One-dimensional Cellular Automata with Memory in Cells of the Most Frequent Recent Value

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Standard cellular automata (CA) rules depend only on the neighborhood configuration of the preceding time step. This article considers an extension to the standard CA framework which implements memory capabilities by featuring each cell with a minimal summary of its past states, the mode of the last three states in particular. A study is made on the spatio-temporal and difference patterns of one-dimensional CA.

1. Introduction

Cellular automata (CA) are discrete, spatially explicit extended dynamic systems. A CA system is composed of adjacent cells or sites arranged as a regular d-dimensional lattice, which evolves in discrete time steps. Each cell is characterized by an internal state whose value belongs to a finite set. The updating of these states is made simultaneously according to a common local transition rule involving only a neighborhood of each cell. Thus, if $\sigma_i^{(T)}$ is taken to denote the value of cell i at time step T, the site values evolve by iteration of the mapping: $\sigma_i^{(T+1)} = \phi(\sigma_j^{(T)} \in \mathcal{N}_i)$, where ϕ is an arbitrary function which specifies the CA rule operating on the neighborhood ($\mathcal{N}$) of cell i. Ilachinski [1] and Wolfram [2] have recently reviewed the subject.

Historic memory can be embedded in CA dynamics by featuring every cell with a summary of its states in the previous time steps. Thus, what we propose is to maintain the rules (ϕ) unaltered, but make them act on the cells of their summary state: $\sigma_i^{(T+1)} = \phi(s_j^{(T)} \in \mathcal{N}_i)$, with $s_i^{(T)}$ being a summary state of the series of states of cell i after iteration T.

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In earlier work[1] we analyzed the effect of memory when cells are featured by a summary of all their previous states: $s_i^{(T)} = s(\sigma_i^{(1)}, \sigma_i^{(2)}, \ldots, \sigma_i^{(T)})$. Here we analyze the effect of minimal trailing memory featuring cells: $s_i^{(T)} = s(\sigma_i^{(T-2)}, \sigma_i^{(T-1)}, \sigma_i^{(T)})$. These automata considering memory are called *historic*, and the standard ones *ahistoric*.

We analyze only one-dimensional ($d = 1$) rules here. Special attention is paid to the simplest scenario using *elementary* rules, that is, CA with two possible values ($k = 2$) at each site ($\sigma \in \{0, 1\}$), operating on their nearest neighbors ($r = 1$). Following Wolfram's notation, these rules are characterized by a sequence of binary values (β) associated with each of the eight possible triplets $\left(\sigma_{i-1}^{(T)}, \sigma_i^{(T)}, \sigma_{i+1}^{(T)}\right)$:

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ (\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8)_{\text{binary}} \end{array} \equiv \left(\sum_{i=1}^{8} \beta_i 2^{8-i} \right)_{\text{decimal}} = R \in [0, 255].$$

The rules are conveniently specified by a decimal integer, to be referred to as their *rule number*, R.

After the elementary rules, we consider those with a *range* of $r = 2$: the value of a given site depends on values of the nearest and next-nearest neighbors. We next analyze *totalistic* rules: the value of a site depends only on the sum of the values of its neighbors, and not on their individual values. Totalistic $k = r = 2$ rules are characterized by a sequence of binary values ($\beta_{\mathtt{s}}$) associated with each of the six possible values of the sum ($\mathtt{s}$) of the neighbors:

$$(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)_{\text{binary}} \equiv \sum_{\mathtt{s}=0}^{5} \beta_{\mathtt{s}} 2^{\mathtt{s}}_{\text{decimal}} = R \in [0, 63].$$

Finally, we explore the effect of memory in the $k = 3$, $r = 1$ scenario. Again, we only analyze totalistic rules in this scenario, coded as:

$$(\beta_6\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)_{\text{ternary}} \equiv \sum_{\mathtt{s}=0}^{6} \beta_{\mathtt{s}} 3^{\mathtt{s}}_{\text{decimal}} = R \in [0, 2186].$$

2. Minimal memory

In the two-state scenario, minimum operative trailing memory is achieved by featuring cells with a mapping of their three most recent values. Thus the 256 elementary CA rules are available. A systematic study of these memory rules is left for future work; here we focus our study into one

[1]See the references by Alonso-Sanz and Alonso-Sanz *et al.*

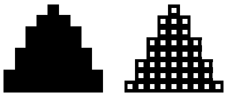
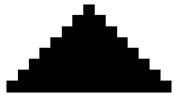
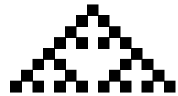
Rule 254		Rule 90	
Historic	Ahistoric	Historic	Ahistoric
			

Table 1. Rules 254 (11111110) and 90 (01011010) starting from a single site live cell until $T = 8$. Historic and ahistoric models. Live cells: ■ last, □ mode of the last three states.

of them, the fairly "natural" way of featuring cells by the most frequent of the their three last states: $s_i^{(T)} = \text{mode}(\sigma_i^{(T-2)}, \sigma_i^{(T-1)}, \sigma_i^{(T)})$.[2]

Memory becomes operative after $T = 3$, so initially $s_i^{(1)} = \sigma_i^{(1)}$, $s_i^{(2)} = \sigma_i^{(2)}$. Consequently the historic and ahistoric evolution patterns are the same until $T = 3$, regardless of the transition rule ϕ. But after the third iteration, the last and most frequent states often differ, and consequently the patterns for the historic and ahistoric automata typically diverge at $T = 4$. We can exemplify the effect of the minimal mode memory mechanism with the elementary Rules 254 (1111110) and 90 (01011010) (both in Table 1) starting with a single site live cell.

Rule 254 progresses as fast as possible (i.e., at the *speed of light*): it assigns a live state to any cell in whose neighborhood there would be at least one live cell. Thus, in the ahistoric model, a single site live cell will grow monotonically, generating segments whose size increases by two units with every time step. The dynamics are slower in the historic mode model: the outer live cells are not featured as live unless they lived twice in the last three iterations, in which case the automaton fires two new outer live cells. Thus the segment of live cells under Rule 254 grows only at odd time steps and remains the same at even time steps: the speed of light in the historic model is half of that of the ahistoric. Rule 90 has a slower dynamic in the historic model starting with a single site live cell. In this case, after $T = 3$ the most frequent state for all the cells is death, so Rule 90 dies out at $T = 4$.

In the standard (ahistoric) scenario, Rules 90 and 150 (together with the trivial 0 and 204) are the only *additive* legal rules, that is, any initial pattern can be decomposed into the superposition of patterns from a single site seed. Each of these configurations can be evolved independently and the results superposed (mod two) to obtain the final complete pattern. As illustrated in Table 2, starting with two adjacent live cells, the additivity of Rules 90 and 150 is lost in the historic model.

It should be emphasized that the memory mechanism considered here is different from that of other CA with memory reported in the liter-

[2]The "majority" Rule 232 (11101000) in Wolfram's notation.

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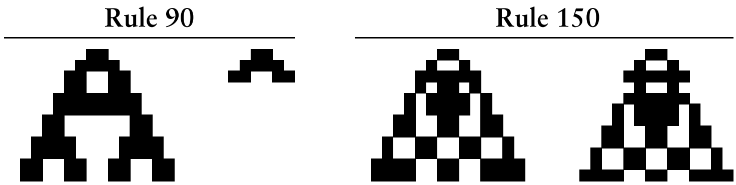

Table 2. Rules 90 and 150 are not additive in the historic formulation. The true evolution pattern starting with two adjacent live cells is shown on the left. The XOR superposed configuration of those evolved independently starting with a single seed is shown on the right. Evolution up to $T = 12$.

ature.[3] Typically, rules with a high time-order incorporate memory into the transition rule, determining the configuration at time $T+1$ in terms of the configurations at previous time steps. Thus, in second-order time (memory of capacity two) rules, the transition rule operates as: $\sigma_i^{(T+1)} = \Phi(\sigma_j^{(T)} \in \mathcal{N}_i, \sigma_j^{(T-1)} \in \mathcal{N}_i)$. Particularly interesting is the reversible formulation based on the exclusive OR (XOR, denoted as $\oplus$): $\sigma_i^{(T+1)} = \phi(\sigma_j^{(T)} \in \mathcal{N}_i) \oplus \sigma_i^{(T-1)}$, reversed as $\sigma_i^{(T-1)} = \phi(\sigma_j^{(T)} \in \mathcal{N}_i \sigma) \oplus \sigma_i^{(T+1)}$. *Double* memory can of course be implemented as: $\sigma_i^{(T+1)} = \Phi(s_j^{(T)} \in \mathcal{N}_i, s_j^{(T-1)} \in \mathcal{N}_i)$. A simple example of double memory is given in section 5 (Table 10), but let us point out here that reversion is infeasible when adding memory to all cells with the ahistoric reversible formulation just indicated as: $\sigma_i^{(T+1)} = \phi(s_j^{(T)} \in \mathcal{N}_i) \oplus s_i^{(T-1)}$.[4] To preserve reversibility, inherent in the XOR operation, the reversible formulation (in the unlimited trailing scenario) with memory must be [6]: $\sigma_i^{(T+1)} = \phi(s_j^{(T)} \in \mathcal{N}_i) \oplus \sigma_i^{(T-1)}$.

Some authors, for example Wolf-Gladrow [15], define rules with *memory* as those with dependence in ϕ on the state of the cell to be updated. So in the $r = 1$ scenario, rules with no memory take the form: $\sigma_i^{(T+1)} = \phi(\sigma_{i-1}^{(T)}, \sigma_{i+1}^{(T)})$. Rule 90: $\sigma_i^{(T+1)} = \sigma_{i-1}^{(T)} + \sigma_{i+1}^{(T)} \bmod 2$, would be an example of a rule with no memory. Our use of the term "memory" is not this.

Every configuration in a standard (ahistoric) CA has a unique successor in time (it may however have several distinct predecessors). A configuration in the historic scenario may have multiple successors:[5] the transition rules operate on the mode state configuration, not on the

[3]See, for example, Wolfram, [13], p. 118; Ilachinski [1], p. 43; or class MEMO in Adamatzky [14], p. 7. The latter shows an example of two-dimensional CA with memory, but to the best of our knowledge the study of the effect of memory on deterministic CA has been rather neglected. Some critics can argue that memory is not in the realm of CA (or even of dynamic systems), but we believe that the subject is worth studying. At least CA with memory can be considered as a (promising) extension of the basic paradigm.

[4]See the evolving patterns of such rules from a single site seed in [9] and [10].

[5]A primary example is given with Rule 254 (Table 1). In the mode historic model, live configurations either reproduce themselves or grow by two units.

last, so the successor of a given configuration depends on the underlying mode configuration. We envisage that this will notably alter the appearance of the state transition diagrams in the historic scenario when compared to ahistoric CA. Quoting a remark by Wuensche and Lesser ([16], p. 15) regarding higher time-order CA, CA with memory in cells "would result in a qualitatively different behavior." A primary effect of memory (described later, but envisaged already in Table 1) is a decrease in the variability of configurations generated by evolution. This would lead to a conjecture of the existence of a higher number of "Garden-of-Eden" nodes (configurations for an automaton which could only exist initially) in the historic scenario.

3. The effect of minimal memory on elementary rules

Figures 1(a) and 1(b) show the effect of the mode memory on elementary rules starting from a single site live cell[6] up to $T = 26$.

Figure 1(a) shows the evolution of the *legal*[7] rules affected by mode memory. History has a dramatic effect on Rules 18, 90, 146, and 218 as the pattern dies out as early as $T = 4$. The case of Rule 22 is particular as two branches are generated at $T = 17$ in the historic model. The patterns of the remaining rules in the historic model are reminiscent of the ahistoric ones, but, let us say, "compressed." Figure 1(b) shows the effect of memory on some relevant[8] quiescent asymmetric rules. Rule 2 "shifts" a single site live cell one space at every time step in the ahistoric model. With memory the shift actuates just up to $T = 3$, and at $T = 4$ the pattern dies. This evolution is common to all rules that just shift a single site cell without increasing the number of living cells at $T = 2$.[9] The patterns generated by Rules 6 and 14 are *rectified*[10] by memory in such a way that the total number of live cells in the historic and ahistoric spatio-temporal patterns is the same.[11] Again, the historic patterns of the remaining rules in Figure 1(b) seem, generally, like the ahistoric ones compressed.

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[6]Rules active from a single site live cell have: $\beta_3 = 1$ or $\beta_4 = 1$ or $\beta_6 = 1$ or $\beta_7 = 1$ or $\beta_8 = 1$.

[7]Rules that are reflection symmetric ($\beta_2 = \beta_5$ and $\beta_4 = \beta_7$) and quiescent ($\beta_8 = 0$). These restrictions leave 32 possible legal rules of the form: $\beta_1\beta_2\beta_3\beta_4\beta_2\beta_6\beta_4 0$. Half of these rules are simple and evolve to the null state from a single site live cell or maintain it unchanged forever in the ahistoric model. Memory has no effect on these Rules: 0, 4, 32, 36, 72, 76, 104, 108, 128, 132, 160, 164, 200, 204 (identity), 232 (majority), and 236.

[8]Either because the effect on them is representative of the effect on many others or because they have been particularly studied in the ahistoric scenario. For example, Rules 30 and 110.

[9]This is the case of the important Rules 184 and 226.

[10]In the sense of having lines in the spatio-temporal pattern with a slower slope.

