Sequences of Preimages in Elementary Cellular Automata

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Regularities are searched for in the sequences of numbers of preimages for elementary cellular automata (CA). For 46 out of 88 "minimal" rules, recognizable patterns are found, usually in the form of second order recurrence equations with constant coefficients. After introducing the concept of asymptotic emulation of CA rules, it is shown how the regularities in the sequences of preimage numbers can be used to find rules emulating identity. It is also shown that the average density of nonzero sites after an arbitrary number of steps (starting from a disordered configuration) can be computed using the sequences of preimage numbers.

1. Introduction

One of the fundamental problems in the theory of cellular automata (CA) is the enumeration of preimages. Preimages for a given spatial sequence are defined as the set of blocks that are mapped to that sequence by the automaton rule. Since the number of preimages for the sequence provides information about the probability distribution associated with the rule, it can be useful for a variety of problems, such as computations of spatial measure entropy, identification of sequences with maximal probability [5], and identification of the Garden of Eden [7].

For one-step preimages, E. Jen showed in [4] that the number of preimages for arbitrary sequences satisfies a system of recurrence relations with coefficients depending on the automaton rule. No analytical results, however, are known for the number of n-step preimages, that is, the number of preimages under the rule iterated n times. In this paper, we will show that the sequences of n-step preimage numbers in many cases follow recognizable patterns, so the expression for the general term of the sequence can be conjectured (and, in some simple cases, proved). We will then present two possible applications of such expressions, in finding asymptotical emulators of CA rules and densities of nonzero sites after an arbitrary number of time steps.

Let $\mathcal{G} = \{0, 1, \ldots N-1\}$ be called *a symbol set*, and let $\mathcal{S}(\mathcal{G})$ be the set of all bisequences over $\mathcal{G}$, where by a bisequence we mean a function on $\mathbf{Z}$ to $\mathcal{G}$. The set $\mathcal{S}(\mathcal{G})$ is a compact, totally disconnected, perfect,

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metric space, and will be called *the configuration space*. Throughout the remainder of this paper we shall assume that $\mathcal{G} = \{0, 1\}$, and the configuration space $\mathcal{S}(\mathcal{G}) = \{0, 1\}^{\mathbf{Z}}$ will be simply denoted by $\mathcal{S}$.

A block of radius r is an ordered set $b_{-r}b_{-r+1}\ldots b_r$, where $r \in \mathbf{N}$, $b_i \in \mathcal{G}$. Let $r \in \mathbf{N}$ and let $\mathcal{B}_r$ denote the set of all blocks of radius r over $\mathcal{G}$. The number of elements of $\mathcal{B}_r$ (denoted by card $\mathcal{B}_r$) equals 2^{2r+1}. The set of all blocks of finite radius will be denoted by $\mathcal{B} = \bigcup_{r=0}^{\infty} \mathcal{B}_r$.

A mapping $f : \{0, 1\}^{2r+1} \mapsto \{0, 1\}$ will be called *a cellular automaton rule of radius r*. Alternatively, the function f can be considered as a mapping of $\mathcal{B}_r$ into $\mathcal{B}_0 = \mathcal{G} = \{0, 1\}$. The set of all mappings of radius r will be denoted by $\mathcal{F}_r$, and the set of all possible CA mappings by $\mathcal{F} = \bigcup_{r=0}^{\infty} \mathcal{F}_r$.

Corresponding to f (also called *a local mapping*) we define a *global mapping* $F : S \to S$ such that $(F(s))_i = f(s_{i-r}, \ldots, s_i, \ldots, s_{i+r})$ for any $s \in S$. The *composition of two rules* $f, g \in \mathcal{F}$ can now be defined in terms of their corresponding global mappings F and G as $(F \circ G)(s) = F(G(s))$, where $s \in S$. We note that if $f \in \mathcal{F}_p$ and $g \in \mathcal{F}_q$, then $f \circ g \in \mathcal{F}_{p+q}$. For example, the composition of two radius-1 mappings is a radius-2 mapping:

$$(f \circ g)(s_{-2}, s_{-1}, s_0, s_1, s_2) = \\ f(g(s_{-2}, s_{-1}, s_0), g(s_{-1}, s_0, s_1), g(s_0, s_1, s_2)). \tag{1}$$

Multiple composition will be denoted by

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}. \tag{2}$$

A *block evolution operator* corresponding to f is a mapping $\mathbf{f} : \mathcal{B} \mapsto \mathcal{B}$ defined as follows. Let $r \geq p > 0$, $a \in \mathcal{B}_r$, $f \in \mathcal{F}_p$, and let $b_i = f(a_{i-p}, a_{i-p+1}, \ldots, a_{i+p})$ for $-r + p \leq i \leq r - p$. Then we define $\mathbf{f}(a) = b$, where $b \in \mathcal{B}_{r-p}$. Note that if $b \in B_1$ then $f(b) = \mathbf{f}(b)$.

In what follows we will consider the case of $\mathcal{G} = \{0, 1\}$ and $r = 1$ rules, that is, *elementary cellular automata*. The set of radius-1 blocks $\mathcal{B}_1$ has then eight elements, which will be denoted by

$$\{\beta_i\}_{i=0}^{i=7} = \{000, 001, 010, 011, 100, 101, 101, 110, 111\}, \tag{3}$$

so that the binary representation of the index i defines the block β_i. Given an elementary rule f, we will try to find the number of n-step preimages of such *basic blocks* under the rule f.

2. Sequences of preimage numbers

The number of n-step preimages of the block b under the rule f is defined as the number of elements of the set $\mathbf{f}^{-n}(b)$. For surjective rules, this number is always easily computed. As proved in [3], under the

Block b	Rule numbers of the rule f
001	0
010	0, 19, 46, 126, 200
011	0, 2, 4, 8, 12, 24, 32, 34
100	0
101	0, 1, 2, 3, 8, 10, 11, 36, 128, 136, 138
110	0, 2, 4, 8, 12, 24, 32, 34
111	0, 2, 4, 6, 8, 10, 12, 14, 18, 24, 28, 32, 34, 40, 42, 50, 56, 72, 76

Table 1. Blocks b which have no preimages under some elementary rules f.

surjective elementary rule every block has exactly four preimages, so $\text{card}[\mathbf{f}^{-n}(b)] = 4^n$ for every block b. For nonsurjective rules, however, sequences of n-step preimage numbers can be highly nontrivial, and no general method for obtaining them without direct counting is known.

