Elementary Cellular Automata with Memory

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Standard cellular automata (CA) are ahistoric (memoryless), that is, the new state of a cell depends on the neighborhood configuration of only the preceding time step. This article introduces an extension to the standard framework of CA by considering automata with memory capabilities. While the update rules of the CA remain unaltered, a history of all past iterations is incorporated into each cell by a weighted mean of all its past states. The historic weighting is defined by a geometric series of coefficients based on a memory factor (α). A study is made of the effect of historic memory on the spatio-temporal and difference patterns of elementary (one-dimensional, two states, nearest neighbors) CA starting with states assigned at random.

1. Introduction

Cellular automata (CA) are discrete, spatially explicit extended dynamical systems. A CA system is composed of adjacent cells[1] or sites arranged as a regular d-dimensional lattice (d is in most cases 1 or 2), which evolves in discrete time steps. Each cell is characterized by an internal state whose value belongs to a finite set. The updating of these states is made simultaneously according to a common local rule involving only a neighborhood of each cell (e.g., [1] for a definition of CA).

CA are discrete *par excellence*, all three of their dimensions (space, time, and state) are discrete. This perfectly fits the features of digital computers, enabling exact computation. The synchronicity of the updating mechanism, the regular topologies, and the locality of interactions[2] make the CA paradigm ideally suited for parallel computers.

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[1]The *bricks* of an oversimplified microworld which do not try to emulate real particles as in Molecular Dynamics.

[2]A feature that relates CA with Markov Random Field models, used in spatial statistical modeling and analysis.

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CA are often described as a counterpart to partial differential equations [2], capable of describing continuous dynamical systems. Roughly speaking, two main levels are studied in a natural system, corresponding to the scale of observation: the microscopic and the macroscopic. Often, the complexity of the macroscopic world is apparently disconnected from that of the microscopic world, although the former is driven by the latter: the microscopic details are lost when the whole system is seen through a macroscopic filter. CA work at the microscopic level which drives the macroscopic behavior: the idea is not to try to describe a complex system from above—to describe it using difficult equations—but to simulate it by the interaction of cells following easy rules. These rules are often formulated in a natural language (with minimal or soft mathematical demands) and are easily translated into a computer programming language, as postulated by the Artificial Intelligence community (particularly when designing Expert Systems). In other words: not to describe a complex system with complex equations, but to let the complexity emerge by interaction of simple individuals following simple rules.

From the theoretical point of view, CA were introduced in the late 1940s by John von Neumann and Stanislaw Ulam. One can say that the *cellular* part comes from Ulam, and the *automata* part from von Neumann. But CA were shunted onto a dead track for a couple of decades [3] and did not reach the general public until 1970, when Martin Gardner published an account of John Conway's Game of "Life" in *Scientific American* [4, 5]. *Life* was destined to become the most famous CA and an inspiration to a generation of Artificial Life researchers.

A number of people at MIT began studying CA beyond *Life* during the 1970s. Probably the most influential figure there was Edward Fredkin, who around 1980 formed the *Information Mechanics Group at MIT* along with Tommaso Toffoli, Norman Margolus, and Gérard Vichniac. By 1984, Toffoli and Margolus had nearly perfected the CAM-6 CA machine, a special computer designed for the lightning-quick execution of CA, and were generating some publicity about it. In addition, in the middle 1980s (perhaps the golden age of CA), Stephen Wolfram was publishing numerous articles about CA [6] which hooked a number of researchers. In these articles, Wolfram suggested that many physical processes that seem random are in fact the deterministic outcome of computations that are simply so convoluted that they cannot be compressed into a shorter form and predicted in advance (he spoke of these computations as *incompressible*).

During the following years, CA were developed and used in many different fields. The requirements for the application of the CA approach to real problems (connecting different levels of detail) enlarged the basic paradigm, leading to systems related to CA (mere extensions in some cases) such as: inhomogeneous,[3] asynchronous[4] continuous state spaces,[5] Lattice-gas[6] or macroscopic[7] automata. Today, some authors use the more comprehensive term *grid-based* models [15] in order to be freed of the restrictions that the CA paradigm imposes.

A vast body of literature has been produced on these topics.[8] To a great extent, the papers on CA relate to many different, unrelated areas of physics.[9]

But the aim of this contribution is not to present a comprehensive review of the avatars of the CA approach but rather to enlarge the paradigm in a relatively unexplored direction: the consideration of all past states (history) in the application of the CA rules, as explained in section 2. To materialize this, we start from the simplest scenario: one-dimensional ($d = 1$) CA with two possible values ($k = 2$) at each site: $a \in \{0, 1\}$, with rules operating on nearest neighbors ($r = 1$). This is the scenario of [28], a landmark review paper largely responsible for the resurgence of interest in CA in the 1980s. The recent book by Wolfram [29] will receive special attention in the CA community (see two early comments by Casti [30] and Giles [31]).

Thus, this paper deals with *elementary rules* ($d = 1, k = 2, r = 1$), which following Wolfram's [28] notation, are characterized by a sequence of binary values (β) associated with each of the eight possible

[3]Or nonuniform: where different cells may be ruled by different transition functions. Nonuniform CA were investigated by Vichniac *et al.* [7] and Sipper [8]. Kauffman's [9] Random Boolean Networks allow arbitrary rules and connections which may be different at each site. Cellular Neural Networks [10] include CA as a special case.

[4]Or lattice Monte Carlo simulations: cells are not updated simultaneously [11].

[5]Such as Coupled Map Lattices (CML), in which time and space are discrete but state is continuous [12].

[6]In which the update is split into two parts: collision and propagation, intended to guarantee propagation of quantities while keeping the proper updating rules (collision) simple [13].

[7]State variables refer to macroscopic quantities; cell dimension is *larger* [14].

[8]The biennial *Conference on* CA *for Research and Industry* (ACRI, from its original Italian acronym) aims to present an international forum for researchers who are active in the CA field, as well as for those interested in evaluating the possibility of applying them in their own fields [16, 19]. Ilachanski [20] has recently reviewed the subject. His book contains an extensive bibliography and provides a listing of CA resources on the World Wide Web. Outstanding, the freeware DDLab [21] is a useful tool, in permanent upgrading, for studying both CA and Random Boolean Networks.

[9]The book by Chopard and Droz [22] might serve as an excellent textbook for physicists. The updated book by Ilachanski [20] is also intended mainly for a physicist audience. Some modern books on statistical mechanics; for example, Wilde and Singh [23], incorporate CA as a tool to study systems that are far from equilibrium. It has been argued that CA , intimately related to discrete statistical models, will play an important part elucidating basic ideas and general principles of statistical mechanics. Conversely, Rujan [24] also studies the usefulness of statistical physics methods to describe the properties of probabilistic CA. The statistical mechanics of probabilistic CA was studied by Lebowitz, *et al.* [25] and Alexander, *et al.* [26]. The book edited by Dieckmann, *et al.* [27] frames CA in the context of (ecological) spatial analysis.

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triplets: $(a_{i-1}^{(T}, a_i^{(T}, a_{i+1}^{(T})$

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ (\beta_1, & \beta_2, & \beta_3, & \beta_4, & \beta_5, & \beta_6, & \beta_7, & \beta_8)_{\text{binary}} \end{array} \equiv \sum_{i=1}^{8} \beta_i 2^{8-i}{}_{\text{decimal.}}$$

The rules are conveniently specified by a decimal integer, to be referred to as their *rule number*, R. The rule numbers for elementary CA range from 0 to 255. We will review in passing the recent literature invoking elementary rules.

Special attention will be paid to *legal rules* [28]: rules that are *reflection symmetric* (so that 100 and 001 as well as 110 and 011 yield identical values), and *quiescent*, that is, they do not transform a dead cell with dead neighbors into a live cell (their binary specification ends in 0). These restrictions leave 32 possible legal rules of the form: $\beta_1\beta_2\beta_3\beta_4\beta_2\beta_6\beta_4 0$. Another set of 32 categorized rules are those *reflection symmetric* but *nonquiescent*, with binary codes of the form: $\beta_1\beta_2\beta_3\beta_4\beta_2\beta_5\beta_4 1$ (the binary specification of any nonquiescent rule ends in a 1). There are 128 *semi-asymmetric* (either $\beta_2 \neq \beta_5$ or $\beta_4 \neq \beta_7$, but not both) rules and 64 *fully asymmetric* ($\beta_2 \neq \beta_5$ and $\beta_4 \neq \beta_7$) rules.