[11]The other quiescent rules affected by memory in such a way are: 14, 20, 38, 52, 46, 84, 116, 134, 142, 148, 166, 174, 180, 212, 244, and 245.

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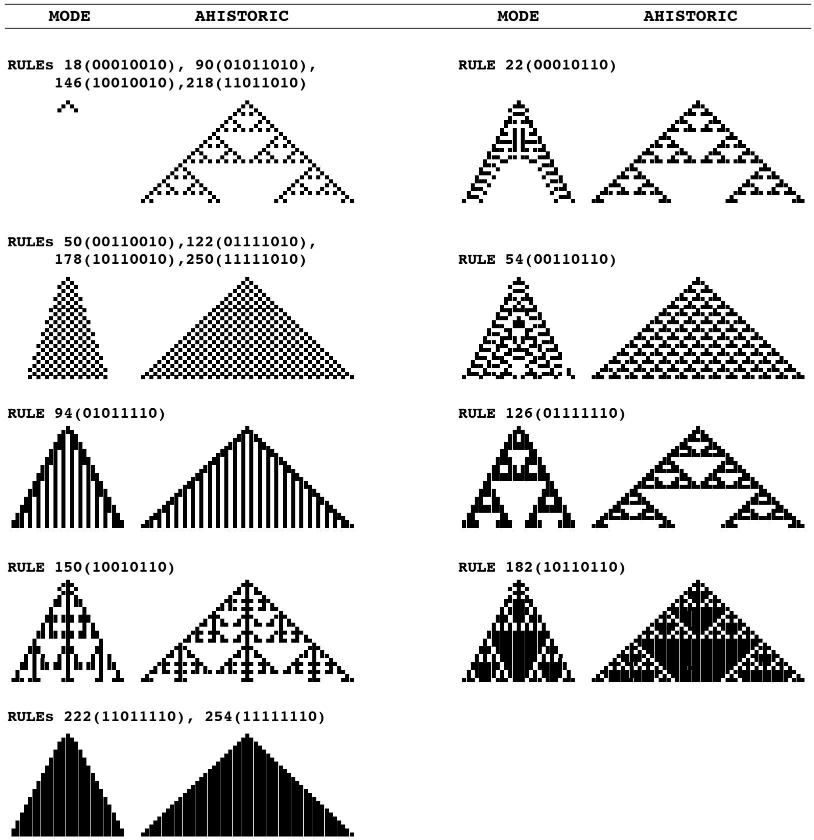

Figure 1(a). Legal $k = 2$, $r = 1$ rules significantly affected by the mode of the three-last-states memory. Evolution from a single site live cell (■) up to $T = 26$.

Figures 2(a), 2(b), and 2(c) show the effect of mode memory on elementary rules starting with the same random[12] initial configuration up to $T = 150$.

Figure 2(a) shows the evolution of the legal rules significantly affected by memory when starting from a random initial configuration. These are the nine legal rules which generate nonperiodic patterns in the ahistoric scenario.[13] Figure 2(a) also shows the difference patterns (DP) produced when the value of the initial center site is reversed. The

[12]The value of each site is initially uncorrelated, and is taken to be 0 or 1 with probability $p = 0.5$.

[13]The legal rules not shown in Figure 2(a) are either unaffected or minimally affected by memory. For example, Rules 32, 128, and 160 which soon die; Rules 250 and 254 which soon blacken the space; or those which serve as *filters* (Wolfram's Class II) exemplified by Rule 36.

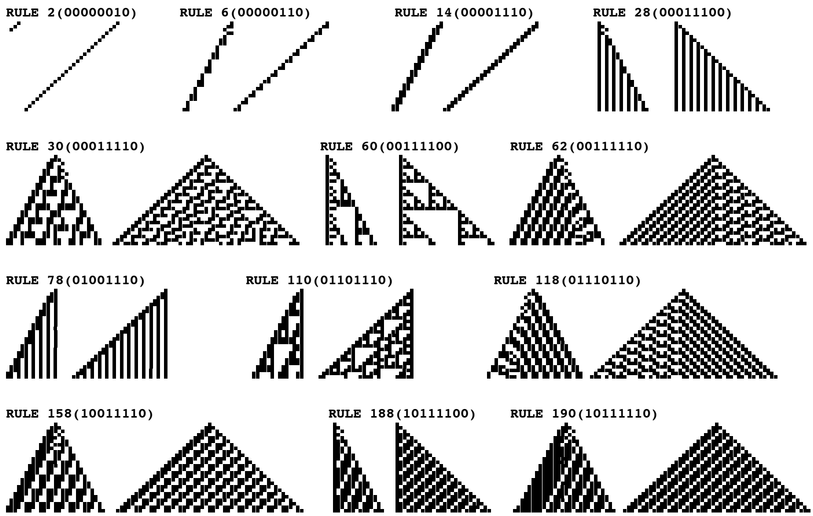

Figure 1(b). Some relevant quiescent asymmetric $k = 2$, $r = 1$ rules affected by the mode of the three-last-states memory. Evolution from a single site live cell (■) up to $T = 26$.

pictures show the "damaged" region as black pixels corresponding to the site values that differed among the patterns generated with the two initial configurations. Again, as in Figure 1(a), history has a dramatic effect on Rule 18. The appearance of its spatio-temporal pattern changes completely: isolated periodic structures are generated, very distinct from the inverted-triangle world of the ahistoric pattern. Reversing the central site cell generates a periodic localized perturbation in the historic model, whereas in the ahistoric model, the perturbation grows at the speed of light: $\lambda_L = \lambda_R = 1$.[14] The effect of memory on Rules 22, 90, 122, 126, and 150 is fairly similar: the spatio-temporal patterns preserve the inverted triangles characteristic of the ahistoric model and the DP are notably restrained (albeit Rule 122 shows an exceptional extinction of the DP in the ahistoric model in Figure 2(a)). Constraint of the DP due to memory is also found in Rule 146 (in fact, extinction) and in Rule 182, but the resulting spatio-temporal patterns are much more altered. Rule 54 also shows an exceptional extinction of the DP

[14] The perturbations in proper chaotic rules propagate to the right and left at a single (maximum) velocity at any time. This behavior illustrates the *butterfly effect*: a small perturbation grows, and finally rules the whole system. The damage-spreading velocity is quantified by means of the left and right Lyapunov exponents (λ_L, λ_R) that measure the rate at which perturbations spread to the left and right. These are given by the slopes of the left and right boundary of the growth of the DP. Thus, zero values indicate periodicity, whereas negative velocity indicates perturbation repair. The maximum λ attainable when $r = 1$ is $\lambda = 1$.

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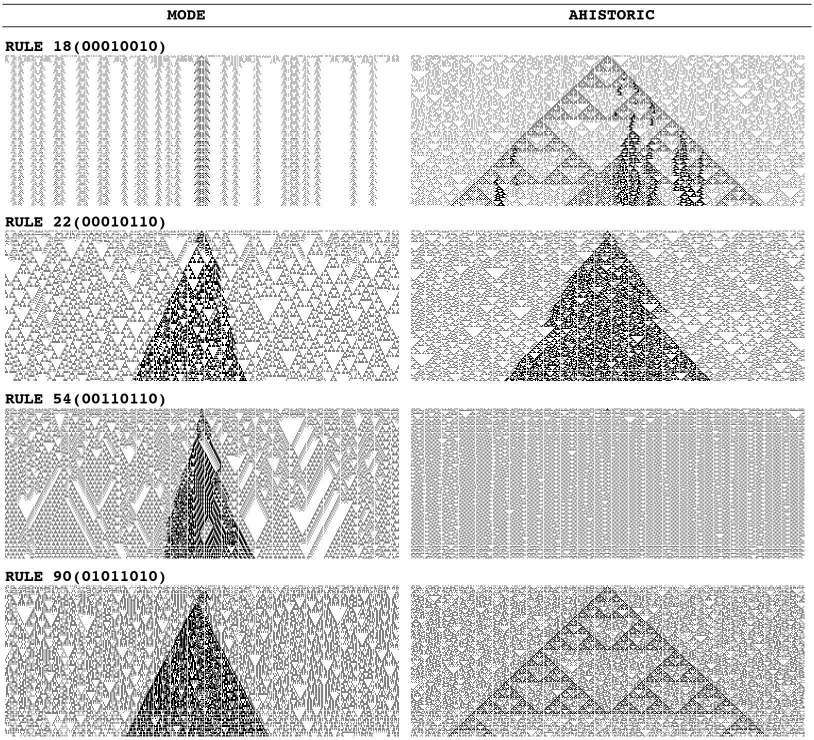

Figure 2(a). Evolution of the legal $k = 2$, $r = 1$ rules significantly affected by the mode of the three-last-states memory model. The values of sites in the initial configuration are chosen at random to be 0 (blank) or 1 (gray) with probability 1/2. The pictures show the evolution of CA with 371 sites for 150 time steps with periodic boundary conditions imposed on the edges. Differences in patterns resulting from reversing the center site value are shown as black pixels.

in the ahistoric model, whereas in the historic one the DP is a distinctive feature, and memory notably alters its pattern.[15]

In order to systematize the analysis of the effect of memory, one can resort to equivalence classes, that is, fundamentally inequivalent rules, formed under the negative, reflection, and negative + reflection transformations [16]. Memory is expected to affect all the rules of an equivalence class in a similar way. Thus the effect of memory on symmetric but not quiescent rules (not shown in this work) can to a great extent be described in terms of the effect on their equivalent legal

[15] The behavior of Rule 54 in the ahistoric model has been featured to some extent as transitional between very simple Wolfram's Class I and II rules and chaotic Class III. Thus, Rule 54 appears among the two one-dimensional rules (with Rule 110) that seem to belong to Wolfram's complex Class IV.

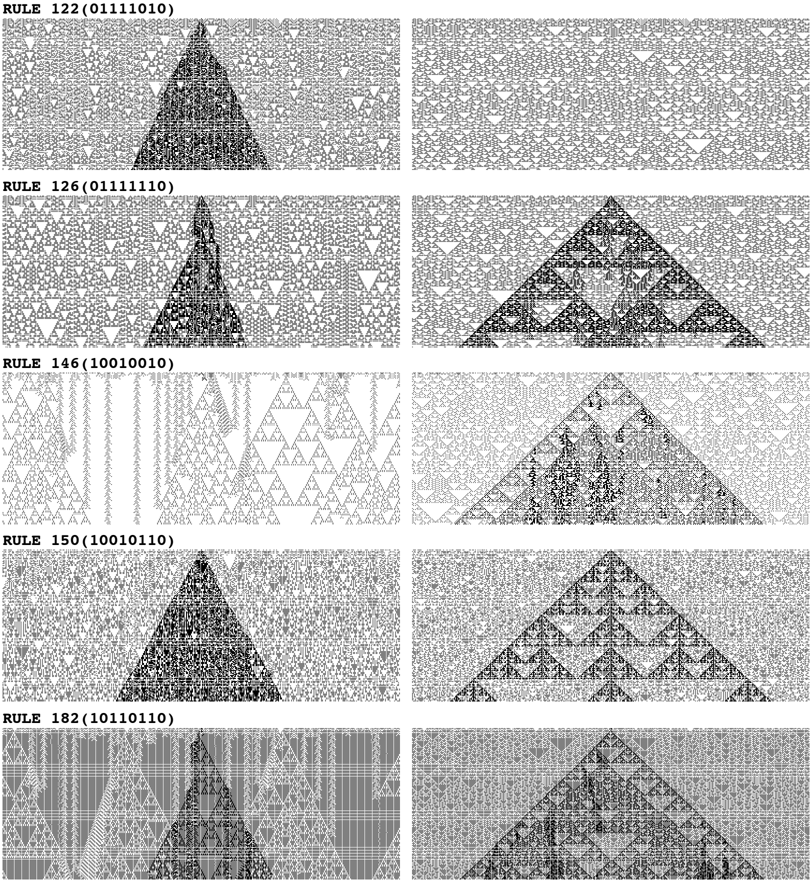

Figure 2(a). (*continued*)

rules. Thus, the pattern of Rule 183 with memory resembles that of its equivalent (under the negative transformation) Rule 18, 151 resembles that of 22, 147 that of 54, 165 that of 90, 161 that of 122, and 129 that of 126 (Rule 150 is its own negative transformed). The parallelism that can be traced between these pairs of rules is of the kind appreciated in Figure 2(a) in equivalent Rules 146 and 182.