Using a simple preimage counting computer program, sequences $a_n = \text{card}\,\mathbf{f}^{-n}(b)$ can be constructed for a given rule f and a block b. For some elementary CA rules and basic blocks, these sequences appear to follow certain recognizable patterns, while for other rules no pattern seems to emerge after computation of the first 10 terms (since the number of possible blocks increases exponentially with the block length, it becomes increasingly difficult to go much beyond $n = 10$ using direct enumeration). The simplest pattern to recognize is the constant sequence $a_n = \text{const}$. Let us first consider the case when $\mathbf{f}^{-n}(\beta_i)$ is empty, that is, $a_n = 0$ for every positive integer n. In order to prove that a given block $b \in \mathcal{B}_1$ has no preimage under a given rule (i.e., $\text{card}\,\mathbf{f}^{-n}(\beta_i) = 0$) one just has to check that for every block $c \in \mathcal{B}_2$ (among 32 possible) the condition $\mathbf{f}(b) \neq c$ is satisfied. We performed this check for all 88 minimal elementary CA rules and all basic blocks. Results are presented in Table 1.

Another type of constant sequence is the case when the set $\mathbf{f}^{-n}(\beta_i)$ has only one element regardless of n, that is, $a_n = 1$. All such cases are shown in Table 2. Although this table was generated with the help of a computer, it is not difficult to prove that $\text{card}\,\mathbf{f}^{-n}(b) = 1$ for a given basic block b. As an example, consider elementary rule 77 (for this rule, $\mathbf{f}^{-1}(0) = \{001, 100, 101, 111\}$). We claim the following.

Proposition 1. For rule 77, both sets $\mathbf{f}^{-n}(000)$ and $\mathbf{f}^{-n}(111)$ have only one element for all positive integers n.

To see this, let us consider a block of ones $11\ldots1$ of radius r, which will be denoted by $\mathbf{1}^{(r)}$ (similarly, a block of zeros of radius r will be denoted by $\mathbf{0}^{(r)}$). It is easy to verify that $\mathbf{f}^{-1}\left(\mathbf{1}^{(r)}\right) = \mathbf{0}^{(r+1)}$. Indeed, if we assume that there exists a block $a \in \mathcal{B}_{r+1}$ such that $\mathbf{f}(a) = \mathbf{1}^{(r)}$, with at least one nonzero site, then block a must include at least one of the subblocks

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Block b	Rule numbers of the rule f
000	77, 178
001	none
010	23, 128, 232
011	128,160, 162, 130, 132
100	none
101	23, 32, 44, 130, 232, 33
110	128, 130, 132, 160, 162,
111	77, 128, 130, 132, 134, 146, 160, 162, 178

Table 2. Basic blocks b and rules f for which $\mathrm{card}[\mathbf{f}^{-n}(b)] = 1$ for every positive integer n.

001, 100, 101, or 111. All of these subblocks belong to $\mathbf{f}^{-1}(0)$, so $\mathbf{f}(a)$ cannot be $\mathbf{1}^{(r+1)}$. Therefore, for the rule 77 $\mathbf{f}^{-1}\left(\mathbf{1}^{(r)}\right) = \mathbf{0}^{(r+1)}$, and similarly $\mathbf{f}^{-1}\left(\mathbf{0}^{(r)}\right) = \mathbf{1}^{(r+1)}$, which implies that $\mathbf{f}^{-n}(000)$ and $\mathbf{f}^{-n}(111)$ are single-element sets for every positive integer n. Similar proofs can be constructed for other entries in Table 2.

Cases with $a_n = \mathrm{const} > 1$ are not numerous. We found only seven of them in all of the "minimal" elementary rules, with the largest possible constant a_n equal to 5. These cases can be summarized in the following conjecture.

Conjecture 1. The only minimal elementary CA rules and the only basic blocks for which the sequence of preimage numbers $a_n = \mathrm{card}[\mathbf{f}^{-n}(\beta_i)]$ is constant (i.e., n-independent) and $a_n > 1$ are:

- $\mathrm{card}[\mathbf{f}_{128}^{-n}(001)] = \mathrm{card}[\mathbf{f}_{128}^{-n}(100)] = 2$
- $\mathrm{card}\,\mathbf{f}_{32}^{-n}(001) = \mathrm{card}[\mathbf{f}_{32}^{-n}(100)] = \mathrm{card}[\mathbf{f}_{58}^{-n}(000)] = 3$
- $\mathrm{card}[\mathbf{f}_{32}^{-n}(010)] = 4$
- $\mathrm{card}[\mathbf{f}_{50}^{-n}(000)] = 5.$

All of these expressions hold for any positive integer n.

The sequence a_n can be, of course, much more complicated than $a_n = \mathrm{const}$. After experimenting with various possibilities, we found that in many cases a_n appears to satisfy a second order difference equation with constant coefficients

$$a_{n+2} = c_1 a_{n+1} + c_2 a_n + c_3. \tag{4}$$

To check whether this is plausible, we performed the following test. Using the first five terms of a_n (obtained using the preimage counting program) we can solve the system of three linear equations for c_1, c_2, and c_3:

$$\begin{aligned} a_3 &= c_1 a_2 + c_2 a_1 + c_3, \\ a_4 &= c_1 a_3 + c_2 a_2 + c_3, \\ a_5 &= c_1 a_4 + c_2 a_3 + c_3. \end{aligned} \tag{5}$$

The solutions c_1, c_2, and c_3 can now be used to generate the next five terms of the sequence $a_6 \ldots a_{10}$. If they agree with the experimental values of $a_6 \ldots a_{10}$, we can conjecture that the sequence a_n is a solution of the difference equation (4).