There are 88 equivalence classes, that is, 88 fundamentally inequivalent rules, formed under the negative, reflection, and negative + reflection transformations [32].

2. Memory

Standard CA are ahistoric (memoryless); that is, memory of only the previous iteration is taken into account to decide the next one. Thus, if $a_i^{(T}$ is taken to denote the value of cell i at time step T, the site values evolve by iteration of the mapping: $a_i^{(T+1} = \phi(\mathcal{N}(a_i^{(T}))$, where ϕ is an arbitrary function which specifies the CA rule operating on the neighborhood $\mathcal{N}$ of the cell i.

In this paper, a variation of the conventional one-dimensional CA is considered, featuring cells by their mean state in all the previous time steps. Thus, what is here proposed is to maintain the rules ϕ unaltered, but make them actuate over cells featured by their mean state: $a_i^{(T+1} = \phi(\mathcal{N}(f_i^{(T}))$, with $f_i^{(T}$ being the mean state of cell i after iteration T. We refer to these automata considering historic memory as *historic* (or $\mathbb{CAM}$ with Toffoli and Margolus' permission), and to the standard ones as *ahistoric*.

It should be emphasized that the memory mechanism considered here is different from that of other CA with memory reported in the literature. Typically, they determine the configuration at time $T+1$ in terms of the configurations at both time T and time $T-1$. Particularly interesting are the second order in time (memory of capacity two) reversible[10] rules due to Fredkin. These rules (referred to by Adamatzky in [41] as the class $\mathbb{MEMO}$), are of the form:

$$a_i^{(T+1} = \phi\big(\mathcal{N}(a_i^{(T)})\big) \oplus a_i^{(T-1},$$

with $\oplus$ denoting the exclusive OR (XOR) operation. As a natural extension of the memory mechanism here proposed, history can be incorporated into these rules in the form [42, 43]:

$$a_i^{(T+1} = \phi\big(\mathcal{N}(f_i^{(T)})\big) \oplus f_i^{(T-1}.$$

But to preserve the reversibility inherent in the XOR operation, the reversible mechanism incorporating memory must be of the form (see [44]):

$$a_i^{(T+1} = \phi\big(\mathcal{N}(f_i^{(T)})\big) \oplus a_i^{(T-1}.$$

Some authors; for example, Wolf-Gladrow [13], define rules with *memory* as those with dependence in ϕ on the state of the cell to be updated. So in the $r = 1$ scenario, rules with no memory take the form: $a_i^{(T+1} = \phi(a_{i-1}^{(T}, a_{i+1}^{(T})$. Rule 90: $a_i^{(T+1} = a_{i-1}^{(T} + a_{i+1}^{(T}$ mod 2, would be an example of a rule with no memory. Our use of the term "memory" is not this.

The effect of memory starting with a single-site seed was reported in [42] regarding one-dimensional CA and in [43] regarding totalistic two-dimensional CA. In an earlier work we analyzed the effect of memory in other CA scenarios: (*i*) Conway's Game of Life [45] and (*ii*) the spatial formulation of the Prisoner's Dilemma [46–50]. These studies become a starting point for the motivation of the present, projected in a more general scenario.

3. Full history at work: Some simple examples

The simplest way to take history into account is to feature cells by their most frequent state, thus not necessarily the last one. In case of a tie, the cell will be featured by its last state. Rule 254 (1111110) progresses as fast as possible (i.e., at the *speed of light*): it assigns a live state to any cell in whose neighborhood there would be at least one live cell. Thus, in the ahistoric model, a single-site live cell will grow

[10]Reversible systems are of interest since they preserve information and energy and allow unambiguous backtracking [33]. They are studied in computer science in order to design computers which would consume less energy [34]. Reversibility is also an important issue in physics [35–38]. Gerald 't Hoof in a speculative paper [39], suggests that a suitably defined deterministic, local reversible CA might provide a viable formalism for constructing field theories at Planck scale. Svozil [40] also asks for changes in the underlying assumptions of current field theories to make their discretizations appear more CA-like.

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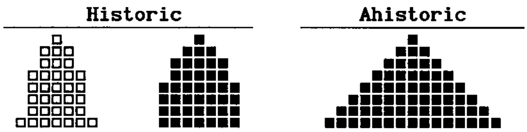

Table 1(a). Rule 254 (11111110) starting with a single-site live cell until $T = 8$. Historic and ahistoric models. Live cells: ■ last, □ most frequent.

Table 1(b). Rule 90 (01011010) starting with a single-site live cell until $T = 4$. Historic and ahistoric models. Live cells: ■ last, □ most frequent.

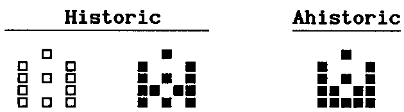

Table 1(c). Rule 233 (11101001) starting with a single-site live cell until $T = 5$. Historic and ahistoric models. Live cells: ■ last, □ most frequent.

monotonically, generating segments whose size increases two units with every time step. The dynamics are slower in the historic model (see Table 1(a)): the outer live cells are not featured as live cells until the number of times they "live" is equal to the number of times they were "dead." Then the automaton fires two new outer live cells.

As long as the local rules of both historic and ahistoric automata remain unaltered, and both the latest and most frequent states coincide after the first two iterations, the historic and ahistoric evolution patterns are the same until $T = 3$. But after the third iteration, the last and the most frequent states often differ, and consequently the patterns for the historic and ahistoric automata typically diverge at $T = 4$.

This is the case in Table 1(a) and in Table 1(b) with rule 90 (01011010), in this case, after $T = 3$ the most frequent state for all the cells is the dead one.

Occasionally the last and the most frequent status may coincide after the third iteration. In this case the historic and ahistoric configurations will still be the same at $T = 4$. This is the case for Rule 233 (see Table 1(c)). But the coincidence in the dynamics of the historic and ahistoric Rule 233 automaton ceases at $T = 5$: the latest and the historic (most frequent) configurations do differ after $T = 4$, as seen in the patterns generated from $T = 5$.

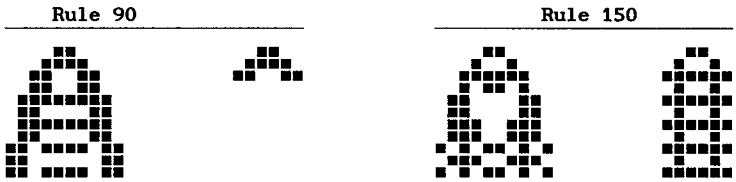

Table 2. Rules 90 and 150 are not additive in the historic formulation. The true evolution pattern starting with two adjacent live cells is shown on the left. The XOR superposed configuration of those evolved independently starting with a single seed is shown on the right. Evolution up to $T = 11$.

In the standard (ahistoric) scenario, Rules 90 and 150 (together with the trivial 0 and 204) are the only *additive legal rules*; that is, any initial pattern can be decomposed into the superposition of patterns from a single-site seed. Each of these configurations can be evolved independently and the results superposed (modulo two) to obtain the final complete pattern. As illustrated in Table 2 starting with two adjacent live cells, additivity of Rules 90 and 150 is lost in the historic model.

Every configuration in a standard (ahistoric) CA has a unique successor in time (it may however have several distinct predecessors). A (last) configuration in the historic scenario may have multiple successors: the transition rules operate on the most frequent state (MFS) configuration, not on the last, so the successor of a given configuration depends on the underlying MFS configuration. A primary example of this is given with Rule 254 (Table 1(a)): in the fully historic model, live configurations reproduce themselves for several iterations (■■■■■ for example, twice) but at a critical time step ($T = 5$ for the example), generate a new (■■■■■■■) one. We envisage that this will notably alter the appearance of the state transition diagrams in the historic scenario as compared to those of ahistoric CA [32]. The irreversibility (a feature necessary for the formation of attractors) of historic rules in Figure 1(a), often leads to the generation of oscillators that appear fairly soon. Thus, the number of different configurations generated by the evolution of a single-site seed is smaller in the historic model. This, generalized to any initial configuration, would lead to conjecture the existence of a higher number of "Garden-of-Eden" nodes (configurations for an automaton which could only exist initially) in the historic scenario.