Figures 2(b) and 2(c) show the spatio-temporal patterns and DP of some of the asymmetric rules affected by memory in the scenario stated in Figure 2(a). Figure 2(b) deals with *semi-asymmetric* (either $\beta_2 \neq \beta_5$ or $\beta_4 \neq \beta_7$, but not both) rules; Figure 2(c) with *fully-asymmetric* ($\beta_2 \neq \beta_5$ and $\beta_4 \neq \beta_7$) rules. Rules are grouped in these figures that belong to the same equivalence class; often showing only the minimal representative rule of the class affected by memory. Rules not affected by memory tend to belong to the same equivalence class. Thus,

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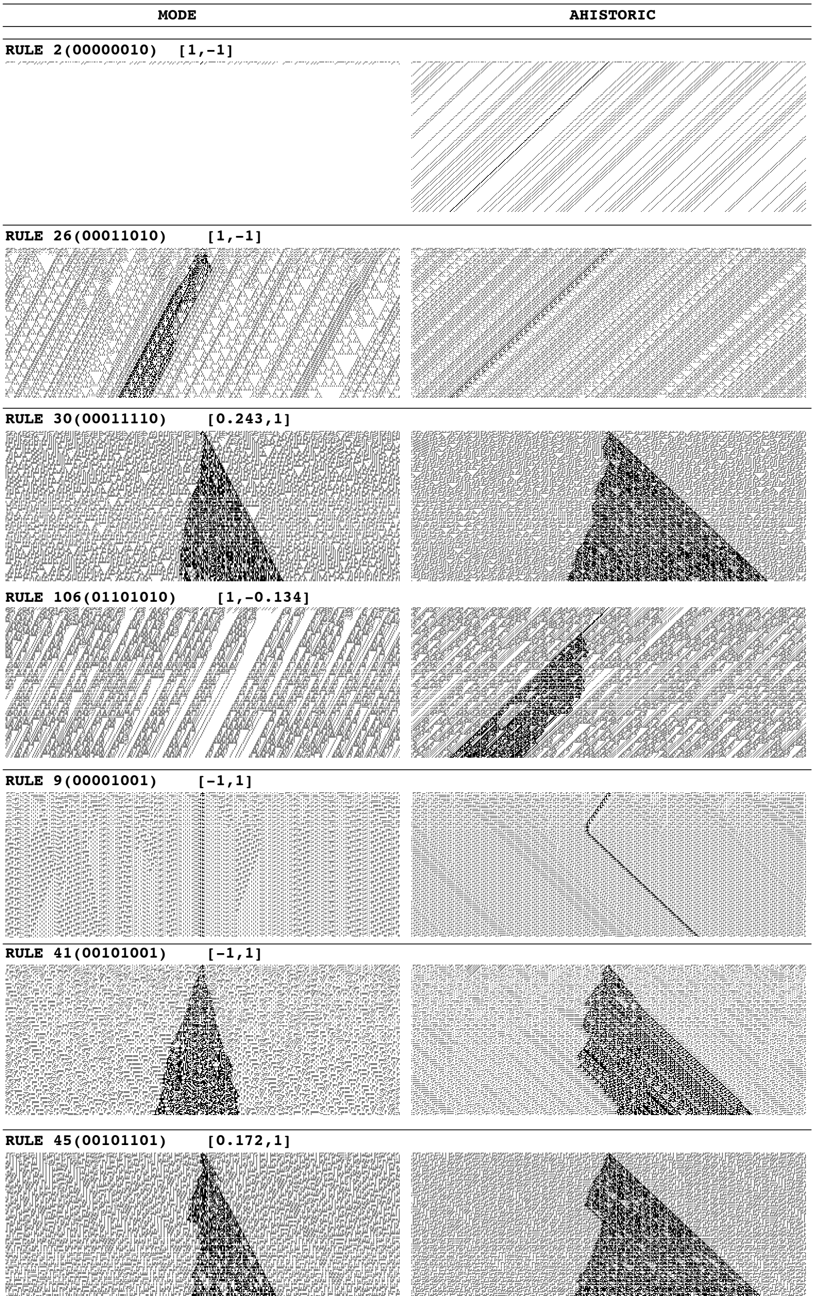

Figure 2(b). Evolution of some semi-asymmetric rules significantly affected by the mode of the three-last-states memory in the scenario of Figure 2(a). The rules are grouped into equivalence classes. The left and right Lyapunov exponents of the lowest rule number of each class (minimal representative) in the ahistoric model are given after the rule codes.

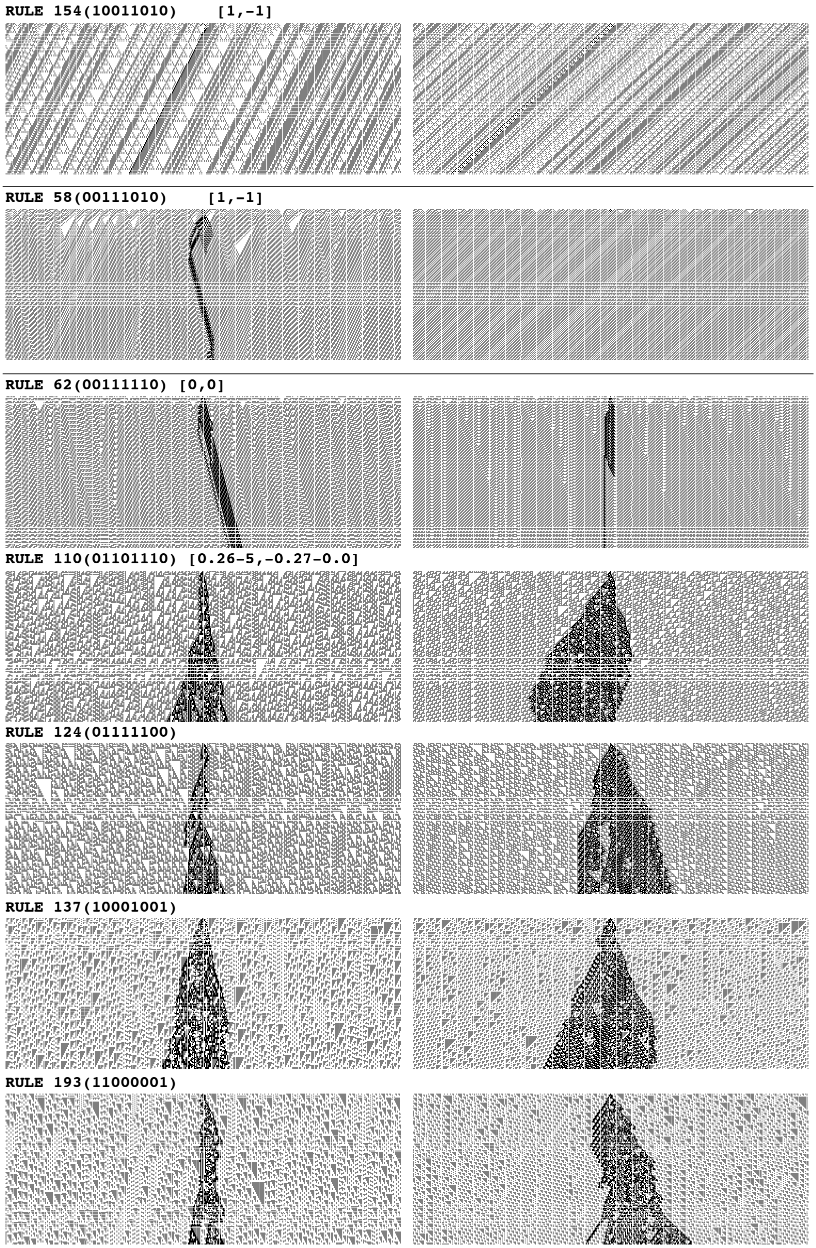

Figure 2(b). (*continued*)

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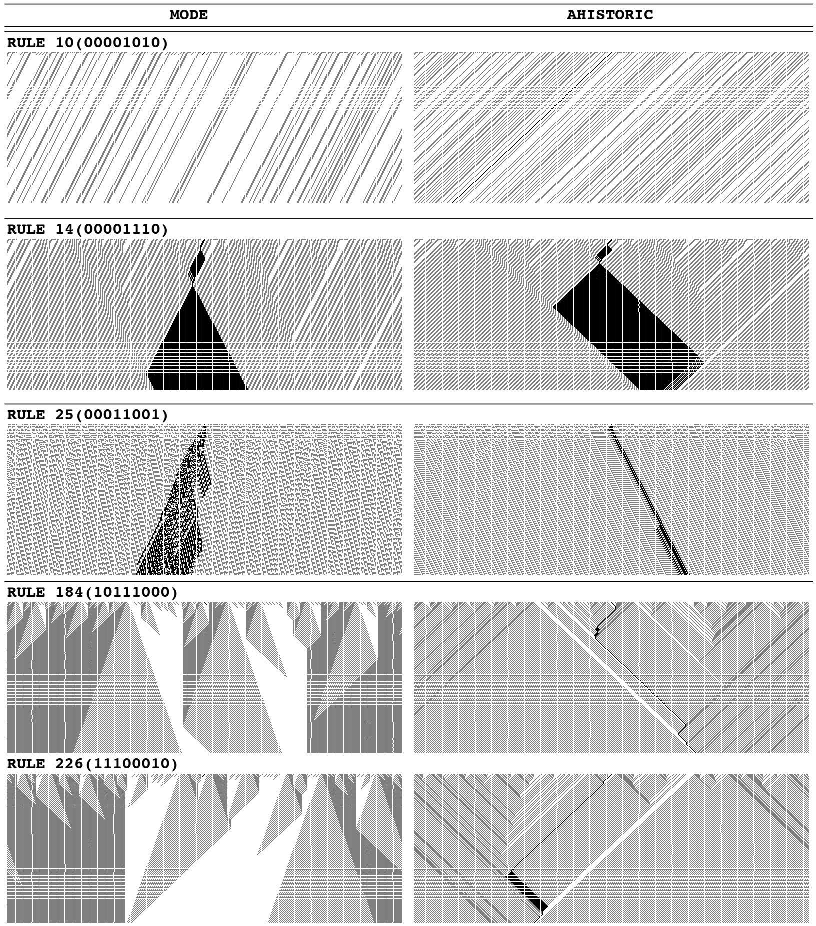

Figure 2(c). Evolution of some fully asymmetric rules significantly affected by the mode of the three-last-states memory in the scenario of Figure 2(a).

for example, memory has no effect on any of the rules of the subclasses: $\{8, 64, 239, 253\}, \{12, 68, 207, 221\}, \{13, 69, 79, 93\}$, and $\{29, 71\}$.

The important Rule 110 and its three equivalent rules under the negative and reflection transformations (Rules 124, 137, and 193), may serve as a paradigmatic example of the expected effect of memory: preservation of the general aspect of the spatio-temporal pattern and constraint in the damage induced by the reversal of a single site. The same applies to the subclass $\{60, 102, 195, 153\}$ in Figure 2(c). History however has an unexpected effect on other rules. This is the case of the important Rule 184 and its equivalent Rule 226 and of Rules 57 and 99; the similar aspect of their ahistoric spatio-temporal patterns are notably

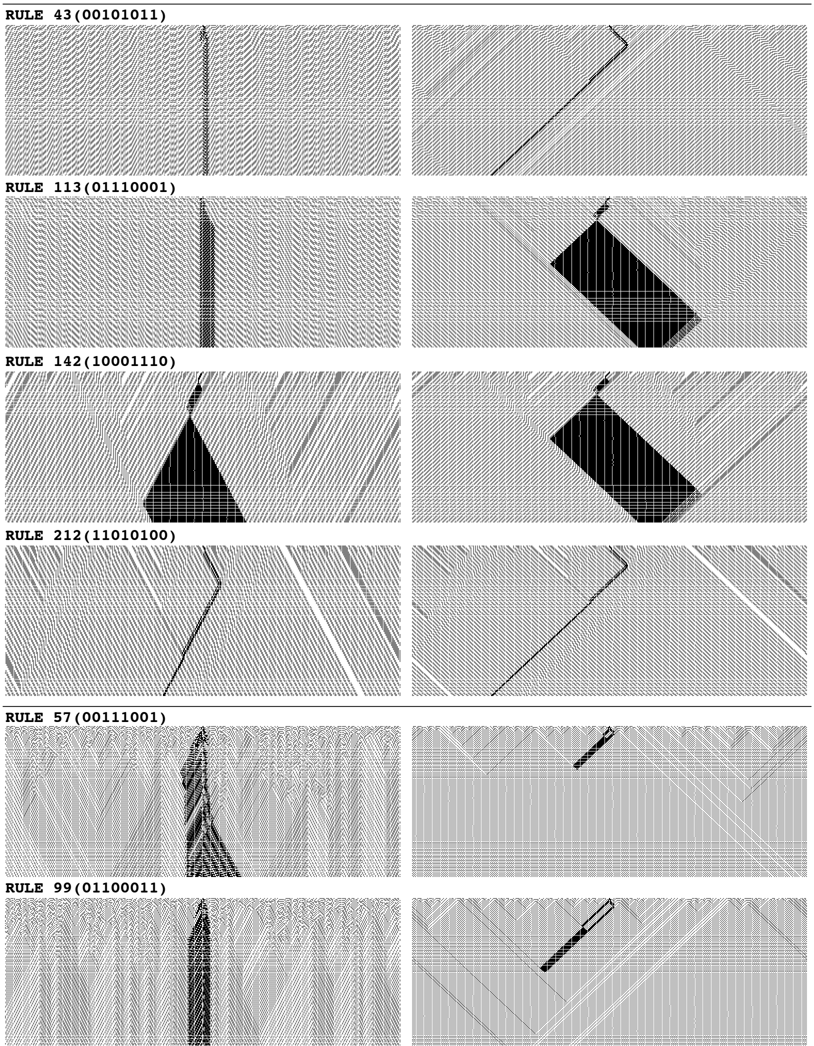

Figure 2(c). (*continued*)

altered by memory. The extinction induced by memory on Rule 2 was predictable from Figure 2(a). Rectification of lines (or more complex patterns) to a slower slope seems to be a common effect of memory in many rules, for example, Rules 26 and 10.