As an example, let us consider the rule 172 and the block 101. This block has four preimages under $\mathbf{f}_{172}$, 12 preimages under $\mathbf{f}_{172}^2$, 40 preimages under $\mathbf{f}_{172}^3$, and so forth. The first few terms of $a_n = \text{card}[\mathbf{f}_{172}^{-n}(101)]$ obtained using the preimage counting program are

$$a_n = \{2, 6, 20, 64, 208, 672, 2176, 7040, 22784 \ldots\}. \tag{6}$$

Solving equation (5) we obtain $c_1 = 2$, $c_2 = 4$, and $c_3 = 0$, that is,

$$a_{n+2} = 2a_{n+1} + 4a_n. \tag{7}$$

Although this difference equation was obtained using $a_1 \ldots a_5$ only, it is easy to check that it is satisfied for all 10 terms listed above. Its solution is

$$a_n = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{8\sqrt{5}}. \tag{8}$$

The same procedure can be applied to other elementary rules, and for many of them expressions similar to equation (8) can be found. A table in the appendix shows all such cases. They are presented as a set of eight expressions, each representing $a_n = \text{card}[\mathbf{f}^n(\beta_i)]$ for all eight basic blocks β_i, $i = 1 \ldots 7$. Only rules for which we were able to conjecture all eight expressions are shown, including cases when $a_n = \text{const}$. Surjective rules (i.e., 15, 30, 45, 51, 60, 90, 105, 106, 150, 170, 204, and 240) are omitted, since for them we always have $a_n = 4^n$.

3. Asymptotic emulation in cellular automata

We say (after [6]) that f *emulates* g in k iterations ($k \geq 0$) or f is *a kth level emulator of* g if

$$f \circ f^k = g \circ f^k. \tag{9}$$

If a CA f emulates g then after k time steps we can replace the rule f by g and we will obtain the same result as if we had kept rule f. For example, many elementary ($r = 1$) rules emulate the identity rule.

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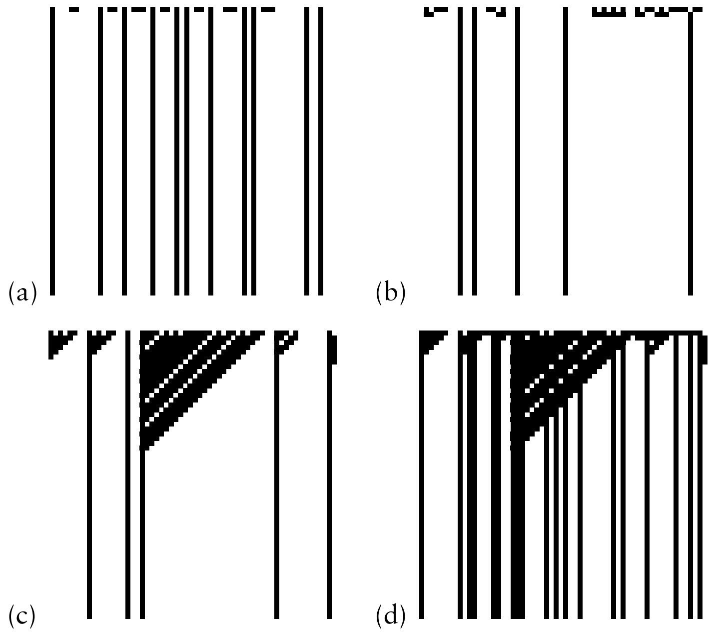

Figure 1. Examples of CA rules emulating identity: (a) rule 4, first-level emulation; (b) rule 36, second-level emulation; (c) rule 172, asymptotic emulation; and (d) rule 164, asymptotic emulation.

As proved in [6], these rules are 0, 4, 8, 12, 36, 72, 76, 200, and 204 (only minimal representatives are listed here), and the level of emulation is always 0, 1, or 2. Spatiotemporal patterns generated by these rules after a few time steps become identical with the pattern generated by the identity rule (vertical strips), as shown in Figures 1(a) and 1(b). Visual examination of patterns generated by elementary CA reveal that more rules than those mentioned earlier produce patterns resembling rule 204 (identity rule). Among 88 "minimal" representatives of elementary rules there are 16 other "identity-like" mappings, namely 13, 32, 40, 44, 77, 78, 104, 128, 132, 136, 140, 160, 164, 168, 172, and 232. Typical patterns produced by these mappings are shown in Figures 1(c) and 1(d). These patterns eventually become vertical strips, but the time required to achieve such a state may be quite long. None of them, of course, emulates identity in the sense of equation (9). We could say, however, that these rules simulate identity "approximately," and that this approximation gets better and better with an increasing number of time steps. A quantitative description of this phenomenon is possible if we introduce a distance between rules. For $f \in \mathcal{F}_p$ and $b \in \mathcal{B}_q$, where $q > p$, we define $f(b) = f(b_{-r}, \ldots, b_i, \ldots, b_r)$. A metric in $\mathcal{F}$ can be constructed as follows.

Proposition 2. Let $f \in \mathcal{F}_m$, $g \in \mathcal{F}_n$, and $k = \max\{m, n\}$. A function $d : \mathcal{F} \times \mathcal{F} \mapsto [0, 1]$ defined by

$$d(f, g) = 2^{-2k-1} \sum_{b \in \mathcal{B}_k} \left| f(b) - g(b) \right| \tag{10}$$

is a metric in $\mathcal{F}$.

Obviously, $d(f, g) \geq 0$ and $d(f, g) = 0 \Leftrightarrow f = g$. Triangle inequality holds too since $|x + y| \leq |x| + |y|$ for all $x, y \in \{0, 1\}$.

A CA rule f *asymptotically emulates rule* g if

$$\lim_{n\to\infty} d(f^{n+1}, g \circ f^n) = 0. \tag{11}$$

Clearly, if f is a kth level emulator of g then f emulates g asymptotically. We may think about asymptotic emulation as infinity-level emulation.

Let us now consider two rules $f, g \in \mathcal{F}$. Their *sum modulo* 2 will be defined as $(f \oplus g)(b) = f(b) + g(b) \bmod 2 = |f(b) - g(b)|$ for any $b \in \mathcal{B}$. Note that $(f \oplus g)(b) = 0$ if $f(b) = g(b)$ and $(f \oplus g)(b) = 1$ if $f(b) \neq g(b)$.