4. Discounting

Historic memory can be weighted by applying a geometric discounting process in which the state $a_i^{(T-\tau}$ obtained τ time steps before the last round is actualized to the value $\alpha^{\tau} a_i^{(T-\tau}$, with α being the *memory factor*. A given cell will be featured by the rounded weighted mean of all its

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past states. This well-known mechanism fully ponders the last round ($\alpha^0 = 1$), and tends to forget the older rounds.

After time step T, the weighted mean of the states of a given cell will be:

$$m_i^{(T}\left(a_i^{(1}, a_i^{(2}, \ldots, a_i^{(T)}\right) = \frac{\sum_{t=1}^{T} \alpha^{T-t} a_i^{(t}}{\sum_{t=1}^{T} \alpha^{T-t}}.$$

Provided that states a are coded as 0 (dead) and 1 (live), the rounded weighted mean state f will be obtained by comparing its weighted mean m to 0.5, so that:

$$f_i^{(T} = \begin{cases} 1 & \text{if} \quad m_i^{(T} > 0.5 \\ a_i^{(T} & \text{if} \quad m_i^{(T} = 0.5 \\ 0 & \text{if} \quad m_i^{(T} < 0.5. \end{cases}$$

This is equivalent to studying the sign of:

$$\partial_i^{(T}(a_i^{(1}, a_i^{(2}, \ldots, a_i^{(T)}) = 2\sum_{t=1}^{T} \alpha^{T-t} a_i^{(t} - \sum_{t=1}^{T} \alpha^{T-t} = \sum_{t=1}^{T} (2a_i^{(t} - 1)\alpha^{T-t}$$

$$f_i^{(T} = \begin{cases} 1 & \text{if} \quad \partial_i^{T} > 0 \\ a_i^{(T} & \text{if} \quad \partial_i^{T} = 0 \\ 0 & \text{if} \quad \partial_i^{T} < 0. \end{cases}$$

After $T = 3$: $\partial(a_i^{(1}, a_i^{(2}, a_i^{(3}) = \alpha^2(2a_i^{(1} - 1) + \alpha(2a_i^{(2} - 1) + 2a_i^{(3} - 1$. Of course, if $a_i^{(1} = a_i^{(2} = a_i^{(3}$, history does not alter the series and it will be $f_i^{(3} = a_i^{(3}$. Neither does history take effect until $T = 3$ if $a_i^{(2} = a_i^{(3}$ nor if $a_i^{(1} = a_i^{(3}$ (see [42]). But the scenario may change if $a_i^{(1} = a_i^{(2} \neq a_i^{(3}$:

$$\partial(0,0,1) = -\partial(1,1,0) = -\alpha^2 - \alpha + 1 \Leftrightarrow \partial$$
$$= 0 \Rightarrow \alpha = \frac{-1+\sqrt{5}}{2} = 0.61805 \equiv \alpha_0.$$

Thus, provided that $\alpha > \alpha_0$, cells with state history 001 or 110 will be featured after $T = 3$ as 0 and 1 respectively instead of as 1 or 0 (last states). Rule 90 starting with a single live cell, as in Table 1(b), provides an excellent example of this: for $\alpha > \alpha_0$ after $T = 3$ all the cells, even the two alive at $T = 3$ but dead at $T = 1$ and $T = 2$, are featured as dead, so that the annihilation of the live pattern happens as early as $T = 4$. But for $\alpha \leq \alpha_0$, the two cells alive at $T = 3$ are featured as live and will regenerate the pattern at $T = 4$ as shown in Table 1(b) for the ahistoric model.

After $T = 4$: $\partial(0,0,0,1) = -\partial(1,1,1,0) = -\alpha^3 - \alpha^2 - \alpha + 1 \Leftrightarrow \partial = 0 \Rightarrow \alpha = 0.5437$.

In general, in the most unbalanced scenario, if $a_i^{(1} = a_i^{(2} = \cdots = a_i^{(T-1} \neq a_i^{(T}$ it holds that:

$$\partial(0,0,\ldots,0,1) = -\partial(1,1,\ldots,1,0) = -\sum_{i=1}^{T-1} \alpha^i + 1$$
$$= -\frac{\alpha^T - \alpha}{\alpha - 1} + 1 \Rightarrow \partial = 0 \Leftrightarrow \alpha^T - 2\alpha + 1 = 0.$$

When T→ ∞, it is:

$$\lim_{T\to\infty} \partial(0,0,\ldots,0,1) = -\lim_{T\to\infty} \partial(1,1,\ldots,1,0)$$
$$= -\frac{\alpha}{1-\alpha} + 1 \Rightarrow \partial = 0 \Leftrightarrow \alpha = 0.5.$$

It is then concluded that memory does not affect the scenario if $\alpha \leq 0.5$. Thus, the value 0.5 for the memory factor becomes a bifurcation point that marks the transition to the ahistoric scenario.

Computationally, there is a saving if instead of calculating $\partial = 2\omega - \Omega$ for every cell, we calculate

$$\omega(a_i, T) = \sum_{t=1}^{T} \alpha^{T-t} a_i^{(t}$$

all across the lattice and compare the ω figures to the factor

$$\frac{1}{2}\Omega(T) = \frac{1}{2}\sum_{t=1}^{T} \alpha^{T-t} = \frac{1}{2}\frac{\alpha^T - 1}{\alpha - 1}.$$

This was done in the computer program written to perform the calculus and generate the figures.

5. Spatio-temporal patterns

Most of the legal rules are either unaffected (e.g., simple rules [28]) or minimally affected (e.g., Rules 250, 254, and 36) by memory, starting from a random initial configuration. In this scenario, only the nine legal rules which generate nonperiodic patterns in the ahistoric scenario [51, 52] are significantly affected by memory. Figure 1(a) shows the evolutive patterns of eight of these rules starting with a random initial configuration with uncorrelated values taken to be 0 or 1 with probability $p(a = 1) = 0.5$; the memory factor varied from 1.0 to 0.5 by 0.1 intervals. The lattice size is $N = 90$ and evolution is shown up to 90 time steps.

Rule 18 was one of the first rules carefully analyzed. History has a dramatic effect on Rule 18. Even at the low value of $\alpha = 0.6$, the appearance of the spatio-temporal pattern fully changes: a number of isolated periodic structures are generated, which is far from the distinctive inverted triangles world of the ahistoric pattern. For $\alpha = 0.7$, the live structures are fewer, advancing the extinction found in [0.8,0.9]. In the fully historic model, only a periodic pattern of live cells, appearing twice, survives.

Rule 146 is affected by memory in much the same way as Rule 18. This is because although their rule numbers are relatively distant, their binary configurations differ only in their β_1 value. The spatio-temporal