4. The effect of minimal memory on *r*=2 and *k*=3 rules

Figures 3 and 4 deal with totalistic CA rules operating on the nearest and next-nearest neighbors ($r = 2$) scenario. Figure 3 shows the spatio-

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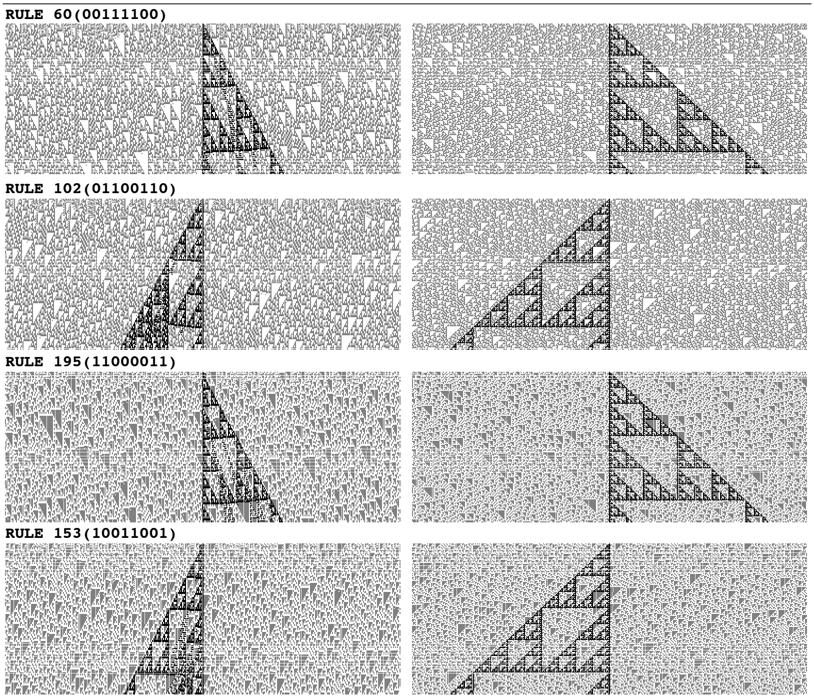

Figure 2(c). (*continued*)

temporal patterns starting from a single live site for all the $r = 2$ totalistic rules sensitive to a sole live cell; that is, the 16 rules from 2 to 62 by intervals of 4, with $\beta_1 = 1$. The patterns shown in Figure 3 are symmetric due to the exclusive consideration of totalistic rules. Figure 4 shows the evolving patterns of the quiescent rules starting at random with the DP produced from a change in the value of its initial center site superimposed. Two conclusions can be derived from Figure 3: (*i*) memory preserves the general aspect of the ahistoric patterns from a single live cell, (*ii*) memory constrains the evolving patterns.

Wolfram classified the behavior of $r = k = 2$ CA rules into four qualitative classes according to the fate of their evolving patterns: Class I: homogeneous state, Class II: simple stable or periodic structures, Class III: chaotic aperiodic pattern, Class IV: complex localized structures, sometimes long-lived. Figure 4 shows the effect of memory on rules of Classes III and IV (Rules 20 and 52[16]). The effects starting from a single site seed (Figure 3) can be extended when starting at random (Figure 4): (*i*) memory preserves the general aspect of the ahistoric patterns, (*ii*) the

[16]These two rules are not active when starting from a single live cell, so are not shown in Figure 3.

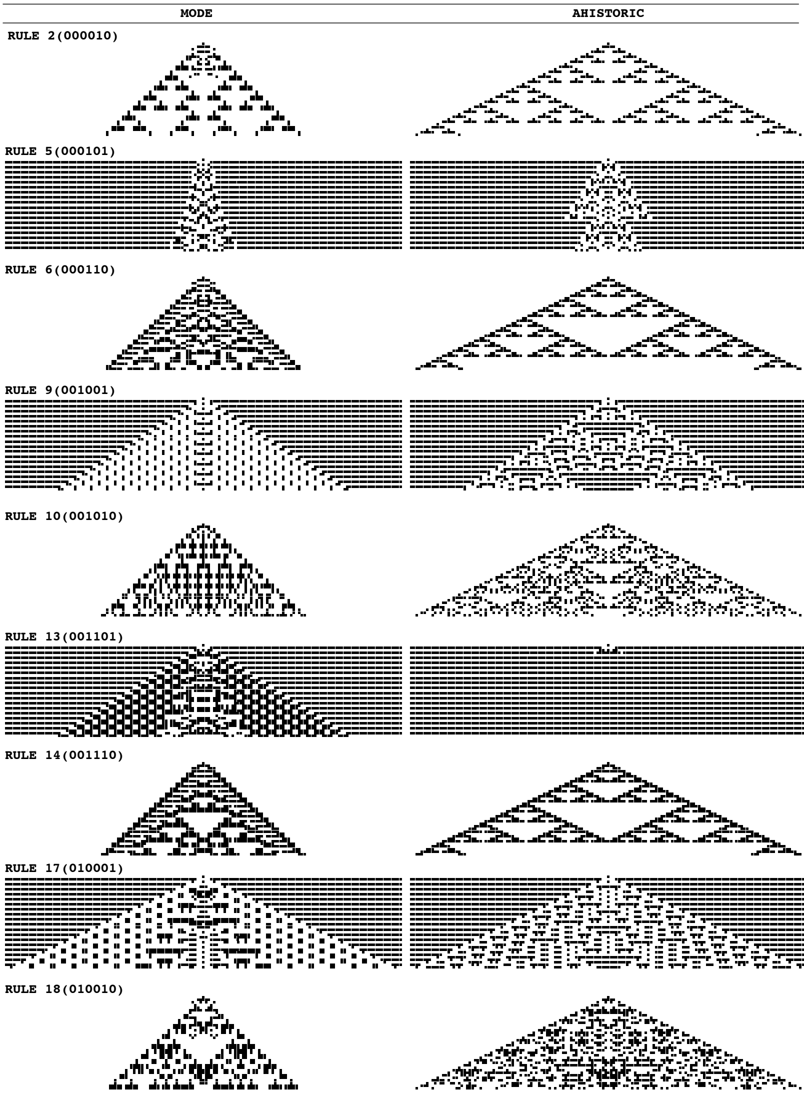

Figure 3. Totalistic $k = r = 2$ rules affected by the mode of the three-last-states memory. Evolution from a single site live cell (■) up to $T = 37$.

damage-spreading becomes constrained when memory enters.[17] The latter conclusion is fully applicable to the chaotic rules, in which the

[17]This result can be expected from Figure 3, in fact both Figures 3 and 4 can be examined in relation to the damage-spreading as two particular cases of the initial alteration of a unique (central) site value.

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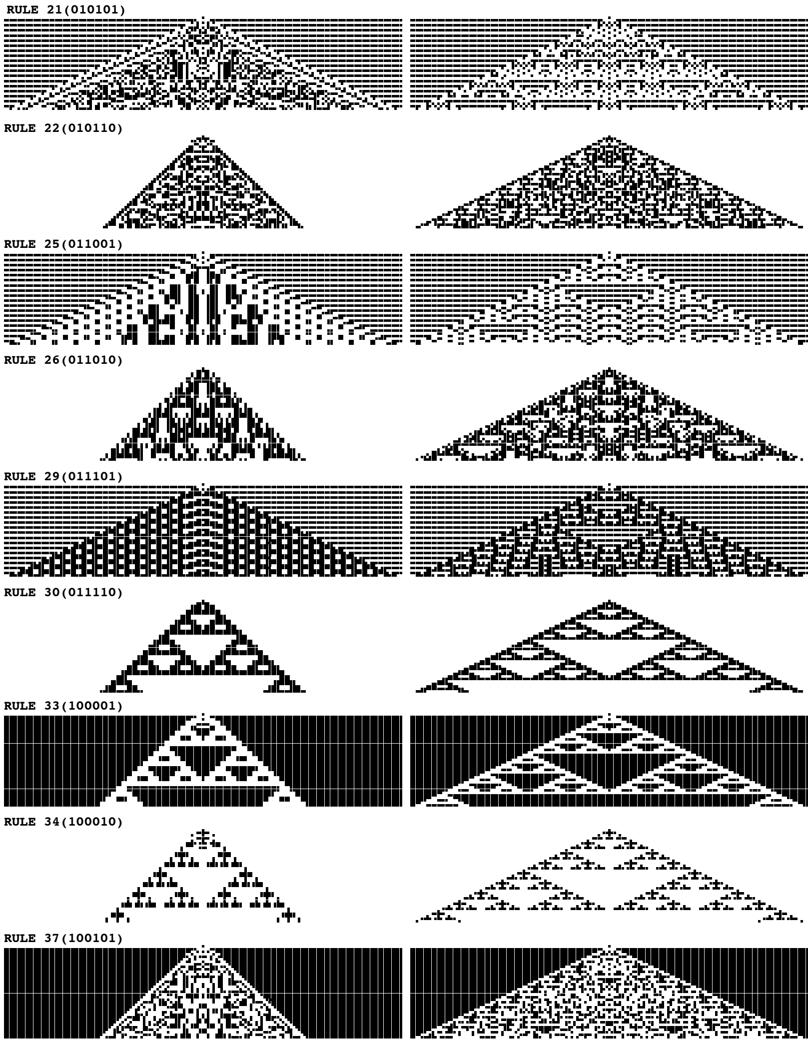

Figure 3. (*continued*)

perturbations in the ahistoric model propagate very rapidly to the right and left at any time.[18] In the particular case of the complex Rules 20 and 52, their singular damage-spreading in the ahistoric model is not actually constrained in the model with memory.

Figures 5 and 6 deal with totalistic $r = 1$ CA rules with three states ($k = 3$). Two memory models are considered in these figures: the already

[18]This behavior also illustrates the butterfly effect.

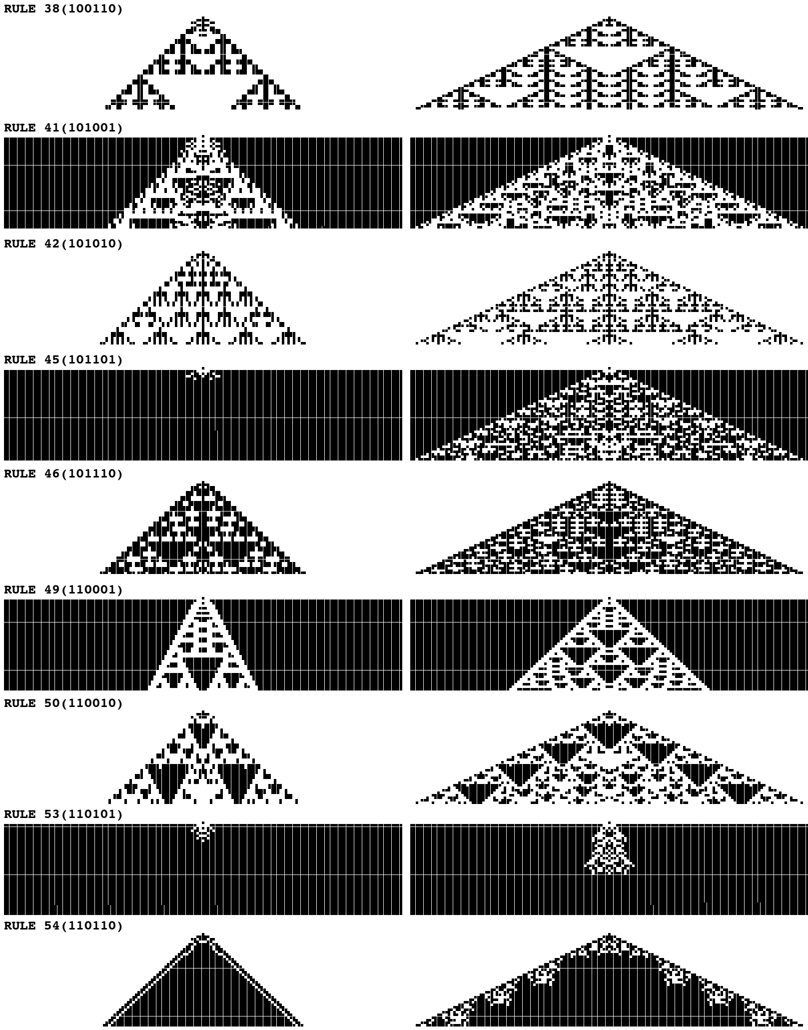

Figure 3. (*continued*)

known mode of the three-last-states memory (mode) and a minimal memory of capacity two (mean) implemented as:

$$m_i^{(T)} = \frac{1}{2}(\sigma_i^{(T-1)} + \sigma_i^{(T)}) \Rightarrow s_i^{(T)} = \begin{cases} 0 & \text{if} \quad m_i^{(T)} < 0.5 \\ \sigma_i^{(T)} & \text{if} \quad m_i^{(T)} = 0.5 \\ 1 & \text{if} \quad 0.5 < m_i^{(T)} < 1.5 \\ \sigma_i^{(T)} & \text{if} \quad m_i^{(T)} = 1.5 \\ 2 & \text{if} \quad m_i^{(T)} > 1.5 \end{cases}$$