Proposition 3. Let $f, g \in \mathcal{F}_1$ and $h = f \oplus g$. Let $A_0 = \mathbf{h}^{-1}(1)$, and let $A_n = \mathbf{f}^{-n}(A_0)$. Then

$$d(f^{n+1}, g \circ f^n) = \frac{\operatorname{card} A_n}{2^{2n+3}}. \tag{12}$$

Proof. Mapping f^{n+1} is a rule of radius $n + 2$, therefore using the definition of the distance equation (10) and properties of block evolution functions we have

$$d(f^{n+1}, g \circ f^n) = 2^{-2n-3} \sum_{b \in \mathcal{B}_{n+1}} \left| \mathbf{f}^{n+1}(b) - \mathbf{g} \circ \mathbf{f}^n(b) \right|, \tag{13}$$

or $d(f^{n+1}, g \circ f^n) = 2^{-2n-3} c_n$, where c_n is a number of blocks $b \in \mathcal{B}_{n+2}$ such that $\mathbf{f}^{n+1}(b) \neq \mathbf{g} \circ \mathbf{f}^n(b)$. Similarly, the set A_0 is a set of all blocks $b \in \mathcal{B}_1$ such that $\mathbf{f}(b) \neq \mathbf{g}(b)$. Let us now consider a block $a \in B_{n+1}$ such that $\mathbf{f}^{n+1}(a) \neq \mathbf{g} \circ \mathbf{f}^n(a)$. The last relation can be written as $\mathbf{f}\,(\mathbf{f}^n(a)) \neq \mathbf{g}\,(\mathbf{f}^n(a))$, and this is possible if and only if $\mathbf{f}^n(a) \in A_0$, which is equivalent to $a \in \mathbf{f}^{-n}(A_0)$. This proves that block $a \in \mathcal{B}_{n+1}$ satisfies $\mathbf{f}^{n+1}(a) \neq \mathbf{g} \circ \mathbf{f}^n(a)$ if and only if $a \in A_n$, so finally $c_n = \operatorname{card} A_n$. ■

Proposition 3 can be useful in finding asymptotical emulators. As an example, consider the case of rule 77 discussed earlier, where we have

$$A_0 = (\mathbf{f}_{77} \oplus \mathbf{f}_{204})^{-1}(1) = \{000, 111\}. \tag{14}$$

We already proved (in Proposition 1) that both $\mathbf{f}_{77}^{-n}(000)$ and $\mathbf{f}_{77}^{-n}(111)$ have only one element for all n. Note that

$$\begin{aligned} \operatorname{card}[\mathbf{f}_{77}^{-n}\{000, 111\}] &= \operatorname{card}[f_{77}^{-n}(000)] + \operatorname{card}[\mathbf{f}_{77}^{-n}(111)] \\ &= 1 + 1 = 2, \end{aligned} \tag{15}$$

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since the preimage of the union of two sets is always the union of the preimages of the sets. This leads to the conclusion that

$$d(f_{77}^{n+1}, f_{204} \circ f_{77}^{n}) = \frac{2}{2^{2n+3}} = 2^{-2n-2}. \tag{16}$$

Of course, the above distance goes to zero with n, therefore rule 77 asymptotically emulates the identity (rule 204). Almost identical reasoning can be presented for rules 128 and 132, both of which asymptotically emulate identity and

$$\begin{aligned} d(f_{128}^{n+1}, f_{204} \circ f_{128}^{n}) &= 3 \cdot 2^{-2n-3} \\ d(f_{132}^{n+1}, f_{204} \circ f_{132}^{n}) &= 2^{-2n-2}. \end{aligned} \tag{17}$$

A slightly different analysis can be performed for rule 32. Here, from Table 1, we read that card$[\mathbf{f}_{32}^{-n}(101)] = 1$. Since

$$(\mathbf{f}_{32} \oplus \mathbf{f}_0)^{-1}(1) = 101,$$

we conclude that $d(f_{32}^{n+1}, f_0 \circ f_{32}^{n}) = 2^{-2n-3}$, and therefore rule 32 emulates the zero rule asymptotically. It also emulates the identity rule asymptotically, as a consequence of the following general property.

Proposition 4. If $f \in \mathcal{F}$ emulates the zero rule asymptotically, then it also emulates the identity rule asymptotically.

Proof. Using the triangle inequality, we have

$$\begin{aligned} 0 &\leq d(f^{n+1}, f_{204} \circ f^{n}) \\ &\leq d(f^{n+1}, f_0 \circ f^{n}) + d(f_0 \circ f^{n}, f_{204} \circ f^{n}). \end{aligned} \tag{18}$$

Since $f_0 \circ f^n = f_0$ and $f_{204} \circ f^n = f^n$, we obtain

$$d(f_0 \circ f^n, f_{204} \circ f^n) = d(f_0, f^n) = d(f^n, f_0 \circ f^{n-1}). \tag{19}$$

Equation (19), and the fact that f asymptotically emulates f_0, implies

$$\lim_{n\to\infty} \left(d(f^{n+1}, f_0 \circ f^n) + d(f_0 \circ f^n, f_{204} \circ f^n) \right) =$$
$$\lim_{n\to\infty} d(f^{n+1}, f_0 \circ f^n) + \lim_{n\to\infty} d(f^n, f_0 \circ f^{n-1}) = 0,$$

so finally $\lim_{n\to\infty} d(f^{n+1}, f_{204} \circ f^n) = 0$, as required for f to emulate identity asymptotically. ■

Of course, we could directly use expressions from the appendix and find that

$$d(f_{32}^{n+1}, f_{204} \circ f_{32}^{n}) = \frac{5}{2^{2n+3}}. \tag{20}$$

For other identity-like rules mentioned at the beginning of this section, the mechanism of emulation is not as simple as in previous cases. Nevertheless, experimental evidence suggests the following conjecture.

f	$d_n = d(f^{n+1}, f_{204} \circ f^n)$	f	$d_n = d(f^{n+1}, f_{204} \circ f^n)$
13	$7 \cdot 2^{-n-4}$	132	2^{-2n-2}
32	$5 \cdot 2^{-2n-3}$	136	2^{-n-2}
40	2^{-n-1}	140	2^{-n-3}
44	$7 \cdot 2^{-2n-3}$	160	$3 \cdot 2^{-n-2} - 4^{-n-1}$
77	2^{-2n-2}	164	$5 \cdot 2^{-n-3} - 4^{-n-1}$
78	4^{-1} if $n = 1$ $15 \cdot 2^{-n-6}$ if $n > 1$	168	$3^{n+1} \cdot 2^{-2n-3}$
104	$163 \cdot 2^{-2n-3}$ if $n > 5$	172	$\frac{-(1-\sqrt{5})^{n+3}+(1+\sqrt{5})^{n+3}}{2^{2n+6}\sqrt{5}}$
128	$3 \cdot 2^{-2n-3}$	232	2^{-2n-2}

Table 3. Distance $d_n = d(f^{n+1}, f_{204} \circ f^n)$ for rules asymptotically emulating identity.