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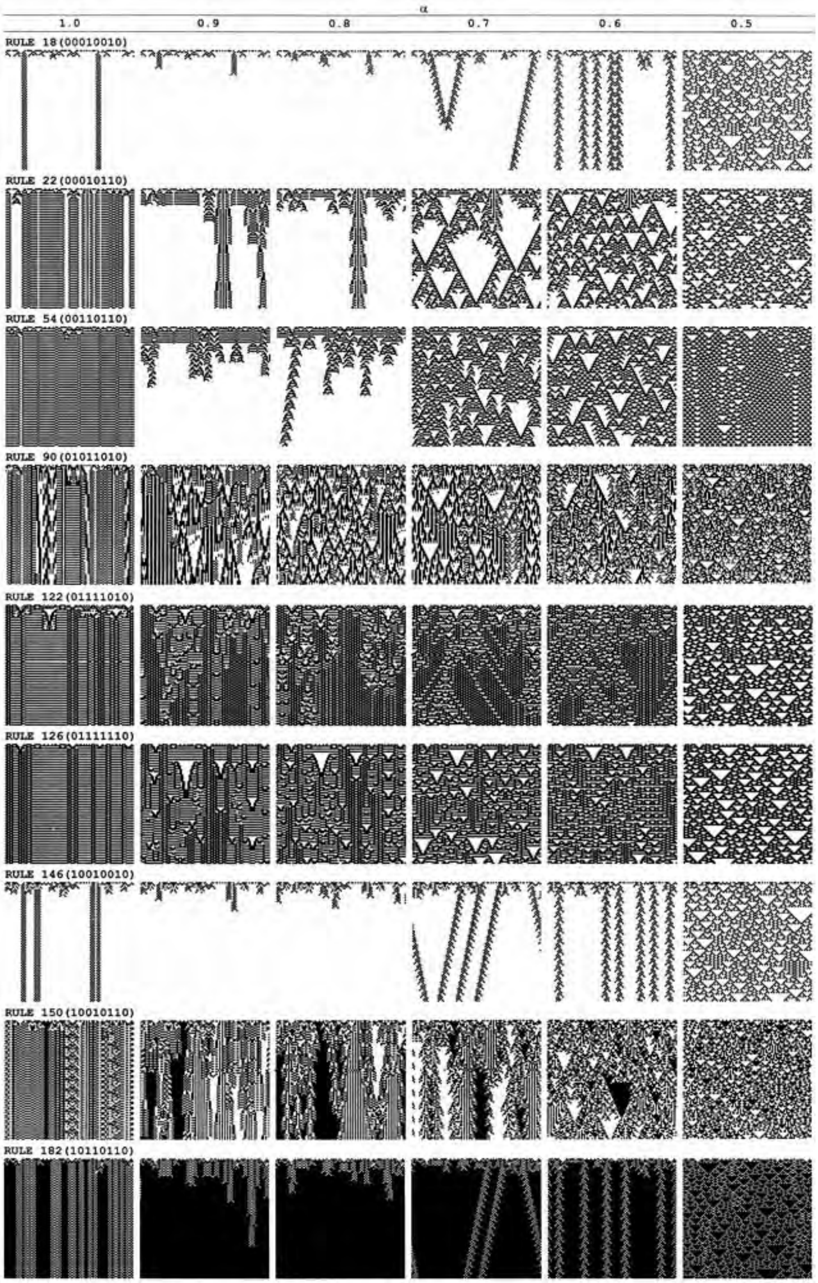

Figure 1(a). Evolution of legal $(\beta_1\beta_2\beta_3\beta_4\beta_2\beta_6\beta_4 0)$ rules significantly affected by memory. The values of sites in the initial configuration are chosen at random to be 0 (*blank*) or 1 (■) with probability 1/2. The pictures show the evolution of CA with 97 sites for 90 time steps. Periodic boundary conditions are imposed on the edges. The evolution of the CA at successive time steps is shown on successive horizontal lines. Memory factor α. Ahistoric model for $\alpha \leq 0.5$, fully historic model for $\alpha = 1.0$ with $k = 2$, $r = 1$.

patterns of Rules 182 and 146 resemble each other, though those of Rule 182 look like negative photograms ($a \to 1-a$) of those of Rule 146.

The effect of memory on Rules 22 and 54 is similar. Their spatio-temporal patterns in $\alpha = 0.6$ and $\alpha = 0.7$ keep the essential feature of the ahistoric, although the inverted triangles become enlarged and tend to be more sophisticated in their basis. A notable discontinuity is found for both rules ascending in the value of the memory factor: in $\alpha = 0.8$ only a periodic structure survives for both rules; in $\alpha = 0.9$ two live structures seem to survive for Rule 22; extinction of live cells is found for Rule 54. But unexpectedly, the patterns of the fully historic scenario differ markedly from the others, showing a high degree of synchronization. The behavior of Rule 54 in the ahistoric model has been featured to some extent as transitional between very simple Wolfram's Class I and II rules and chaotic Class III. Thus, Rule 54 appears among the two one-dimensional rules (with Rule 110) that seem to belong to Wolfram's complex Class IV.

The four remaining chaotic legal rules (90, 122, 126, and 150) show a much smoother evolution from the ahistoric to the historic scenario: no pattern evolves either to full extinction or to the preservation of only a few isolated persistent propagating structures (solitons). Rules 122 and 126 (close in terms of rule number), evolve in a similar form (particularly when comparing the ahistoric and fully historic patterns), showing a high degree of synchronization in the fully historic model.

Some spatio-temporal patterns of rules in the ahistoric model are reminiscent of the patterns of pigmentation observed on the shells of certain mollusks [29, 53]. Wolfram [6] illustrates this idea by showing a natural cone shell with a pigmentation intended to be reminiscent of the pattern generated by Rule 90 in the ahistoric scenario. But the clearings in the shell would suggest Rule 22 for some value of α between 0.6 and 0.7: with $\alpha = 0.6$ the clearings seem to be scarce, with $\alpha = 0.7$ the clearings appear to be excessive.

Figure 1(b) shows the evolution of the average fraction of sites with value 1 at time T, density ρ_T, up to 200 time steps, starting with an initial density $\rho_0 = 0.5$. The simulation is implemented for the same rules as in Figure 1(a), but with a notably wider lattice of $N = 401$. A visual inspection of the plots in Figure 1(b), ratifies the general features observed in the patterns in Figure 1(a).

Thus, starting with a disordered configuration of any nonzero density, the evolution of the density ρ_T according to Rule 18 in the ahistoric model yields an asymptotic density $\rho_\infty = 1/4$. Figure 1(b) illustrates how this value is reached soon, and how history induces a depletion of the asymptotic density, null for $\alpha = 0.8$ and $\alpha = 0.9$. In the fully historic model, a smooth period two density oscillator ($0.105 \to 0.115$) is generated as early as $T = 8$.

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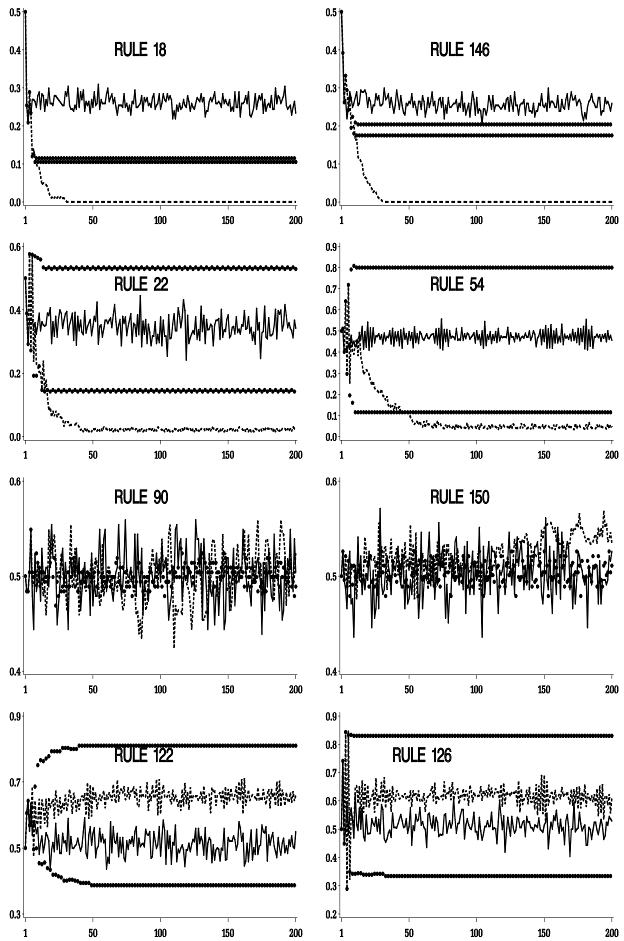

Figure 1(b). Evolution of the average fraction of sites with value 1 at time step T, density ρ_T, starting with an initial density $\rho_0 = 0.5$, up to 200 time steps. Results concerned with the same rules of Figure 1(a) are shown when operating in a lattice with size $N = 401$. Note that in order to improve plot resolution, y-axes scopes have been tailored. Plots code: unjoined dots → fully historic model, dashed line → $\alpha = 0.8$, continuous line → ahistoric model.