This kind of mean memory mechanism, that can be implemented with

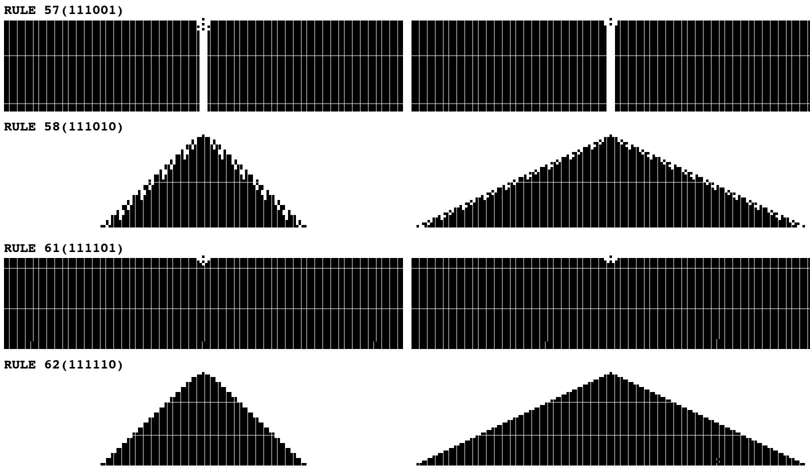

Figure 3. (*continued*)

higher memory depth [5], actually only changes the cell featured as 1 after the sequences of states 02 and 20. Figure 5 shows the effect of memory on the spatio-temporal patterns starting from a single live site seed with $\sigma = 1$, in all the totalistic quiescent ($\beta_0 = 0$) *parity* rules, that is, rules with β_1, β_3, and β_5 nonnull, and $\beta_2 = \beta_4 = \beta_6 = 0$. Again, the patterns in Figure 5 are symmetric due to the consideration of totalistic rules. Rule 273 (0101010) is not a proper $k = 3$ but a $k = 2$ rule, because no β is equal to two in its ternary description. For this reason, its spatio-temporal pattern in the mean memory model coincides with the ahistoric one. In the mode memory model, Rule 273 shows the characteristic inhibition of growth in Figure 5 already seen for its equivalent rule in the $k = 2$ scenario, that is, the parity Rule 150. Rule 516 (0201010) evolves like Rule 273 starting with a single $\sigma = 1$ seed because in this scenario the sum of neighbors is never five and only $\beta_5 = 2$. Mode memory acts on Rules 300 and 543 also inhibiting growth towards the formation of two branches. The remaining rules in Figure 5 (276, 519, 303, and 546) are unaffected by mode memory: they extinguish at $T = 3$, as in the ahistoric model. But these rules show a rich dynamic in the mean memory model.[19] This activation

[19]These quiescent parity ($\beta_0 = \beta_2 = \beta_4 = \beta_6 = 0$) rules have $\beta_1 = 2$, thus their evolution is truncated at $T = 3$ in the ahistoric model. In the mode memory model the truncation happens after: $\begin{matrix} 1 & & \rightarrow 222 \\ \downarrow & \uparrow & \downarrow \\ 1 & \rightarrow & 222 \end{matrix}$. Extinction is avoided in the mean memory model as the featuring after $T = 2$ is: $\begin{matrix} 1 & & \rightarrow 222 \\ \downarrow & \uparrow & \downarrow \\ 1 & \rightarrow & 121 \end{matrix}$.

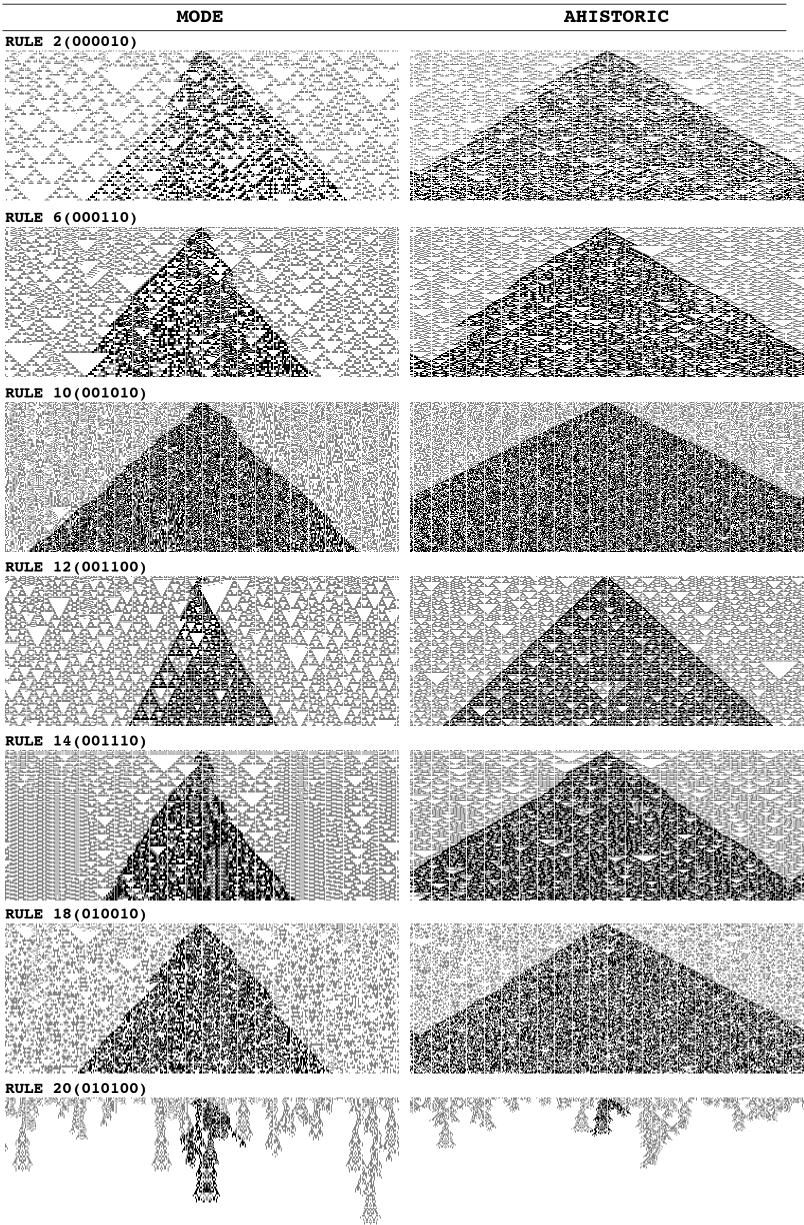

Figure 4. Evolution of the totalistic $k = r = 2$ quiescent rules significantly affected by the mode memory model in the scenario of Figure 2(a).

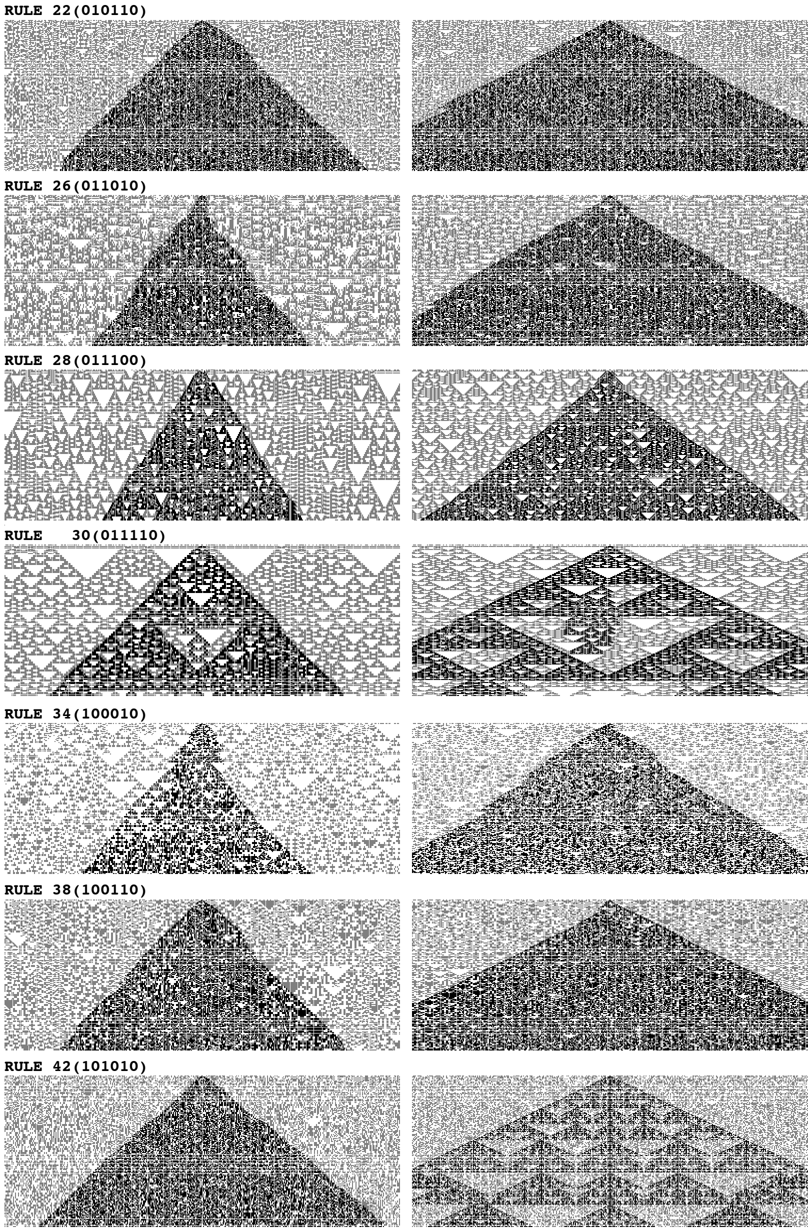

Figure 4. (*continued*)

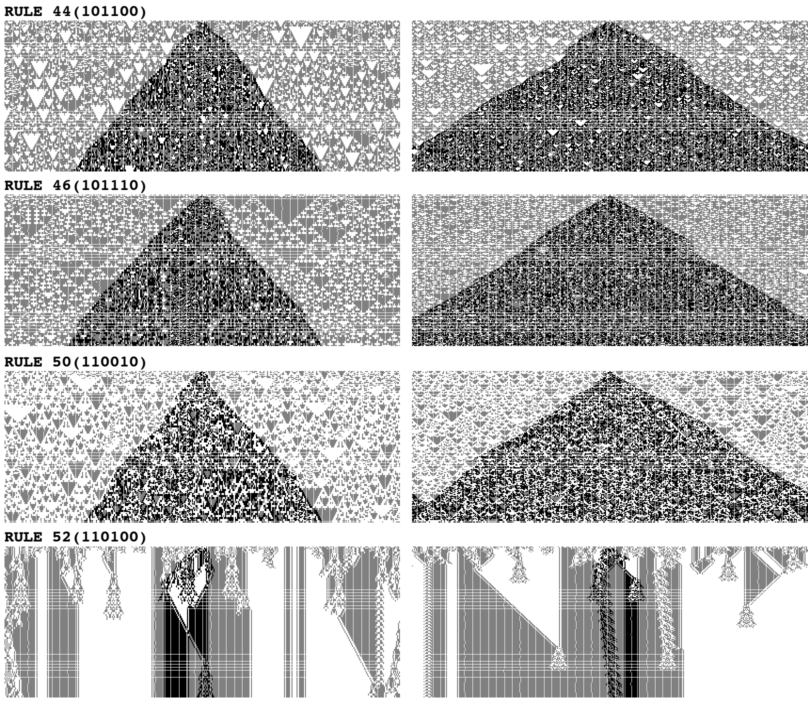

Figure 4. (*continued*)

under (mean) memory of rules that die at $T = 3$ in the ahistoric model is infeasible in the $k = 2$ scenario. Figure 6 shows the evolving patterns of the $k = 3$ parity rules starting from the same initial configuration with values chosen at random as 0, 1, or 2 with probability 1/3. Again mode and mean memory models are considered. The similitudes in the evolving patterns starting from a single seed in Figure 5 are qualitatively reflected in Figure 6. With mode memory the evolving patterns of Rules 273 and 516 and of Rules 276 and 519 are similar, the patterns of Rules 300 and 543 are distinctive, and Rules 276, 303, 519, and 546 are unaffected or minimally affected by mode memory. All the rules in Figure 6 show a rich dynamic in the mean memory model, so no rule with memory dies out and mean memory fires the pattern of the rules that die out (or nearly so) in the ahistoric model.