Conjecture 2. Among the 88 "minimal" elementary CA rules, only rules 14, 40, 44, 78, 104, 136, 140, 160, 164, and 172 asymptotically emulate the identity rule.

Postulated expressions for the distance $d(f^{n+1}, f_{204} \circ f^n)$ are shown in Table 3. For completeness, rules for which the proof is known (i.e., 32, 77, 128, and 132) are included as well.

4. Density of nonzero sites

The simplest statistical quantity characterizing a configuration is the average fraction of sites with value 1 at time t, denoted by c_t. The question we want to address now is as follows: If we start from a disordered configuration with $c_0 = 0.5$ (i.e., equal probability of 0 and 1), what is the density c_t at a later time t? When $c_0 = 0.5$, a disordered configuration contains all eight possible basic blocks with equal probability. Applying a CA rule to this initial state yields a configuration in which the fraction of sites with value 1 is given by

$$c_1 = \frac{\text{card}[\mathbf{f}^{-1}(1)]}{8}, \tag{21}$$

or in other words, by the fraction of the eight possible basic blocks which yield 1 according to the CA rule [8]. Similarly, the density of ones after two time steps will be given by the fraction of the 32 blocks of radius 2 which yield 1 when f^2 is applied. In general, we can write

$$c_t = \frac{\text{card}[\mathbf{f}^{-t}(1)]}{2^{2t+1}}, \tag{22}$$

where card$[f^{-t}(1)]$, as usual, denotes the number of preimages of 1 under

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Rule	c_t	Rule	c_t
1	$7/16 + \frac{5}{16}(-1)^t$	2	$1/8$
3	$7/16 + \frac{3}{16}(-1)^t$	4	$1/8$
5	$7/16 + \frac{3}{16}(-1)^t$	7	$15/32 + \frac{3}{32}(-1)^t - (-2)^{-t-4} - 2^{-t-4}$
8	0	10	$1/4$
12	$1/4$	13	$7/16 - (-2)^{-t-3}$
19	$1/2 + \frac{3}{32}(-1)^t$	23	$1/2$
24	$3/16$	27	$17/32 + \frac{1}{32}(-1)^t$
28	$1/2 + \frac{1}{48}(-1)^t - \frac{5}{24}2^{-t}$	29	$1/2$
32	2^{-1-2t}	34	$1/4$
36	$1/16$	38	$9/32$
40	2^{-t-1}	42	$3/8$
44	$1/6 + \frac{5}{6}2^{-2t}$	46	$3/8$
50	$1/2 - 2^{-2t-1}$	72	$1/8$
76	$3/8$	77	$1/2$
78	$9/16$	108	$5/16$
128	2^{-1-2t}	130	$1/6 + \frac{1}{3}2^{-2t}$
132	$1/6 + \frac{1}{3}2^{-2t}$	136	2^{-t-1}
138	$3/8$	140	$1/4 + 2^{-t-2}$
156	$1/2$	160	2^{-t-1}
162	$1/3 + \frac{1}{6}4^{-t}$	164	$1/12 - \frac{1}{3}4^{-t} + \frac{3}{4}2^{-t}$
168	$3^t 2^{-2t-1}$	172	$1/8 + \frac{(10-4\sqrt{5})(1-\sqrt{5})^t+(10+4\sqrt{5})(1+\sqrt{5})^t}{40\cdot 2^{2t}}$
178	$1/2$	200	$3/8$
232	$1/2$		

Table 4. Density of ones for a disordered initial state with $c_0 = 0.5$.

$\mathbf{f}^t$. To make use of the table in the appendix, we can rewrite the last equation as

$$c_t = 2^{-2t-1} \sum_{\mathbf{f}(\beta_i)=1} \text{card}[\mathbf{f}^{-t+1}(\beta_i)], \tag{23}$$

where the sum runs over all radius-1 blocks β_i which yield 1 according to the CA rule, which can also be written as

$$c_t = 2^{-2t-1} \sum_{i=0}^{7} f(\beta_i)\, \text{card}[\mathbf{f}^{-t+1}(\beta_i)]. \tag{24}$$

Applying this procedure to rules listed in the appendix, we obtain expressions for c_t, as shown in Table 4. The following three kinds of c_t behavior can be observed in this table.

Rule	Approximate c_∞	Exact c_∞
7	0.469 ± 0.001	15/32
13	0.437 ± 0.001	7/16
27	0.531 ± 0.001	17/32
44	0.167 ± 0.001	1/6
78	0.562 ± 0.001	9/16
130	0.167 ± 0.001	1/6
162	0.333 ± 0.001	1/3
164	0.083 ± 0.001	1/12

Table 5. Rules for which exact values of asymptotic density can be found using n-step preimage counting.

1. c_t is constant, like in rule 4.
2. c_t oscillates and the asymptotic density is undefined, like in rule 5.
3. c_t converges exponentially to the final density like in rule 44, sometimes oscillating like in rule 13.

Note that no rule listed in Table 4 converges to the final density slower than exponentially. This is due to the fact that all rules for which we were able to conjecture exact expressions for the number of n-step preimages were either class 1 or class 2 rules according to Wolfram's classification. It is well known that some class 3 and class 4 rules (e.g., rule 18) exhibit power law relaxation to the final state, but we failed to find any patterns in their n-step preimage sequences, thus no expressions for c_t could be postulated.

However, even for "simple" rules like those listed in Table 4, our method yields some interesting results. For example, [8] lists asymptotic densities for all "minimal" elementary rules, but for many of them only experimental (i.e., computer simulation) values are given. For eight such rules we were able to find exact values of c_∞, simply by computing the limit of c_t as $t \to \infty$. These rules are presented in Table 5, along with experimental values of c_∞ quoted from [8]. We also verified some exact values of c_∞ given in [8]. For example, the density of nonzero sites for rule 132 is

$$c_t = \frac{1}{6} + \frac{2^{-2t}}{3}, \tag{25}$$

hence $c_\infty = 1/6$, not 1/8 as [8] suggests.

5. Conclusion and remarks

Some experimental results are presented regarding sequences of numbers of n-step preimages under elementary cellular automata (CA) rules.

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Many of these sequences exhibit apparent regularities, and the expressions for the general term of the sequence can be conjectured for 46 out of 88 "minimal" CA rules. Expressions obtained this way can be used to find asymptotic emulators of rules as well as the density of nonzero sites.