Rule 146 density plots resemble those of Rule 18, but with a sharper period two oscillator generated in the fully historic model ($0.175 \rightarrow 0.204$). Rule 182 (not shown in Figure 1(b)) yields $\rho_\infty = 3/4$ in the ahistoric model, the shape of its evolution density curves resembling the complement to 1 of its equivalent Rule 146.

Wolfram [28] proved that for Rule 90, $\rho_\infty = 1/2$, independent of the initial density ρ_0 (so long as $\rho_0 \neq 0$). When historic memory is taken into account, ρ_T varies erratically around 0.5 as shown in Figure 1(b) for Rules 90 and 150, without either periodic evolution or tendency to a fixed point. Not even in the fully historic model.

Quoting Grassberger [51], Rule 22 is the only one of the elementary rules whose long-time behavior is not yet understood. Wolfram [28] resorts to simulations to report $\rho_\infty = 0.35 \pm 0,02$ for evolution with Rule 22. Rule 54, very notably absent in Wolfram's [28] considerations, again resembles Rule 22 with regard to Figure 1(b). For example, both rules present a very low density in the [0.8,0.9] interval, null for $\alpha = 0.9$ in Rule 54.

Wolfram [6, Table 6] reports $\rho_\infty = 1/2$ for Rules 122 and 126. In our simulation, ρ_T oscillates around values slightly over 1/2 in the ahistoric model. Figure 1(b) makes apparent how the effect of memory differs in both groups of rules: memory sharply depletes the density curves of Rules 18 and 146, but leads them over the starting 0.5 value for Rules 122 and 126. In the fully historic model, these rules do not lead to low density values, but to synchronous behavior.

The high degree of synchronization visually appreciated for Rules 22, 54, 122, and 126 in the fully historic model in Figure 1(a), stands out also in Figure 1(b) (unjoined dot plots). Synchronization[11] in CA is not a trivial task since synchronous oscillation is a global property, whereas CA typically employ only local interactions; but the phenomenon of synchronous oscillations occurs in nature in fairly striking forms.[12]

6. Difference patterns

Figure 2 shows the difference patterns (DP) produced by evolution from the disordered initial configuration taken in Figure 1(a) resulting from changing the value of its initial center site value. The pictures show the *damaged* region as black squares corresponding to the site values that differed among the patterns generated with the two initial configurations.

The perturbations in proper chaotic rules propagate to the right and left at a single (maximum) velocity at any time. This behavior illustrates the *butterfly effect*: a small perturbation grows, and finally rules the whole system. The velocity in the *damage spreading* is quantified by means of the left and right Lyapunov exponents (λ_L, λ_R) which measure the rate at which perturbations spread to the left and right, and are given by the slopes of the left and right boundary of the growth of the DP. Thus, zero values indicate periodicity, whereas negative velocity indicates perturbation repair. The chaotic Class III rules in Figure 2

[11] Given any initial configuration, the CA must reach a final configuration, within a finite number of time steps, that oscillates between all zeros and all ones at successive time steps.

[12] The synchronization task has been investigated by Das, *et al.* [54] and Sipper [8], who conclude that Rules 21 and 22 are the best performing synchronizer rules.

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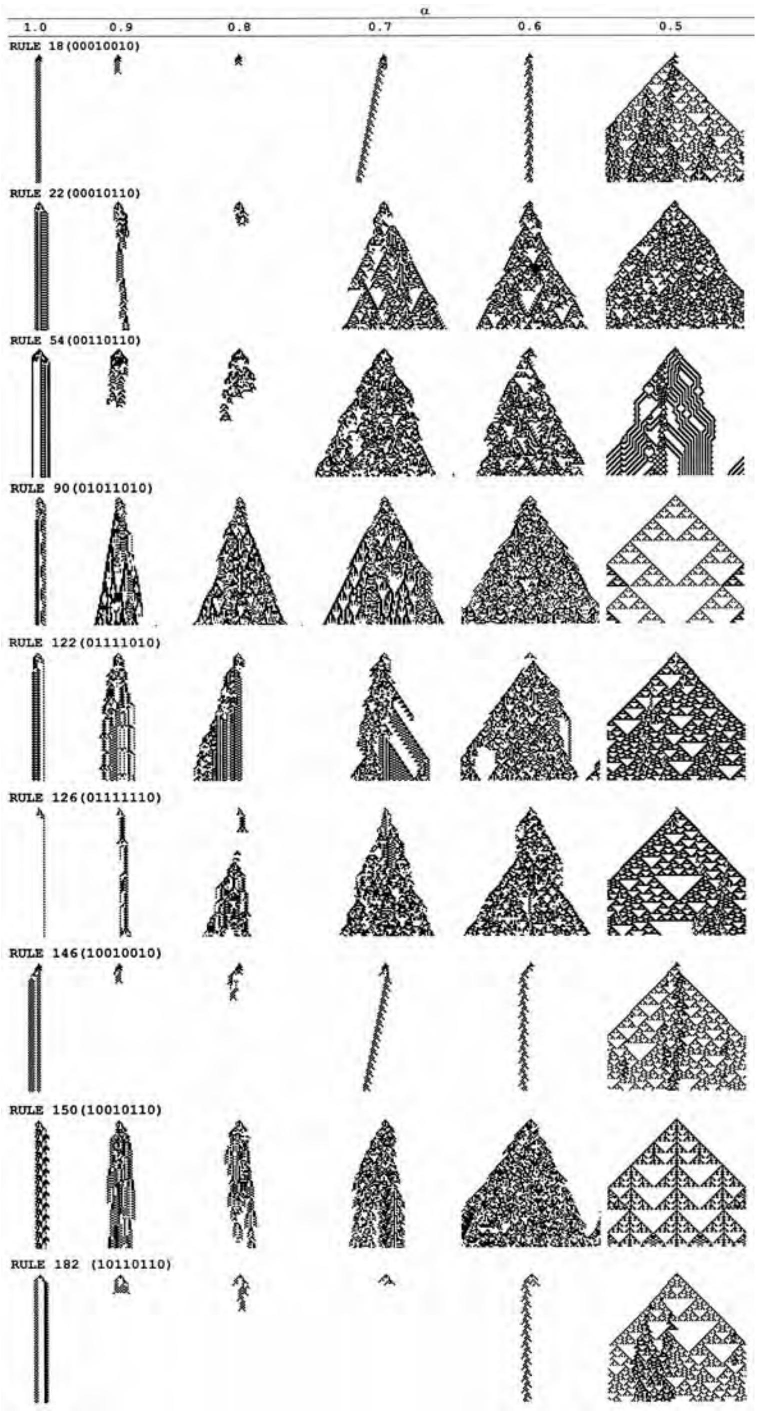

Figure 2. DP produced by evolution from the disordered initial configuration taken in Figure 1(a) and that resulting from reversing its center site value. The pictures show as black squares the site values that differed. Evolution up to 90 time steps. Memory factor α. Ahistoric model for $\alpha \leq 0.5$, fully historic model for $\alpha = 1.0$ with $k = 2$, $r = 1$.

reach the maximum λ attainable when $r = 1, \lambda = 1$. The Lyapunov exponents of ahistoric elementary CA are tabulated in [6].

As a rule, the effect of memory on the DP mimics that of spatio-temporal patterns, so that the rule parallelisms found for the spatio-temporal patterns are again applicable to the DP.

Rule 18 may serve as an example: a periodic structure (with only two elements) appears in the fully historic scenario; the differences die out when $\alpha \in [0.8, 0.9]$, the perturbation remains localized (again in the form of periodic structures) when $\alpha \in [0.6, 0.7]$. Finally, in the ahistoric model, the perturbation grows at the speed of light: $\lambda_L = \lambda_R = 1$. The pattern shown in Figure 2 conforms to this property, although the picture appears delimited in growth. This is because the lattice size $N = 97$ is not large enough for free expansion during $T = 90$ time steps (which would be feasible with $N = 181$; recall that with $r = 1$, after T time steps, an initial sole mutation may affect the values of at most $2T$ sites). This distortion operates on every ahistoric pattern in Figure 2 and is found, to a lesser degree in some rules and in $\alpha > 0.5$ values. Examples are Rule 90, 122, and 150 for $\alpha = 0.9$. The DP of Rule 146 resembles that of Rule 18, and evolves in a similar way. In turn, the DP of Rule 182, fully resembles that of its equivalent Rule 146. The DP of Rules 22 and 54 are again similar:[13] the DP is constrained when history begins to actuate [0.6,0.7], and ceases, or is localized near the central site, at higher α values. The group of Rules 90, 122, 126, and 150 show a fairly gradual evolution from the ahistoric to the historic scenario, so that the DP appear more constrained as more historic memory is retained, with no extinction for any α value.

