

Both of our theorems allow the use of results about Erdős–Rényi random graphs to obtain lower bounds on probabilities or expected subgraph counts in more general random graph models. This may lead to bounds that would be otherwise hard to obtain via direct analysis of the model. We have described an example in Section 3.

Let us mention here another interpretation that exhibits interesting extremal properties of the Erdős–Rényi random graph model. We state them below in two theorems, which directly follow from Theorems 1 and 2.

Theorem 3. Let Q be a graph property, which is assumed monotone. Then the Erdős–Rényi random graph $G_{n,p}$ has the smallest probability to have property Q among all p -robust random graphs on n vertices.

Theorem 4. Let Q be an arbitrary graph property. Then among all random graphs on n vertices with the positive correlation property and with edge probability p , the Erdős–Rényi random graph $G_{n,p}$ has the smallest expected Q -subgraph count.

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