All rules discussed in this paper were either class 1 or class 2 according to Wolfram's classification. Sequences of preimage numbers for chaotic rules (except surjective rules) appear to be much more complex, and no patterns seem to appear. If any regularities exist, their detection will most certainly require computation of many more terms of the sequence, and a more efficient algorithm may be necessary. P. Grassberger proposed such an algorithm in [1], but even with his method going beyond $n = 20$ becomes impractical. Another method proposed in [2], called the statistical inverse iteration, is unfortunately only approximate, thus not very usable for the purpose of exact enumeration.

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Appendix

Table of preimage sequences

The following table shows the sequence numbers of n-step preimages for some elementary cellular automata rules. They are presented as a set of eight expressions (although not all of them are independent), each representing $a_n = \text{card}[\mathbf{f}^n(\beta_i)]$ for all eight basic blocks $i = 1 \ldots 7$. Only rules for which the author was able to conjecture all eight expressions are shown.

$$\text{Rule } 0: 32 \cdot 4^{n-1}, 0, 0, 0, 0, 0, 0, 0$$

$$\text{Rule } 1: \frac{-5 \cdot (-4)^n + 7 \cdot 4^n}{2}, 2 \cdot 4^{n-1}, \frac{-3 \cdot (-4)^n + 5 \cdot 4^n}{32}, \frac{3 \cdot (-4)^n + 11 \cdot 4^n}{32},$$
$$2 \cdot 4^{n-1}, 0, \frac{3 \cdot (-4)^n + 11 \cdot 4^n}{32}, \frac{77 \cdot (-4)^n + 85 \cdot 4^n}{32}$$

$$\text{Rule } 2: 20 \cdot 4^{n-1}, 4^n, 4^n, 0, 4^n, 0, 0, 0$$

$$\text{Rule } 3: \frac{-3 \cdot (-4)^n + 5 \cdot 4^n}{2}, 4^n, \frac{-(-4)^n + 3 \cdot 4^n}{8}, \frac{(-4)^n + 5 \cdot 4^n}{8}, 4^n,$$
$$0, \frac{(-4)^n + 5 \cdot 4^n}{8}, \frac{11 \cdot (-4)^n + 15 \cdot 4^n}{8}$$

$$\text{Rule } 4: 21 \cdot 4^{n-1}, 3 \cdot 4^{n-1}, 4^n, 0, 3 \cdot 4^{n-1}, 4^{n-1}, 0, 0$$

$$\text{Rule } 5: \frac{-6 \cdot (-4)^n + 9 \cdot 4^n}{4}, \frac{-3 \cdot (-4)^n + 9 \cdot 4^n}{16}, 5 \cdot 4^{n-1}, \frac{3 \cdot (-4)^n + 7 \cdot 4^n}{16},$$
$$\frac{-3 \cdot (-4)^n + 9 \cdot 4^n}{16}, \frac{3 \cdot (-4)^n + 9 \cdot 4^n}{8}, \frac{3 \cdot (-4)^n + 7 \cdot 4^n}{16}, \frac{9 \cdot (-4)^n + 11 \cdot 4^n}{8}$$

$$\text{Rule 7}: \frac{-3\cdot(-4)^n}{4}+\frac{3\cdot(-2)^n}{8}-\frac{17\cdot 2^n}{8}+\frac{13\cdot 4^n}{4}, \frac{-(-2)^n}{8}+\frac{3\cdot 2^n}{8}+\frac{4^n}{2}, 3\cdot 2^{n-1}, \frac{(-2)^n}{4}+\frac{3\cdot 2^n}{4}+\frac{4^n}{2}, \frac{-(-2)^n}{8}+\frac{3\cdot 2^n}{8}+\frac{4^n}{2}, \frac{3\cdot\left((-2)^n+5\cdot 2^n\right)}{8}, \frac{(-2)^n}{4}+\frac{3\cdot 2^n}{4}+\frac{4^n}{2}, \frac{3\cdot(-4)^n}{4}-(-2)^n-\frac{7\cdot 2^n}{2}+\frac{11\cdot 4^n}{4}$$

$$\text{Rule 8}: 8\cdot 4^n-12\mathrm{I}_n, 4\mathrm{I}_n, 4\mathrm{I}_n, 0, 4\mathrm{I}_n, 0, 0, 0$$

$$\text{Rule 10}: 12\cdot 4^{n-1}, 6\cdot 4^{n-1}, 4^n, 2\cdot 4^{n-1}, 6\cdot 4^{n-1}, 0, 2\cdot 4^{n-1}, 0$$

$$\text{Rule 12}: 10\cdot 4^{n-1}, 6\cdot 4^{n-1}, 8\cdot 4^{n-1}, 0, 6\cdot 4^{n-1}, 2\cdot 4^{n-1}, 0, 0$$

$$\text{Rule 13}: \frac{-3\cdot(-2)^n}{4}+\frac{7\cdot 2^n}{4}, \frac{-(-2)^n}{2}+4^n, \frac{-(-2)^n}{8}-\frac{21\cdot 2^n}{8}+\frac{7\cdot 4^n}{2}, \frac{3\cdot(-2)^n}{8}+\frac{7\cdot 2^n}{8}, \frac{-(-2)^n}{2}+4^n, \frac{3\cdot(-2)^n}{4}-\frac{7\cdot 2^n}{4}+\frac{5\cdot 4^n}{2}, \frac{3\cdot(-2)^n}{8}+\frac{7\cdot 2^n}{8}, \frac{3\cdot(-2)^n}{8}+\frac{7\cdot 2^n}{8}$$

$$\text{Rule 19}: 3\cdot(-4)^{n-1}+5\cdot 2^{2n-1}, 3\cdot 4^{n-1}, 0, 3\cdot 4^{n-1}+\mathrm{I}_n, 3\cdot 4^{n-1}, \mathrm{I}_n, 3\cdot 4^{n-1}+\mathrm{I}_n, 5\cdot 2^{2n-1}-3\cdot(-4)^{n-1}-3\mathrm{I}_n$$

$$\text{Rule 23}: -2+\frac{1+2\cdot 4^{n+1}}{3}, \frac{1+2\cdot 4^n}{3}, 1, \frac{1+2\cdot 4^n}{3}, \frac{1+2\cdot 4^n}{3}, 1, \frac{1+2\cdot 4^n}{3}, -2+\frac{1+2\cdot 4^{n+1}}{3}$$