The patterns with inverted triangles dominate the scene in the ahistoric DP in Figure 2 (the exception is the peculiar DP of Rule 54), but history destroys this common appearance (and that of Rule 54), even at the lowest value of α with memory effect in the figure: $\alpha = 0.6$. Thus, there is a sort of discontinuity implied in the consideration of historic memory (perhaps with the exception of Rule 22) regarding the DP, which Rule 90 might exemplify: memory, at the low rate $\alpha = 0.6$, destroys the structures characteristic of ahistoric DP. To avoid coined terms such as "chaotic" or "random," the DP generated for $\alpha = 0.6$ could be described as "helter-skelter." Regarding the central site, for Rules 22, 54, 90, 122, 126, and 150, the disruption induced by its initial reversion, is in some manner more unpredictable in the historic model with $\alpha = 0.6$ than in the ahistoric. Extreme examples are: Rule 90 (after initial reversion, the original evolution is restored in the ahistoric model) and Rule 150 (the initial reversion remains forever).

[13]This coincidence in behavior agrees with their complexity: they are the only legal rules with no extreme Lyapunov exponents: $\lambda_L = \lambda_R \cong 0.75$ for Rule 22 and $\lambda_L = \lambda_R \cong 0.55$ for Rule 54 (see Table 6 in [6]).

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The conclusions drawn from Figure 2 are supported by the $N = 401$ simulations run for Figure 1(b). Again, the central site has been reversed and two types of plots are implemented to feature the DP: (*i*) the Hamming distance ($\mathcal{H}$, the number of nonzero site values of the DP), and (*ii*) the width of these patterns. From these plots (not included here to avoid graphical overloading) the overall conclusion drawn from Figure 2 remains valid: memory implies a depletion in the damaged region (*i*) and in the speed of propagation of perturbation (*ii*).

In order to systematize the analysis of the DP, one can resort to equivalence classes. Memory is expected to affect all the rules of an equivalence class in a similar way.

This was already observed in the legal class {146, 182}. Figure 3 shows the DP of the remaining rules equivalent to the legal ones. A fairly consistent correspondence in the appearence of the DP is observed between equivalent rules: {18, 183}, {22, 151}, {54, 147}, {90, 165}, {122, 161}, and {126, 129}. The DP of Rule 150 resembles that of its complementary Rule 105. This is so despite the fact that Rules 150 and 105 do no belong to the same equivalence class (some classes are formed by a single rule, as in the case of Rule 150 [55]).

We have also explored the effect of memory in asymmetric rules. Figure 4 shows the DP of some such rules affected by memory in the scenario stated in Figure 2(a).

Rules in which almost all changes in initial configuration die out, and rules with $\lambda_{L,R} = 0$ are not greatly affected by memory (e.g., Rules 156 and 100). A considerable number of equivalence classes of asymmetric rules have $|\lambda_L| = |\lambda_R| = \lambda$ with $\lambda = \pm 1$ or $\lambda = -1/2$ as their minimal representative rule. These rules present a diagonal as DP in the ahistoric model, which is rectified (i.e., both Lyapunov exponents evolve to zero) and/or led to extinction by memory. But either extinction or rectification of the trajectory of the perturbation are not always achieved in a uniform way. Numerous examples of unexpected evolution have been found in this context.

The important Rule 110[14] and the others of its equivalence class, may serve as a paradigmatic example of the expected effect of memory: the damage induced by the reversal of the initial central site value becomes more constrained as the memory factor increases, with no discontinuities in the preserving effect. The same applies to all the rules of the equivalence classes of the three that have one irrational Lyapunov exponent:[15] Rules 30,[16] 45, and 106.

[14]This rule shows highly complex properties of information transmission, associated with particle-like structures [56].

[15]In Rules 22 and 54 (Figure 2(a)) both the Lyapunov exponents are irrational.

[16]Wolfram uses Rule 30 for random sequence generation in [57].

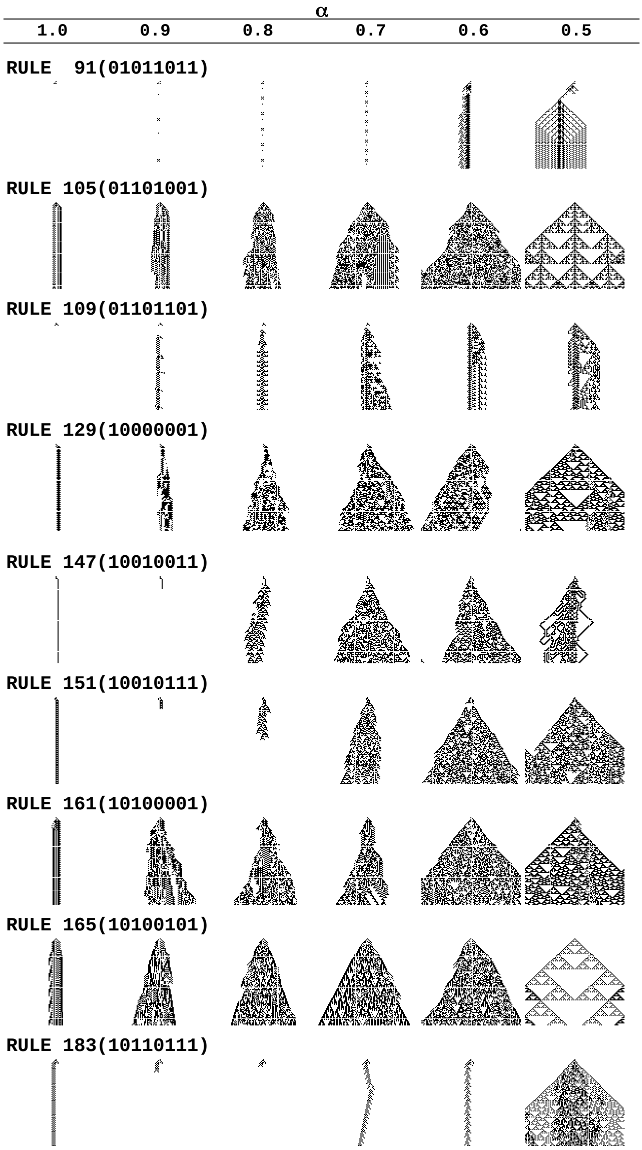


Figure 3. DP of reflection symmetric but nonquiescent rules $(\beta_1\beta_2\beta_3\beta_4\beta_2\beta_6\beta_4 1)$ affected by memory in the scenario of Figure 2(a).