5. Other short-range memories and contexts

A natural extension to the memory mechanism considered here results in increasing the length of the trailing memory (capacity or depth τ) to beyond the minimal distinctive $\tau = 3$ in $k = 2$. Thus, $s_i^{(T)} =$ mode $(\sigma_i^{(T-\tau+1)}, \ldots, \sigma_i^{(T-1)}, \sigma_i^{(T)})$, adopts the mode criterion into the initial iter-

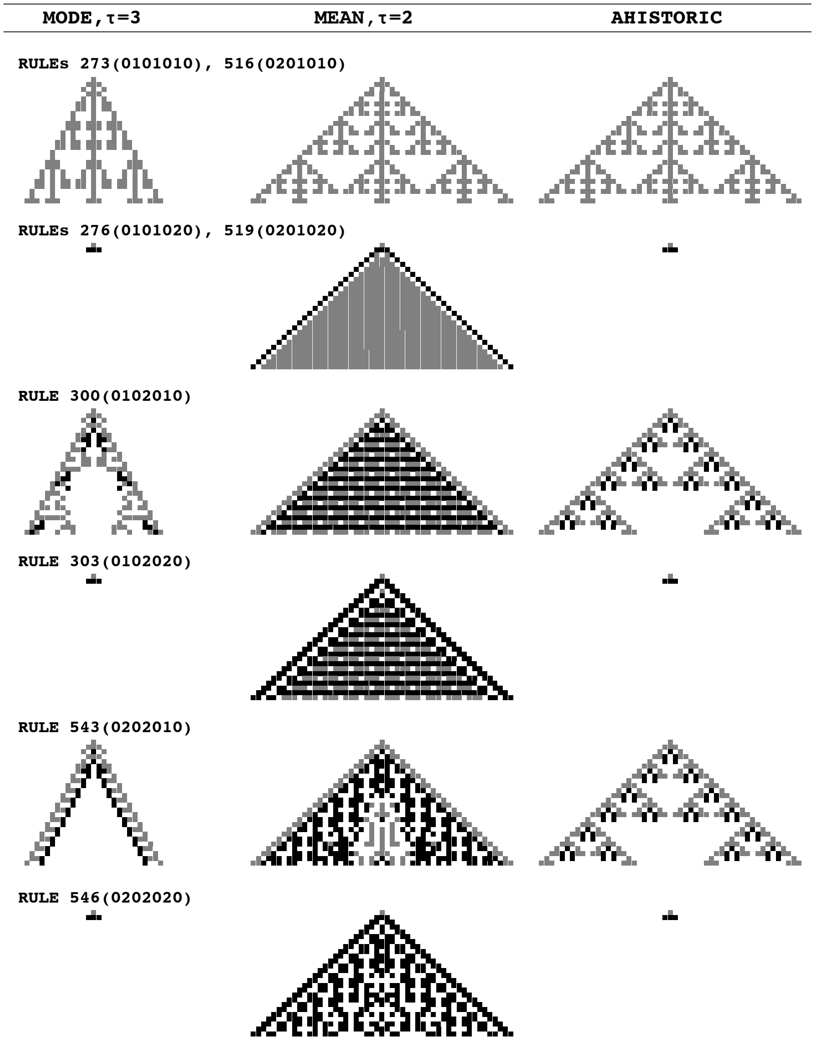

Figure 5. Evolving patterns of the parity totalistic, $k = 3$, $r = 1$ rules from a single site seed with $\sigma = 1$. Evolution up to 26 time steps. Ahistoric, mode of the three-last-states, and mean memory models.

ations when $T < \tau$. To keep short-range memory, we consider $\tau = 4$: $s_i^{(T)} = \text{mode}(\sigma_i^{(T-3)}, \sigma_i^{(T-2)}, \sigma_i^{(T-1)}, \sigma_i^{(T)})$. In this scenario, of two states and τ even, the mode criterion can lead to an ambiguity: the number of time steps in which $\sigma = 0$ can be equal to the number of steps in which $\sigma = 1$. In such a case, the cell will be featured by its last state.[20] In

[20]This problem already appeared in the mode-3 model after $T = 2$. The indefinition was solved there as here, by assigning the last state: $s_i^{(2)} = \sigma_i^{(2)}$.

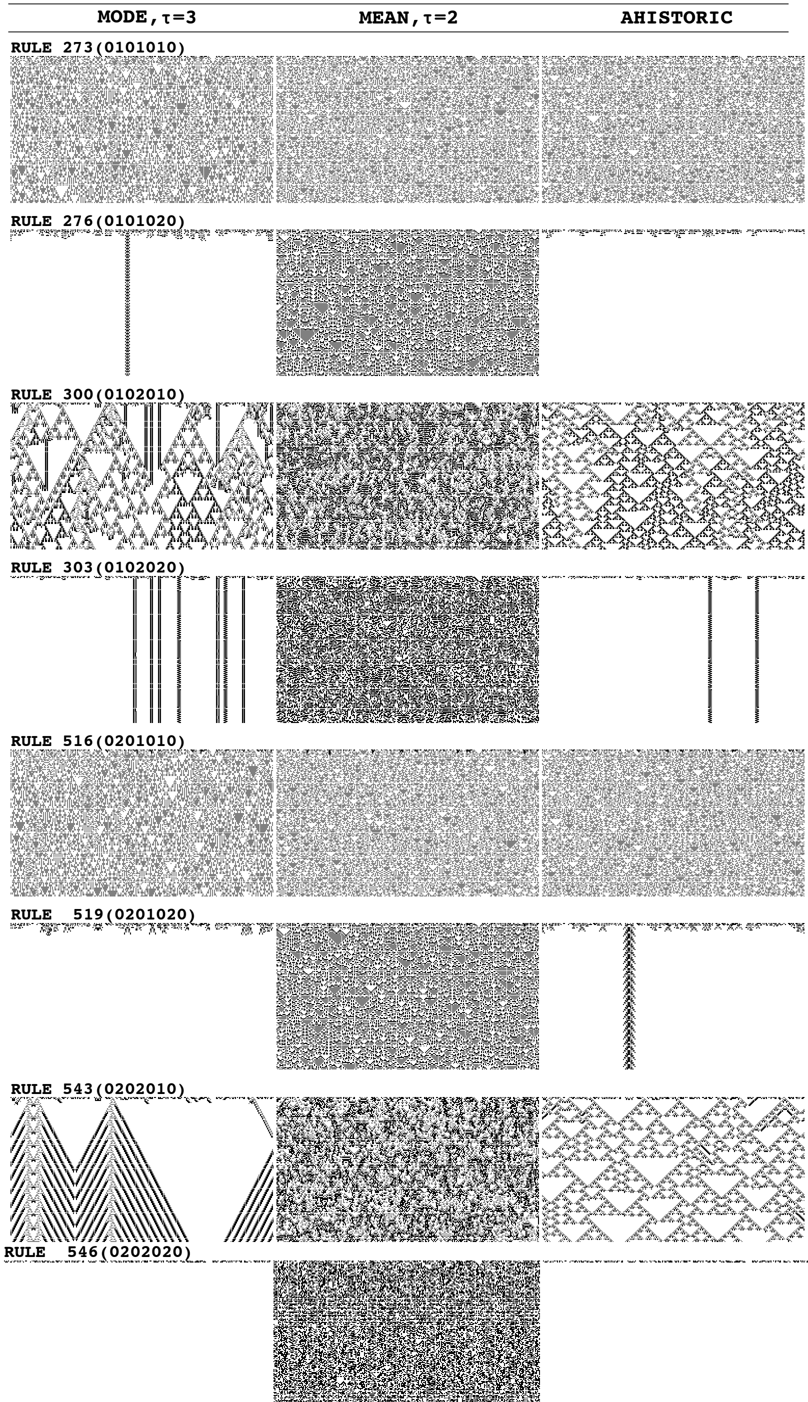

Figure 6. Evolving patterns of the parity totalistic, $k = 3$, $r = 1$ rules starting with the same initial configuration with values chosen at random as 0, 1, or 2 with probability 1/3. Evolution up to 150 time steps in a lattice with 253 sites. Periodic boundary conditions are imposed on the edges. Ahistoric, mode of the three-last-states, and mean memory models.

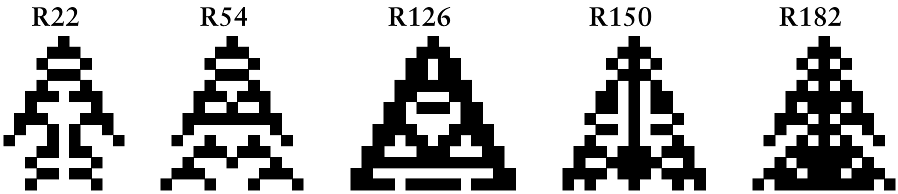

Table 3. Legal rules affected by $\tau = 4$ mode memory in a different way from that of Figure 1(a). Evolving patterns from a single site seed up to $T = 14$.

this way, Rule 254 evolves in the $\tau = 4$ scenario as it does when $\tau = 3$ (Table 1): the outer live cells are featured as live at even time steps after two living and two dead steps. Table 3 shows the legal elementary rules affected by mode memory of depth $\tau = 4$ in a different way from the $\tau = 3$ scenario. The most remarkable case may be that of Rule 22 which does not bifurcate when $\tau = 4$ as it does when $\tau = 3$ (Figure 1(a)).

Historic memory can be weighted by applying δ_t to the state at time step t, so that:

$$m_i^{(T)}(\sigma_i^{(T-\tau+1)}, \ldots, \sigma_i^{(T-1)}, \sigma_i^{(T)}) = \frac{1}{\Delta(T)} \sum_{t=\top}^{T} \delta_t \sigma_i^{(t)},$$

$$\Delta(T) = \sum_{t=\top}^{T} \delta_t,$$

with $\top = \max(1, T - \tau + 1)$.[21] Provided that states are coded as 0 and 1, the weighted mean state (s) of a given cell is obtained by comparing its *memory charge* $m_i^{(T)} = \omega_i^{(T)}/\Delta(T)$ to 0.5, or, to facilitate computing, by comparing $\omega_i^{(T)}$ to $1/2\Delta(T)$ so that:

$$s_i^{(T)} = \begin{cases} 1 & \text{if} \quad \omega_i^{(T)} > \dfrac{\Delta(T)}{2} \\ \sigma_i^{(T)} & \text{if} \quad \omega_i^{(T)} = \dfrac{\Delta(T)}{2} \\ 0 & \text{if} \quad \omega_i^{(T)} < \dfrac{\Delta(T)}{2} \end{cases}$$

Thus, for example, increasing the value of the weighting factor can be implemented in an *exponential* way as $\delta_t = e^{-\beta(T-t)}$, with β ranging in $\mathbb{R}^+$; or in a *geometric* form with $\delta_t = \alpha^{T-t}$, using $0 \leq \alpha \leq 1$ as the memory factor. These mechanisms fully ponder the last round ($e^0 = 1$, $\alpha^0 = 1$), and tend to forget the older rounds. We have analyzed the effect of the latter model in the unlimited trailing memory scenario.[22]

[21] $\top = 1$ in the unlimited trailing memory scenario.

[22] Not only on conventional CA (e.g., [7]) but also on the spatial formulation of the Prisoner's Dilemma game [12]. This weighting is also adopted by Oprisan [17].

$\sigma(T)$	$\omega(T)$	$\Delta(T)/2$	$s(T)$
■	1.000	0.500	□
■ ■	1.000 0.600 1.000	0.800	□ □
■ ■	1.000 0.600 0.360 0.600 1.000	0.980	□ □
■ ■ ■ ■	1.000 0.600 1.360 0.216 1.360 0.600 1.000	1.088	□ □
■ ■	0.600 1.360 0.816 0.816 1.360 0.600	1.088	□ □
■ ■ ■ ■	1.360 0.816 1.360 1.360 0.816 1.360	1.088	□ □ □ □
■ ■	1.000 0.816 0.360 0.816 0.816 0.360 0.816 1.000	1.088	

Table 4. Evolving pattern of Rule 90 (01011010) with memory of depth $\tau = 4$ and discount factor $\alpha = 0.6$ starting from a single site live cell up to its extinction at $T = 8$. Live cells are noted as squares.

In the two-state scenario, for $\tau = 3$,[23] if $\alpha > 0.61805$[24] the memory mechanism becomes that of selecting the mode of the last three states. In the $\tau = 4$ scenario, the memory factor α must be at least 0.5437 to have an effect on the evolving patterns; if so, a cell with state series 0001 will be featured as 0 instead of 1.[25] This is the case of the two outer live cells after $T = 4$ in Table 4, which marks the first difference in the evolution of Rule 90 from its ahistoric scenario. After $T = 7$, all the cells, including the two live cells at this time step, are featured as dead, so the pattern dies.

In order to work only with integers (*á la* CA) previous states can be pondered with the weight $\delta_t = t^c$,[26] where $c \in \mathbb{N}$ is a free parameter. An integer-based implementation is achieved by comparing $2\omega^{(T)}$ to $\Delta(T)$. For $c = 0$, the mode memory model is recovered; for $c = 1$ it is $\Delta(T) = \sum_{t=1}^{T} t = T(T+1)/2$. The larger the value of c, the more heavily the recent past is taken into account, and is consequently closer to the ahistoric scenario. Arithmetic with integers has a clear computational advantage of the model with weights $\delta_t = t^c$ over the geometric or exponential weights just considered. Nevertheless, the weight t^c has a serious drawback: it numerically "explodes" at high values of t, even for $c = 2$. Merely limiting the trailing memory as $\omega_i^{(T)} = \sum_{t=\top}^{T} t^c \sigma_i^{(t)}$ does not avoid the explosion of t^c. This can achieved by translating the first τ weights ($\delta_t = t^c, t \leq \tau$) to the further assignation of the weighting

[23] Recall that in the $k = 2$ context, τ must be at least three in order for memory to have an effect. In the $k = 3$ scenario, memory can be as little as two, that is, only the last two states can be effectively weighted, in such a way that cells with transitions $0 \to 2$ are featured as 1 if $\alpha > 1/3$ (see [5]).