$$\text{Rule 24}: 14\cdot 4^{n-1}-4\mathrm{I}_n, 6\cdot 4^{n-1}, 6\cdot 4^{n-1}+2\mathrm{I}_n, 0, 6\cdot 4^{n-1}, 2\mathrm{I}_n, 0, 0$$

$$\text{Rule 27}: (-4)^{n-1}+5\cdot 4^{n-1}, 4^n, 2\cdot 4^{n-1}, 4^n, 4^n, 2\cdot 4^{n-1}, 4^n, -(-4)^{n-1}+7\cdot 4^{n-1}$$

$$\text{Rule 28}: \frac{5\cdot 2^n}{2}, \frac{-(-4)^n}{6}+\frac{(-2)^n}{2}+\frac{5\cdot 2^n}{3}+\frac{4^n}{2}, \frac{-(-4)^n}{6}-(-2)^n-\frac{10\cdot 2^n}{3}+3\cdot 4^n, \frac{(-4)^n}{6}+\frac{(-2)^n}{2}+\frac{5\cdot 2^n}{6}+\frac{4^n}{2}, \frac{-(-4)^n}{6}+\frac{(-2)^n}{2}+\frac{5\cdot 2^n}{3}+\frac{4^n}{2}, \frac{(-4)^n}{6}-(-2)^n-\frac{25\cdot 2^n}{6}+3\cdot 4^n, \frac{(-4)^n}{6}+\frac{(-2)^n}{2}+\frac{5\cdot 2^n}{6}+\frac{4^n}{2}, 0$$

$$\text{Rule 29}: 3\cdot 4^{n-1}, 3\cdot 4^{n-1}, 7\cdot 4^{n-1}, 3\cdot 4^{n-1}, 3\cdot 4^{n-1}, 7\cdot 4^{n-1}, 3\cdot 4^{n-1}, 3\cdot 4^{n-1}$$

$$\text{Rule 32}: -11+32\cdot 4^{n-1}, 3, 4, 0, 3, 1, 0, 0$$

$$\text{Rule 34}: 10\cdot 4^{n-1}, 6\cdot 4^{n-1}, 8\cdot 4^{n-1}, 0, 6\cdot 4^{n-1}, 2\cdot 4^{n-1}, 0, 0$$

$$\text{Rule 36}: 26\cdot 4^{n-1}-10\mathrm{I}_n, 2\cdot 4^{n-1}+2\mathrm{I}_n, 2\cdot 4^{n-1}, 2\mathrm{I}_n, 2\cdot 4^{n-1}+2\mathrm{I}_n, 0, 2\mathrm{I}_n, 2\mathrm{I}_n$$

$$\text{Rule 38}: 47\cdot 4^{n-2}-\frac{11}{4}\mathrm{I}_n, 21\cdot 4^{n-2}-\frac{1}{4}\mathrm{I}_n, 3\cdot 4^{n-1}, 3\cdot 4^{n-1}, 21\cdot 4^{n-2}-\frac{1}{4}\mathrm{I}_n, 3\cdot 4^{n-2}+\frac{1}{4}\mathrm{I}_n, 3\cdot 4^{n-1}, 3\mathrm{I}_n$$

$$\text{Rule 40}: -9\cdot 2^n+32\cdot 4^{n-1}, 4\cdot 2^{n-1}, 4\cdot 2^{n-1}, 2^n, 4\cdot 2^{n-1}, 2^n, 2^n, 0$$

$$\text{Rule 42}: 7\cdot 4^{n-1}, 5\cdot 4^{n-1}, 4^n, 4^n, 5\cdot 4^{n-1}, 3\cdot 4^{n-1}, 4^n, 0$$

$$\text{Rule 44}: -7+4^{n+1}, \frac{1}{3}(4^{n+1}-1), \frac{1}{3}(4^{n+1}-1)-3+\mathrm{I}_n, 4-\mathrm{I}_n, \frac{1}{3}(4^{n+1}-1), 1, 4-\mathrm{I}_n, 2+\mathrm{I}_n$$

$$\text{Rule 46}: 38\cdot 4^{n-2}-7\cdot\mathrm{I}_n/2, 18\cdot 4^{n-2}-\mathrm{I}_n/2, 0, 6\cdot 4^{n-1}, 18\cdot 4^{n-2}-\mathrm{I}_n/2, 6\cdot 4^{n-2}+\mathrm{I}_n/2, 6\cdot 4^{n-1}, 4\mathrm{I}_n$$

$$\text{Rule 50}: 5, 3+\frac{2\cdot(-1+4^n)}{3}, \frac{8\cdot(-1+4^n)}{3}, \frac{2\cdot(-1+4^n)}{3}, 3+\frac{2\cdot(-1+4^n)}{3}, -3+\frac{8\cdot(-1+4^n)}{3}, \frac{2\cdot(-1+4^n)}{3}, 0$$