History however has an unexpected effect on most of the rules whose damage propagation direction alternates: $\lambda_L = (-1, 1)$, $\lambda_R = (1, -1)$. Figure 4 shows the case of fully asymmetric Rules 43, 57, and 184. Particularly curious are the DP generated for $\alpha > 0.6$ in Rules $\{43, 113\}$. The DP of Rules $\{184, 226\}$ are rectified when history is taken into account (that of Rule 184 becomes extinct for α in [0.6,0.8]). Rules 184 and 226 have proved particularly effective in solving the density problem

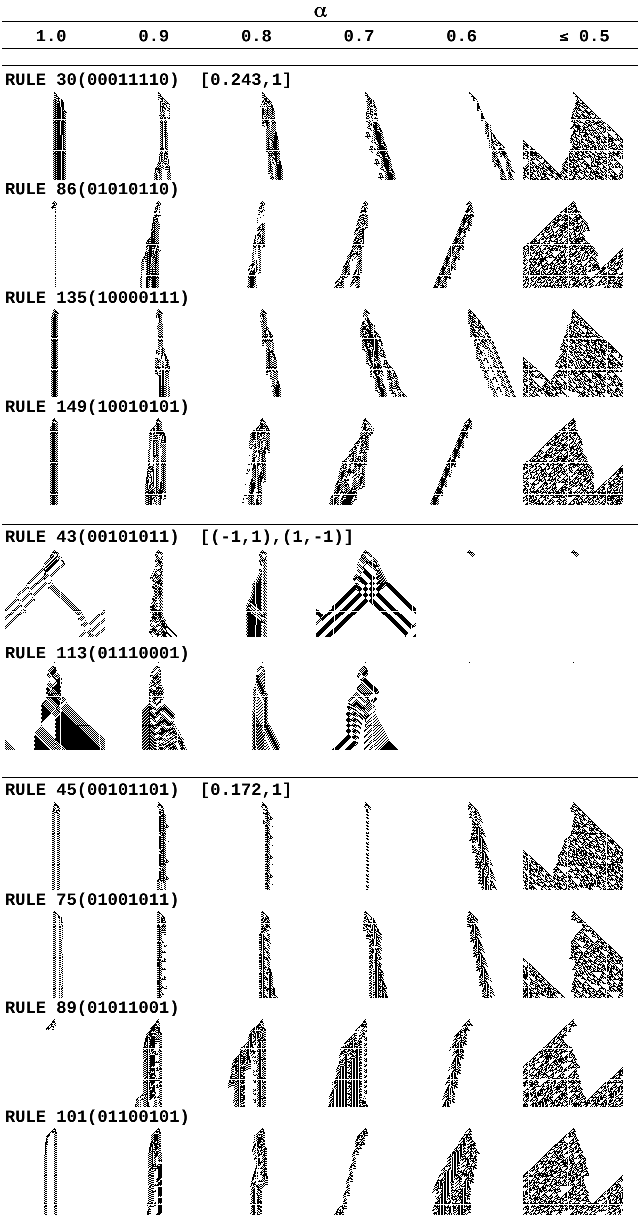


Figure 4. DP of some asymmetric rules affected by memory in the scenario of Figure 2. The rules are grouped by equivalence classes. The left and right Lyapunov exponents of the lowest rule number of each class (minimal representative) in the ahistoric model are given after rule codes.

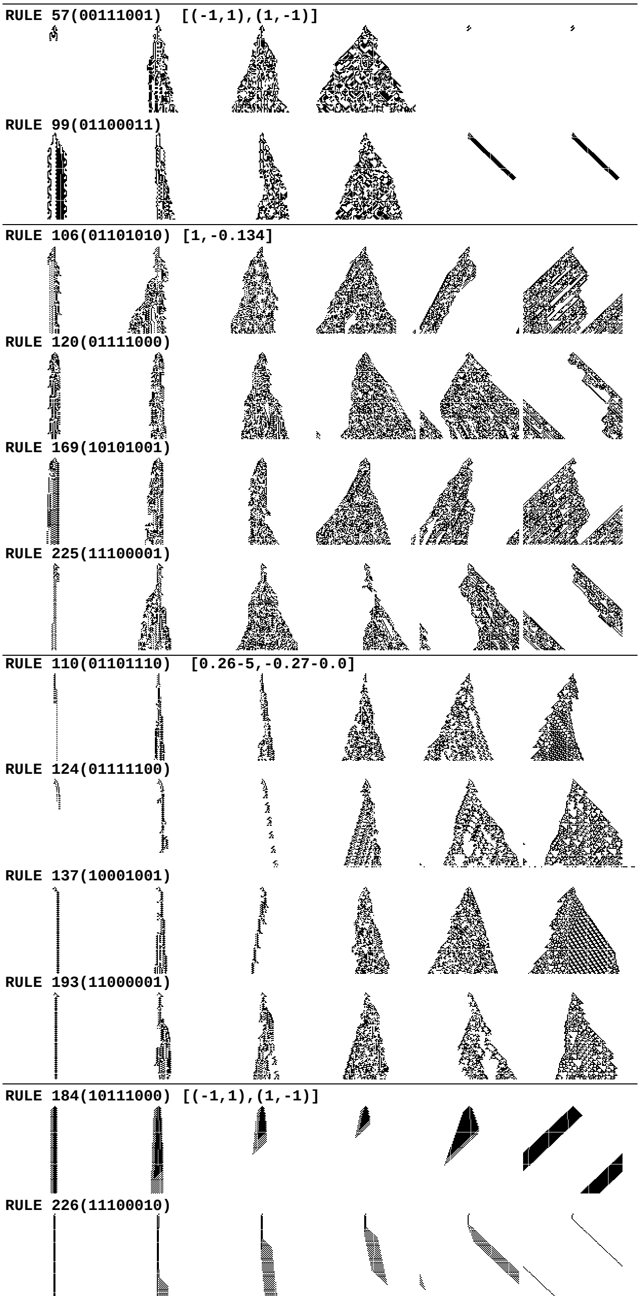


Figure 4. *(continued)*

[58]: to decide whether an arbitrary initial configuration contains a density of ones above or below ρ_c. As in synchronization, the density task comprises a nontrivial computation for small-radius CA: again, density is a global property of a configuration, where small-radius CA rely solely on local interactions.

7. Other memories

This work deals only with the memory mechanism described in section 4. Nevertheless, some alternative memory mechanisms are reported in this section. We expect to feature them in a subsequent work.

7.1 Exponential weighting

The decay in the weighting factor can be implemented in an exponential way instead of in the geometric one:

$$m_i^{(T}\left(a_i^{(1}, a_i^{(2}, \ldots, a_i^{(T)}\right) = \frac{\sum_{t=1}^{T} \exp\left(-\beta(T-t)\right) a_i^{(t}}{\sum_{t=1}^{T} \exp\left(-\beta(T-t)\right)}.$$

This is equivalent to a geometric discount model with $\alpha = e^{-\beta}$, so $\beta = -\ln\alpha$. Thus, the fully historic model ($\alpha = 1$) corresponds to $\beta = 0$. The parameter β ranges in $\mathbb{R}$, but as has been seen in section 4, below $\alpha = 1/2$ memory has no effect, therefore history has no effect for β beyond $\ln 2 = 0.6931$.

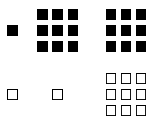
7.2 Inverse memory

The weighting mechanism may be designed in an inverse way, so that the older rounds are remembered more than the more recent ones. The idea is implemented in the following way:

$$m_i^{(T}(a_i^{(1}, a_i^{(2}, \ldots, a_i^{(T)}) = \frac{\sum_{t=1}^{T} \alpha^{t-1} a_i^{(t}}{\sum_{t=1}^{T} \alpha^{t-1}}$$

$$f_i^{(T} = \begin{cases} 1 & \text{if} \quad m_i^{(T} > 0.5 \\ 1 - a_i^{(T} & \text{if} \quad m_i^{(T} = 0.5 \\ 0 & \text{if} \quad m_i^{(T} < 0.5. \end{cases}$$

Recall that in the case of an equality, the opposite to the most recent state is registered, so the configuration at $T = 3$ will be the same as at $T = 2$ regardless of the rule ϕ. This will alter the evolution patterns in the inverse memory mechanism as compared to the standard (direct) one as from $T = 3$, even in the fully historic model. For example, starting with a single-site live cell, the pattern at $T = 3$ for any quiescent, $\beta_1 = 1$ rule will be again, as at $T = 2$, a 3×3 square. After $T = 3$ these cells are featured as live:



7.3 Accumulative memory

The weighting memory mechanism described in section 4 is holistic in its information demands: for evaluating the weighted mean of a cell, it is necessary to know the entire series of states in time. In order to avoid this demand, previous states can be pondered in the following way:

$$m_i^{(T}\left(a_i^{(1}, a_i^{(2}, \ldots, a_i^{(T}\right) = \frac{\sum_{t=1}^{T} t^k a_i^{(t}}{\sum_{t=1}^{T} t^k}.$$

The rounded weighted mean state f will be obtained, as in section 4, by comparing

$$\omega\left(a_i^{(t}, t \leq T\right) = \sum_{t=1}^{T} t^k a_i^{(t}$$

to

$$\Omega(T) = \frac{1}{2} \sum_{t=1}^{T} t^k,$$

or, in order to work only with integers, by comparing 2ω to 2Ω. This mechanism is not holistic but accumulative in its demand of knowledge of past history: for evaluating $\omega(a_i, T)$ it is not necessary to know the whole $\{a_i^{(t}\}$ series because it is determined by the accumulation of the contribution of the last state $(T^k a_i^{(T})$ to the already accumulated $\omega(a_i, T-1) : \omega(a_i^{(t}, t \leq T) = \omega(a_i^{(t}, t \leq T-1) + T^k a_i^{(T}$. In fact we have already considered two such accumulative mechanisms, (i) inverse memory: $\omega(a_i^{(t}, t \leq T) = \omega(a_i^{(t}, t \leq T-1) + \alpha^{T-1} a_i^{(T}$ and (ii) the full memory scenario described in section 3: $\omega(a_i^{(t}, t \leq T) = \omega(a_i^{(t}, t \leq T - 1) + a_i^{(T}$.