[24] So that a cell with state series 001 will be featured as 0 instead of 1:
$$m(0,0,1) = \frac{1}{\alpha^2 + \alpha + 1} \Leftrightarrow m(0,0,1) = 0.5 \Leftrightarrow \alpha = 0.61805.$$

[25] $$m(0,0,0,1) = \frac{1}{\alpha^3 + \alpha^2 + \alpha + 1} \Leftrightarrow m(0,0,0,1) = 0.5 \Leftrightarrow \alpha = 0.5437.$$

[26] See [8] regarding the effect of this memory mechanism in the unlimited trailing memory scenario. Another alternative weight, with the same integer-based property, is c^t. But it is not operative in the two-state scenario. Memory can be effectively implemented using this weight when $k > 2$ (see [5] for $k = 3$).

factors beyond τ. So, for $T > \tau$: $\delta_t = (t - \top + 1)^c$, $\tau \leq t \leq T$. This mechanism for stabilizing the δ_t: $t - \tau + 1 \leq t \leq T$, stabilizes the value of $\Delta(T) = \sum_{t=1}^{\tau} t^c$, $T \geq \tau$. The effect of memory with weights $\delta_t = t^c$ is illustrated in Table 5. The evolving patterns of Rule 90 are shown in the $c = 1$ scenario, in the unlimited trailing memory scenario ($\delta_t = t$), and when the trailing memory is limited to $\tau = 4$. In the latter case, the initial four weights $1, 2, 3, 4$ are kept to the rest of the process, so its sum $\Delta(T)$ remains equal to 10 from $T = 4$. Starting from a single site live cell, Rule 90 evolves in the $c = 1$, $\tau = 4$ scenario (Table 5) as in the $\alpha = 0.6$, $\tau = 4$ scenario (Table 4). Again, the first difference in the evolution of Rule 90 compared to the ahistoric scenario is given in the characterization of the two outer live cells after $T = 4$ which are featured as dead instead of live.[27] It is remarkable that these memory mechanisms are accumulative in their demand of knowledge of past history, not holistic. In fact, to calculate the memory charge $m_i^{(T)} = \omega_i^{(T)}/\Delta(T)$ even in the unlimited trailing memory scenario, it is not necessary to know the whole $\{\sigma^{(t)}\}$ series but, to (sequentially) calculate $\omega_i^{(T)}$ it suffices to accumulate the contribution of the last state to the already accumulated $\omega_i^{(T-1)}$. Thus, with $\delta_t = \alpha^{T-t}$ it is $\omega_i^{(T)} = \alpha\omega_i^{(T-1)} + \sigma_i^{(T)}$, and with $\delta_t = t^c$ it is $\omega_i^{(T)} = \omega_i^{(T-1)} + T^c$.

The standard CA and CA with memory models can be combined by considering two types of cell characterization of the neighborhood $\mathcal{N}$. Thus for a subset of $\mathcal{N}$, let it be $\mathcal{S}$, the cells can be featured by their last state and the remaining $\overline{\mathcal{S}} = \mathcal{N} - \mathcal{S}$ by their summary states. For example, history can feature the cells of the strict neighborhood but not the cell to be updated: $\sigma_i^{(T+1)} = \phi(s_{i-1}^{(T)}, \sigma_i^{(T)}, s_{i+1}^{(T)})$; or, the contrary, the cell to be updated but not the outer cells: $\sigma^{(T+1)} = \phi(\sigma_{i-1}^{(T)}, s_i^{(T)}, \sigma_{i+1}^{(T)})$. Table 6 shows some of the legal rules affected by limiting the trailing memory in the former scenario.

Departing from the weighted mean memory scheme, cells can be featured (i) by the *parity* with the sum of previous states: $s_i^{(T)} = (\sum_{t=\top}^{T} \sigma_i^{(t)}) \bmod 2$, ($ii$) by the *interest* of the previous series: a cell is featured as dead if all its previous states are equal (zero interest means boring), and as alive if any of them are different: $s_i^{(T)}(\sigma_i^{(1)}, \sigma_i^{(2)}, \ldots, \sigma_i^{(T)}) = 0$ iff $\sigma_i^{(1)} = \sigma_i^{(2)} = \cdots = \sigma_i^{(T)}$,[28] ($iii$) by some form of the Heaviside step

[27]Their state series 0001 implies $\omega^{(4)} = 4$ so $2\omega^{(4)} < \Delta(4) = 10 \Leftrightarrow s = 0$. It is remarkable that this change in the cell featuring is not activated after $T = 3$: $001 \Rightarrow \omega^{(3)} = 3$ so $2\omega^{(4)} = \Delta(4) = 6 \Rightarrow s = 1$ (recall that in case of a tie the last state features the cell). In fact the weight t^c is not operative in the $\tau = 3$ scenario regardless of the $c > 0$ chosen: $\omega^{(3)} = 3^c$ and, provided that $c > 1$, $\Delta^{(3)} = 1 + 2^c + 3^c$ so $2\omega^{(3)}\ 3^c + 3^c > 1 + 2^c + 3^c$ because $3^c = (1+2)^c = 1 + \sum_{j=1}^{c-1} \binom{c}{j} 2^{c-j} + 2^c 1 + 2^c$.

[28]Toffoli and Margolus [19] use the "interest" idea when considering the time-tunnel, a reversible mechanism based on the two-dimensional rule: $\phi(\sigma_j^{(T)} \in \mathcal{N}_i) = 0$ if all the states in $\mathcal{N}$ are equal, and 1 in the contrary case.

Unlimited Trailing Memory

$\sigma(T)$	$2\omega(T)$	$\Delta(T)$	$s(T)$
■	2	1	□
■ ■	4 2 4	3	□ □
■ ■	6 4 2 4 6	6	□ □
■ ■ ■ ■	8 6 12 2 12 6 8	10	□ □
■ ■	8 16 12 2 12 16 8	15	□ □
■ ■ ■ ■	20 16 24 2 24 16 20	21	□ □
■ ■	20 30 24 2 24 30 20	28	□ □
■ ■ ■ ■	36 30 40 2 40 30 36	36	□ □ □ □
■ ■	18 36 30 40 2 40 30 36 18	45	

$\tau = 4$

$\sigma(T)$	$2\omega(T)$	$\Delta(T)$	$s(T)$
■	2	1	□
■ ■	4 2 4	3	□ □
■ ■	6 4 2 4 6	6	□ □
■ ■ ■ ■	8 6 12 2 12 6 8	10	□ □
■ ■	6 12 8 8 12 6	10	□ □
■ ■ ■ ■	12 8 12 12 8 12	10	□ □ □ □
■ ■	8 8 4 8 8 4 8 8	10	

Table 5. Evolving pattern of Rule 90 (01011010) with memory based on weights $\delta_t = t$. Evolution from a single site live cell. Live cells are noted as squares.

Rule 22 (00010110)		Rule 54 (0011010)		Rule 150 (10010110)	
UTM	$\tau = 3$	UTM	$\tau = 3$	UTM	$\tau = 3$

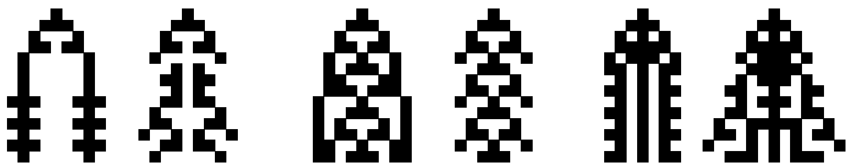

Table 6. Evolving patterns of some legal rules in the mode memory scenario with no memory of the central cell. Unlimited trailing memory (UTM) and trailing memory of the last-three-states scenarios. Evolution up to $T = 14$.

function $s_i^{(T)} = \mathcal{H}(\sum_{t=\top}^{T} \sigma_i^{(t)} - \theta)$.[29] These memory mechanisms admit an operative implementation just with $\tau = 2$: (i) $s_i^{(T)} = (\sigma_i^{(t-1)} + \sigma_i^{(t)}) \bmod 2$ (cells featured by the parity of the sum of their two latest states), (ii) $\sigma_i^{(T)} = 1$ if $\sigma_i^{(T-1)} \neq \sigma_i^{(T)}$ (a cell will be featured as live if its two latest states are different), (iii) $s_i^{(T)} = \mathcal{H}(\sigma_i^{(t-1)} + \sigma_i^{(t)} - 1)$ (a cell will be featured as live if at least in one of its two latest states it was alive). Ta-

[29]A study of the (neural) automaton $x_{t+1} = \mathcal{H}(\sum_{t=\top}^{T} \delta_t x_t - \theta)$ is made in [18]. In the paper by Layman [20], the weights (coupling factors) of the also neural (but unlimited) type of memory are $\delta_t = K^{T-t}$, $0 < K < 1$.

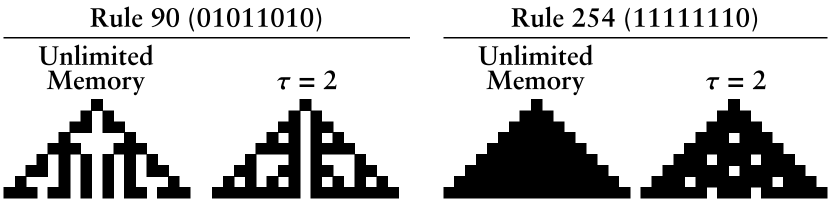

Table 7. Rules 90 and 254 in the parity memory scenario. Evolution from a single site live cell up to $T = 9$.

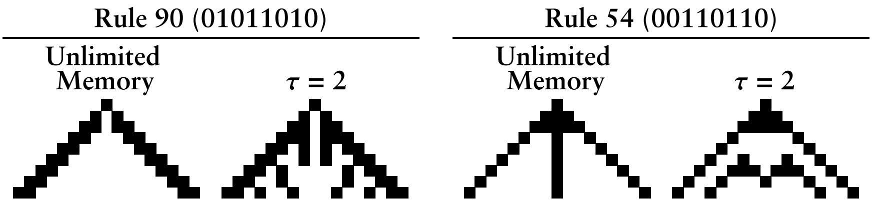

Table 8. Rules 90 and 54 in the interest memory scenario. Evolution from a single site live cell up to $T = 9$.

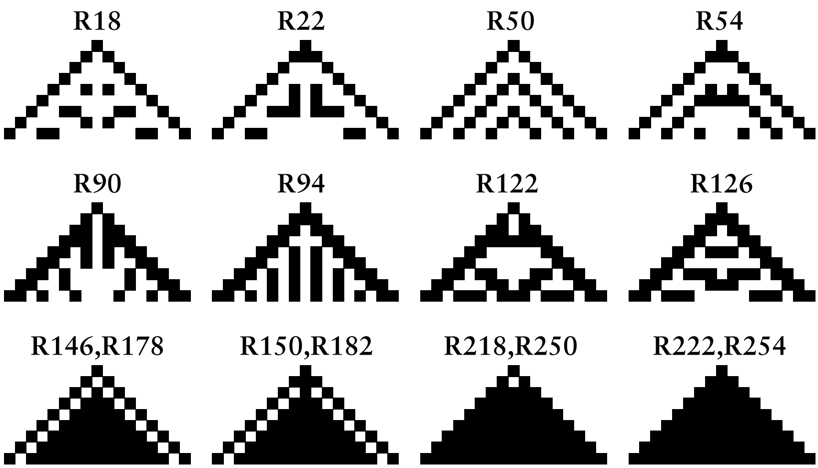

Table 9. Evolving patterns of the legal rules when featuring cells as: $s_i^{(T)} = \mathcal{H}(\sigma_i^{(T-1)} + \sigma_i^{(T)} - 1)$. Evolution up to $T = 9$ from a single site live cell.

bles 7 through 9 show the effect of these memory mechanisms with the initial assignment $s_i^{(1)} = \sigma_i^{(1)}$. (Rule 254 is unaffected by any limiting of the trailing memory in the interest memory scenario.) It is remarkable that the border of the patterns in these tables progresses at the speed of light, only the inside varies.

A simple (totalistic) example of memory in the rule transition can be implemented by assigning the parity of the sum of the two latest neighbor

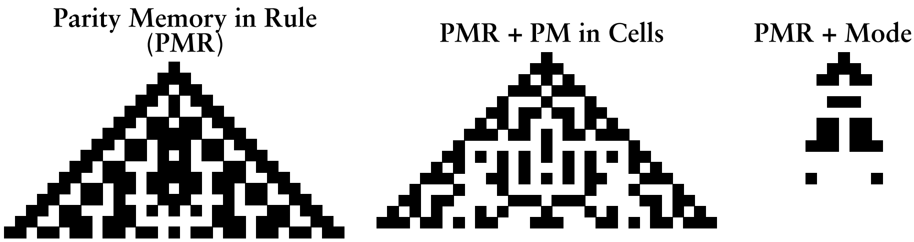

Table 10. Evolving patterns with only parity of the last two neighborhoods memory in the rule (left), and also with memory in cells: parity of the two last states (center) and mode of the three-last-states (right). Evolution up to $T = 16$ from a single site live cell.

values: $\sigma_i^{(T+1)} = (\sigma_{i-1}^{(T)} + \sigma_i^{(T)} + \sigma_{i+1}^{(T)} + \sigma_{i-1}^{(T-1)} + \sigma_i^{(T-1)} + \sigma_{i+1}^{(T-1)}) \bmod 2$. Table 10 shows an example of the dynamics of this rule with the initial assignment $\sigma_i^{(2)} = (\sigma_{i-1}^{(1)} + \sigma_i^{(1)} + \sigma_{i+1}^{(1)}) \bmod 2$. If memory in cells is also embedded in such a scenario, the rule becomes: $\sigma_i^{(T+1)} = (s_{i-1}^{(T)} + s_i^{(T)} + s_{i+1}^{(T)} + s_{i-1}^{(T-1)} + s_i^{(T-1)} + s_{i+1}^{(T-1)}) \bmod 2$. In Table 10 two types of memory are embedded in cells: the parity of the two last states and the mode of the last three states. In the latter scenario, the pattern extinguishes soon (at $T = 13$) after four *cataleptic* (i.e., no cell alive) episodes.