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$$\begin{aligned}
\text{Rule 72}: &\ 97\cdot 4^{n-2}-41\mathrm{I}_n/4, 7\cdot 4^{n-2}+9\mathrm{I}_n/4, 4\mathrm{I}_n, 2^{2n-1}, 7\cdot 4^{n-2}+9\mathrm{I}_n/4,\\
&\ 4^{n-2}+7\mathrm{I}_n/4, 2^{2n-1}, 0\\
\text{Rule 76}: &\ 5\cdot 4^{n-1}, 5\cdot 4^{n-1}, 8\cdot 4^{n-1}, 2\cdot 4^{n-1}, 5\cdot 4^{n-1}, 5\cdot 4^{n-1}, 2\cdot 4^{n-1}, 0\\
\text{Rule 77}: &\ 1, \frac{1+2\cdot 4^n}{3}, -2+\frac{1+2\cdot 4^{n+1}}{3}, \frac{1+2\cdot 4^n}{3}, \frac{1+2\cdot 4^n}{3},\\
&\ -2+\frac{1+2\cdot 4^{n+1}}{3}, \frac{1+2\cdot 4^n}{3}, 1\\
\text{Rule 78}: &\ 3\cdot 2^{n-1}, 3\cdot 2^{n-1}, 5\cdot 2^{2n-1}-9\cdot 2^{n-1}+4\mathrm{I}_n, 3\cdot 2^{n-1}+4^n-2\mathrm{I}_n, 3\cdot 2^{n-1},\\
&\ 7\cdot 2^{2n-1}-9\cdot 2^{n-1}+2\mathrm{I}_n, 3\cdot 2^{n-1}+4^n-2\mathrm{I}_n, 3\cdot 2^{n-1}-2\mathrm{I}_n\\
\text{Rule 108}: &\ 32\cdot 4^{n-2}-3\mathrm{I}_n, 26\cdot 4^{n-2}-3\mathrm{I}_n/2, 24\cdot 4^{n-2}-3\mathrm{I}_n, 6\cdot 4^{n-2}+3\mathrm{I}_n/2,\\
&\ 26\cdot 4^{n-2}-3\mathrm{I}_n/2, 4^{n-1}, 6\cdot 4^{n-2}+3\mathrm{I}_n/2, 4^{n-1}+6\mathrm{I}_n\\
\text{Rule 128}: &\ -8+32\cdot 4^{n-1}, 2, 1, 1, 2, 0, 1, 1\\
\text{Rule 130}: &\ -3+4^{n+1}, \frac{-1+4^{n+1}}{3}, \frac{-1+4^{n+1}}{3}, 1, \frac{-1+4^{n+1}}{3}, 1, 1, 1\\
\text{Rule 132}: &\ -4+\frac{17\cdot 4^n}{4}, \frac{2}{3}+\frac{13\cdot 4^n}{12}, \frac{-1+4^{n+1}}{3}, 1, \frac{2}{3}+\frac{13\cdot 4^n}{12}, 4^{n-1}, 1, 1\\
\text{Rule 136}: &\ -8\cdot 2^n+8\cdot 4^n, 4\cdot 2^{n-1}, 2^n, 2^n, 4\cdot 2^{n-1}, 0, 2^n, 2^n\\
\text{Rule 138}: &\ 8\cdot 4^{n-1}, 6\cdot 4^{n-1}, 3\cdot 4^{n-1}, 3\cdot 4^{n-1}, 6\cdot 4^{n-1}, 0, 3\cdot 4^{n-1}, 3\cdot 4^{n-1}\\
\text{Rule 140}: &\ -2\cdot 2^n+\frac{5\cdot 4^n}{2}, 6\cdot 4^{n-1}, -2^n+2\cdot 4^n, 2^n, 6\cdot 4^{n-1}, 2\cdot 4^{n-1},\\
&\ 2^n, 2^n\\
\text{Rule 156}: &\ 2^n, 2^n+\frac{4^n}{2}, -3\cdot 2^n+3\cdot 4^n, 2^n+\frac{4^n}{2}, 2^n+\frac{4^n}{2}, -3\cdot 2^n+3\cdot 4^n,\\
&\ 2^n+\frac{4^n}{2}, 2^n\\
\text{Rule 160}: &\ 3-10\cdot 2^n+8\cdot 4^n, -1+2^{n+1}, -3+4\cdot 2^n, 1, -1+2^{n+1}, -1+2^{n+1}, 1, 1\\
\text{Rule 162}: &\ \frac{-1+4^{n+1}}{3}, \frac{-1+4^{n+1}}{3}, -2+\frac{1+2\cdot 4^{n+1}}{3}, 1, \frac{-1+4^{n+1}}{3}, \frac{-1+4^{n+1}}{3}, 1, 1\\
\text{Rule 164}: &\ 4-9\cdot 2^n+6\cdot 4^n, -\frac{2}{3}+2^n+\frac{2\cdot 4^n}{3}, \frac{1+2\cdot 4^n}{3}, -1+2^{n+1},\\
&\ -\frac{2}{3}+2^n+\frac{2\cdot 4^n}{3}, 2^n, -1+2^{n+1}, -1+2^{n+1}\\
\text{Rule 168}: &\ -7\cdot 3^n+8\cdot 4^n, 3^n, 3^n, 3^n, 3^n, 3^n, 3^n, 3^n\\
\text{Rule 172}: &\ 20\cdot 4^{n-1}-\frac{-\left(1-\sqrt{5}\right)^{n+2}+\left(1+\sqrt{5}\right)^{n+2}}{8\cdot\sqrt{5}}-\frac{-\left(1-\sqrt{5}\right)^{n+3}+\left(1+\sqrt{5}\right)^{n+3}}{4\cdot\sqrt{5}},\\
&\ 4^n, 4^n-\frac{-\left(1-\sqrt{5}\right)^{n+1}+\left(1+\sqrt{5}\right)^{n+1}}{2\cdot\sqrt{5}},\\
&\ \frac{-\left(-\left(1-\sqrt{5}\right)^{n+2}+\left(1+\sqrt{5}\right)^{n+2}\right)}{8\cdot\sqrt{5}}+\frac{-\left(1-\sqrt{5}\right)^{n+3}+\left(1+\sqrt{5}\right)^{n+3}}{8\cdot\sqrt{5}},\\
&\ 4^n, \frac{-\left(1-\sqrt{5}\right)^{n+2}+\left(1+\sqrt{5}\right)^{n+2}}{8\cdot\sqrt{5}},
\end{aligned}$$

$$\frac{-\left(-\left(1-\sqrt{5}\right)^{n+2}+\left(1+\sqrt{5}\right)^{n+2}\right)}{8\cdot\sqrt{5}}+\frac{-\left(1-\sqrt{5}\right)^{n+3}+\left(1+\sqrt{5}\right)^{n+3}}{8\cdot\sqrt{5}},$$
$$\frac{-\left(1-\sqrt{5}\right)^{n+3}+\left(1+\sqrt{5}\right)^{n+3}}{8\cdot\sqrt{5}}$$

Rule 178 : $1, \frac{1+2\cdot 4^n}{3}, -2+\frac{1+2\cdot 4^{n+1}}{3}, \frac{1+2\cdot 4^n}{3}, \frac{1+2\cdot 4^n}{3},$
$-2+\frac{1+2\cdot 4^{n+1}}{3}, \frac{1+2\cdot 4^n}{3}, 1$

Rule 200 : $13\cdot 4^{n-1}, 3\cdot 4^{n-1}, 0, 4^n, 3\cdot 4^{n-1}, 4^{n-1}, 4^n, 4^n$

Rule 232 : $-2+\frac{1+2\cdot 4^{n+1}}{3}, \frac{1+2\cdot 4^n}{3}, 1, \frac{1+2\cdot 4^n}{3}, \frac{1+2\cdot 4^n}{3}, 1, \frac{1+2\cdot 4^n}{3},$
$-2+\frac{1+2\cdot 4^{n+1}}{3}$

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