For $k = 0$, we have the full historic model of section 3; for $k = 1$ it is

$$2\Omega(T) = \sum_{t=1}^{T} t = \frac{T(T+1)}{2}.$$

The larger the value of k is, the more heavily the recent past is taken into account and consequently closer to the ahistoric scenario.

Choosing integer k values allows working only with integers (*á la* CA). There is a clear computational advantage of this model over that of section 4. But the accumulative memory mechanism just described has a serious drawback: t^k "explodes," even for $k = 2$, when t grows (see [59]).

7.4 Parity memory

Cells can be featured by the parity with the sum of previous states:

$$p_i^{(T} = \Big(\sum_{t=1}^{T} a_i^{(t}\Big) \bmod 2.$$

7.5 Interest memory

In the interest memory scenario, a cell will be featured as dead if all its previous states are equal (zero interest means boring), and as alive if any of them are different.

Toffoli and Margolus [1987] use the interest idea when considering the *time-tunnel*, a reversible mechanism based on the rule: $\phi(\mathcal{N}(a_i^{(T}))=0$ if all the states in $\mathcal{N}$ are equal, 1 in the contrary case.

7.6 Continuous valued memory

Historic memory can be embedded in Coupled Map Lattices, in which the state variable ranges in $\mathbb{R}$, by considering m instead of f in the application of the updating rule: $a_i^{(T+1} = \phi(\mathcal{N}(m_i^{(T}))$.

7.7 Fuzzy rules

Fuzzy CA, with states ranging in the real [0,1] interval, may be obtained by *fuzzification* of the disjunctive normal form of boolean CA rules by replacing: $a \vee b$ by $\min(1, a+b)$, $a \wedge b$ by ab, and $\neg a$ by $1-a$ [60]. Featuring cells by the unrounded weighted mean of their past states; that is, m, will provide *fuzzy* historic CA. An illustration of the effect of memory in fuzzy CA, starting with a single crisp live cell, is given in [42]. The illustration operates on Rule 90:

$$a_i^{(T+1} = \Big(a_{i-1}^{(T} \wedge \neg a_{i+1}^{(T}\Big) \vee \Big(\neg a_{i-1}^{(T} \wedge a_{i+1}^{(T}\Big),$$

after fuzzification:

$$a_i^{(T+1} = a_{i-1}^{(T} + a_{i+1}^{(T} - 2a_{i-1}^{(T}a_{i+1}^{(T},$$

which gives the historic formulation:

$$a_i^{(T+1} = m_{i-1}^{(T} + m_{i+1}^{(T} - 2m_{i-1}^{(T}m_{i+1}^{(T}.$$

7.8 Imperfect memory transmission

The standard CA and the CA with memory models can be combined by considering two types of cell characterization of the neighborhood $\mathcal{N}$.

Thus for a subset of $\mathcal{N}$, let it be $\mathcal{S}$, the cells can be featured by their last state and the remaining $\overline{\mathcal{S}} = \mathcal{N} - \mathcal{S}$ by their weighted mean states. For example, history can feature the cells of the strict neighborhood but not the cell to be updated: $a_i^{(T+1} = \phi(f_{i-1}^{(T}, a_i^{(T}, f_{i+1}^{(T})$, or, the contrary, the cell to be updated but not the outer cells: $a_i^{i(T+1} = \phi(a_{i-1}^{(T}, f_i^{(T}, a_{i+1}^{(T})$.

The different memories mentioned can be considered to implement new reversible mechanisms. For example, by using partial memory transmission, $a_i^{(T+1} = \phi(a_{i-1}^{(T}, f_i^{(T}, p_{i+1}^{(T}) \oplus a_i^{(T-1}$.

8. Conclusion

The consideration of historic memory of past states has been found to produce an inertial (or conservation) effect. According to this principle, historic memory:

1. Leads chaotic Class III rules in the ahistoric model to turn to Class II-like behavior (separate simple stable or periodic structures) in the fully historic model.

2. Induces a preserving effect regarding the damage caused by the reversal of a single site value. Thus, memory tends to confine the disruption measured by the difference patterns (DP).

   Discounting memory, as implying less historic information retained, implies an approach to the ahistoric model. Considering discount,

3. On increasing the value of the memory factor from $\alpha \leq 0.5$, for which the evolution corresponds to the standard (ahistoric) model, to $\alpha = 1.0$ (fully historic), there is usually a gradual variation of the effect of memory. Nevertheless,

4. An appreciable number of exceptions (and discontinuities) in such a gradual effect have been found.

These conclusions agree qualitatively with those found when starting with a single-site live cell [42, 43], which could be considered as a special case of reversion of a single site.

In general, it is possible to distinguish two main approaches to cellular automata (CA): *forward* and *backward*. The forward (theoretical) approach implies the study of transition rules of a given cellular space in order to establish its intrinsic properties (dynamical behavior, pattern growth, and so on). The backward (practical) approach involves the design of sets of transition rules to match the "correct" behavior of the CA system of a given complex system (physical, biological, social, and so on).

This paper adopts the forward approach: it introduces a kind of CA with memory ($\mathbb{CAM}$), and surveys its properties in the simplest (elementary) scenario in a fairly qualitative (pictorial) form. Borrowing an expression from Vichniac [1990], this paper deals with the $\mathbb{CAM}$ "zoology," that is, the study of $\mathbb{CAM}$ for its own sake. Following this metaphor, $\mathbb{CAM}$ increases the CA-(*bio*)diversity. A more complete analysis of the class $\mathbb{CAM}$ is left for future work. For example, the dynamics of $\mathbb{CAM}$ in the state (phase) space is to come under scrutiny. Another interesting open question is the analysis of the effect of memory on solving computational tasks such as synchronization, density, and ordering. The study of $r = 2$ $\mathbb{CAM}$ will inform on the effect of memory in complex (Wolfram's Class IV) rules. The study of two-dimensional $\mathbb{CAM}$ will follow. In this future work, the pioneer papers by Wolfram [62, 63] will again be an important reference.

The forward and backward (or inverse) approaches are obviously interrelated. A major impediment in the backward approach stems from the difficulty of utilizing complex CA behavior to exhibit a particular behavior or perform a particular task. This has severely limited their applications and hindered computing and modeling with CA [64].

$\mathbb{CAM}$ structurally exhibit a growth inhibition feature found exceptionally in standard CA [28]. The extent of the growth inhibition can be modulated by varying the memory factor α. This could mean a potential advantage of $\mathbb{CAM}$ over standard CA as a new tool for modeling slow diffusive growth from small regions, a common phenomenon in nature.

The question of how errors spread and propagate in cooperative systems has been studied in a variety of fields. Given the difficulty of creating analytical models for any but the simplest systems, most investigations have been conducted by computer simulations, especially in the area of statistical physics of many-body systems [8, 65]. Errors reverse the state in elementary CA. In $\mathbb{CAM}$, the damage induced by a single error is generally confined to the proximity of the site where it occurred. From this point of view, $\mathbb{CAM}$ can be featured as resilient in the face of errors. Fault tolerance is an important issue when considering systems with a large number of components, in which faults will be highly probable. The robust $\mathbb{CAM}$ could play a role in this (nanotechnological) scenario.

Acknowledgments

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