Historic memory can be embedded in continuous-valued CA in which the state variable ranges in $\mathbb{R}$, just by considering m instead of s when applying the update rule: $\sigma_i^{(T+1)} = \Phi(m_j^{(T)} \in \mathcal{N}_i)$, where Φ is a continuous function. In the very simple case of a coupled map lattice (CML), the new state level of a cell can be the average of its own state level and the states of its neighbors: $\sigma_i^{(T+1)} = 1/3(\sigma_{i-1}^{(T)} + \sigma_i^{(T)} + \sigma_{i+1}^{(T)})$, which becomes with memory: $\sigma_i^{(T+1)} = 1/3(m_{i-1}^{(T)} + m_i^{(T)} + m_{i+1}^{(T)})$. Memory can be implemented in this way in *fuzzy* CA, a sort of continuous CA with states ranging in the real [0, 1] interval. Thus, for example, the fuzzified Rule 90: $\sigma_i^{(T+1)} = \sigma_{i-1}^{(T)} + \sigma_{i+1}^{(T)} - 2\sigma_{i-1}^{(T)}\sigma_{i+1}^{(T)}$ becomes $\sigma_i^{(T+1)} = m_{i-1}^{(T)} + m_{i+1}^{(T)} - 2m_{i-1}^{(T)}m_{i+1}^{(T)}$ incorporating memory.[30] Last but not least, memory can be embedded in quantum CA. Thus, for example, the simple one-dimensional quantum models introduced by Grössing and Zeilinger [21] in which the complex amplitudes vary as: $\sigma_j^{(T+1)} = 1/N^{1/2}(i\rho\sigma_{j-1}^{(T)} + \sigma_j^{(T)} + i\rho^*\sigma_{j+1}^{(T)})$ would become with memory: $\sigma_j^{(T+1)} = 1/N^{1/2}(i\rho m_{j-1}^{(T)} + m_j^{(T)} + i\rho^* m_{j+1}^{(T)})$. Again these memory models admit an operative implementation just with $\tau = 2$: $m_i^{(T)} = 1/(\delta_1 + \delta_2)\,(\delta_2\sigma_i^{(t-1)} + \delta_1\sigma_i^{(t)})$, as in Tables 11 through 13, in which the *mixing factors* are $\delta_1 = \delta_2 = 1$, so $m_i^{(T)}$ is the arithmetic mean of the two last state values.

[30] An illustration of the effect of unlimited trailing memory on the fuzzified Rule 90 is given in [9].

Ahistoric

								.01	.01
						.01	.01	.02	.02
				.01	.02	.03	.04	.04	.04
			.04	.05	.06	.07	.07	.08	.08
		.11	.11	.12	.12	.12	.12	.12	.12
	.33	.22	.22	.20	.19	.17	.16	.15	.15
1.0	.33	.33	.26	.23	.21	.19	.18	.17	.16
	.33	.22	.22	.20	.19	.17	.16	.15	.15
		.11	.11	.12	.12	.12	.12	.12	.12
			.04	.05	.06	.07	.07	.08	.08
				.01	.02	.03	.04	.04	.04
						.01	.01	.02	.02
								.01	.01

Historic ($T > 2$)

						.01	.01
			.01	.01	.02	.02	.03
	.01	.03	.04	.05	.06	.06	.07
.06	.11	.11	.12	.12	.12	.12	.12
.28	.22	.22	.21	.20	.19	.18	.17
.33	.31	.27	.25	.23	.21	.20	.19
.28	.22	.22	.21	.20	.19	.18	.17
.06	.11	.11	.12	.12	.12	.12	.12
	.01	.03	.04	.05	.06	.06	.07
			.01	.01	.02	.02	.03
						.01	.01

Table 11. Evolving patterns of the CML: $\sigma_i^{(T+1)} = 1/3(m_{i-1}^{(T)} + m_i^{(T)} + m_{i+1}^{(T)})$ up to $T = 10$ from a single site live cell. In the historic formulation m is the mean value of the two last states; in the ahistoric model m is replaced by σ.

									1.0									
								1.0		1.0								
							.50	.50	.50	.50	.50							
						.25	.75	.38	.38	.38	.75	.25						
					.13	.63	.45	.52	.49	.52	.45	.63	.13					
				.06	.44	.59	.49	.51	.49	.51	.49	.59	.44	.06				
			.03	.28	.60	.49	.50	.50	.50	.50	.50	.49	.60	.28	.03			
		.02	.17	.52	.53	.50	.50	.50	.50	.50	.50	.50	.53	.52	.17	.02		
	.01	.10	.40	.55	.50	.50	.50	.50	.50	.50	.50	.50	.50	.55	.40	.10	.01	
	.06	.29	.53	.51	.50	.50	.50	.50	.50	.50	.50	.50	.50	.51	.53	.29	.06	
.03	.20	.47	.52	.50	.50	.50	.50	.50	.50	.50	.50	.50	.50	.50	.52	.47	.20	.03

Table 12. Evolving patterns of the historic formulation of the fuzzified Rule 90: $\sigma_i^{(T+1)} = (m_{i-1}^{(T)} + m_{i+1}^{(T)} - 2m_{i-1}^{(T)}m_{i+1}^{(T)})$ up to $T = 11$ from a single site crisp cell; m is the mean value of the two last states.

Ahistoric

							.01		.03
				.01		.04		.06	
			.05		.10		.12	.01	.14
		.17		.23		.24	.01	.23	.01
	.50		.45	.01	.39	.01	.34	.01	.31
1.0		.66		.51	.01	.42	.01	.36	.02
	.50		.45	.01	.39	.01	.34	.01	.31
		.17		.23		.24	.01	.23	.01
			.05		.10		.12	.01	.14
				.01		.04		.06	
							.01		.03

Historic $T > 2$

			.01	.01		.03	.03
	.01	.05	.02	.02	.10	.04	.02
.10	.14		.14	.20	.01	.14	.22
.22	.07	.43	.18	.04	.38	.18	.03
.36	.56	.03	.30	.45	.02	.23	.39
.22	.07	.43	.18	.04	.38	.18	.03
.10	.14		.14	.20	.01	.14	.22
	.01	.05	.02	.02	.10	.04	.02
			.01	.01		.03	.03

Table 13. Evolving patterns of the probabilities $|\sigma_i|^2$ of the quantum CA described in the text with $\rho = 20$ up to $T = 11$ from a single site live cell. In the historic formulation, m is the mean value of the two last states; in the ahistoric model m is replaced by σ.

In probabilistic CA (PCA), deterministic state-transitions are replaced by specifications of the probabilities of cell-value assignments. That is to say, the β values in rules ϕ are replaced by probabilities: prob($\sigma_i^{(T+1)} = 1/\sigma_j^{(T)} \in \mathcal{N}_i$). Memory can readily be embedded in PCA by considering prob($\sigma_i^{(T+1)} = 1/s_j^{(T)} \in \mathcal{N}_i$). We have explored the effect of memory in PCA in [3].

6. Conclusions

This paper introduces a kind of cellular automata (CA) with minimal memory in cells ($\mathbb{CAMM}$), and surveys its properties in a fairly qualitative (pictorial) form by comparing the spatio-temporal and difference patterns with and without memory. As a rule, CA with minimal memory show spatio-temporal patterns that preserve the main features of the ahistoric ones, and the region damaged due to the reversion of a single site is depleted by memory. Nevertheless, these overall effects are not universal, so some rules (no totalistic ones in particular) are notably altered by minimal memory.

A more complete analysis of the class $\mathbb{CAMM}$ is left for future work. For example, the dynamics of $\mathbb{CAMM}$ in the state (phase) space or the potential fractal features are to come under scrutiny [22]. Another interesting open question is the analysis of the effect of memory on solving computational tasks such as synchronization, density, and ordering. The study of two-dimensional and probabilistic $\mathbb{CAMM}$ will follow. Embedding memory in states broadens the spectrum of CA as a tool for modeling. It is likely that in some contexts, a transition rule with memory could match the "correct" behavior of the CA system of a given complex system (e.g., physical, biological, or social). A major impediment in modeling with CA stems from the difficulty of utilizing their complex behavior to exhibit a particular behavior or perform a particular function. Memory in CA tends to inhibit complexity, inhibition that can be modulated by varying the memory depth. This could mean a potential advantage of $\mathbb{CAMM}$ over standard CA as a tool for modeling slow diffusive growth from small regions, a common phenomenon in nature.

The question of how errors spread and propagate in cooperative systems has been studied in a variety of fields. Given the difficulty of creating analytical models for any but the simplest systems, most investigations have been conducted by computer simulations, especially in the area of statistical physics of many-body systems. Errors reverse the state in $k = 2$ CA. In $\mathbb{CAMM}$, the damage induced by a single error tends to be confined to the proximity of the site where it occurred. From this point of view, $\mathbb{CAMM}$ can be featured as resilient in the face of errors. Fault tolerance is an important issue when considering systems with a large number of components, in which faults will be highly probable. The robust $\mathbb{CAMM}$ could play a role in nanotechnological scenarios.

Let us conclude this work by pointing out the possibility of incorporating (minimal) memory into *discrete dynamical systems*: $x_{n+1} = f(x_n)$ by means of $x_{n+1} = f(m_n)$ with

$$m_n(x_{n-1}, x_n) = \frac{1}{\alpha + 1}(\alpha x_{n-1} + x_n), \qquad 0 \leq \alpha \leq 1.$$

Figure 7 deals with the canonical example, the logistic map: $x_{n+1} = x_n + 3.0 x_n(1 - x_n)$.[31] A low level of memory ($\alpha = 0.1$) smooths the characteristic chaotic dynamics of the standard (ahistoric, $\alpha = 0$) formulation; α in the $[0.2, 0.3]$ interval generates period-two oscillators, and α in $[0.4, 0.8]$ leads to a fixed point ($x_n \to 1$). Unexpectedly, period-two oscillators of broad amplitude appear again for $\alpha \geq 0.9$. Anyway, we transit from chaos to order by varying not the parameter of the model ($\lambda = 3.0$ in Figure 7) but the degree of memory incorporated.

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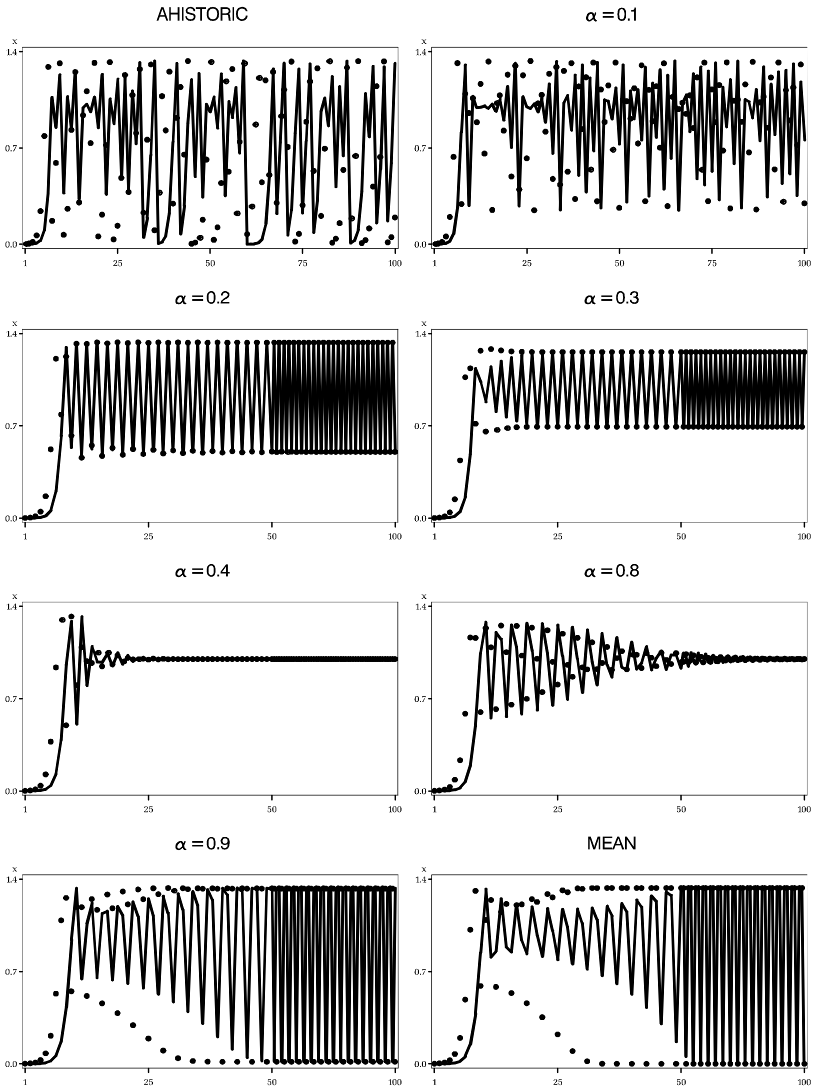

Figure 7. The effect of memory on the logistic map with memory (m_n): $x_{n+1} = m_n + 3m_n(1 - m_n)$ up to $T = 100$. Two simulations are plotted: the one starting with $x_1 = 0.0001$ is shown with its unmarked values joined; the values of the simulation starting with $x_1 = 0.001$ are unjoined and marked with dots. In order to enhance the first iterations of nonchaotic graphics the abscissas (t) of the simulations with $\alpha > 0.1$ are evenly spaced.

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