Dynamics of the Cellular Automaton Rule 142

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We investigate dynamics of the cellular automaton Rule 142. This rule possesses additive invariant of the second order, namely it conserves the number of blocks "10". Rule 142 can be alternatively described as an operation on a binary string in which we simultaneously flip all symbols which have dissenting right neighbors. We show that the probability of having a dissenting neighbor can be computed exactly using the fact that the surjective Rule 60 transforms Rule 142 into Rule 226. We also demonstrate that the conservation of the number of 10 blocks implies that these blocks move with speed −1 or stay in the same place, depending on the state of the preceding site. At the density of blocks 10 equal to 0.25, Rule 142 exhibits a phenomenon similar to the jamming transitions occurring in discrete models of traffic flow.

1. Introduction

Let s be a binary string of L symbols, that is, $s = s_0 s_1 \ldots s_{L-1}$, where $s_i \in \{0, 1\}$ for $0 \leq i \leq L$, $L \in \mathbb{N}$. We will say that the symbol s_i has a *dissenting right neighbor* if $s_{i-1} = s_i \neq s_{i+1}$. By *flipping* a given symbol s_i we will mean replacing it by $1 - s_i$.

Consider now the following problem. Suppose that we simultaneously flip all symbols which have dissenting right neighbors as follows:

$$\begin{array}{ccccccccccccccc} \cdots & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ & & \big\downarrow & & & \big\downarrow & & & \big\downarrow & & & & & \\ \cdots & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & \cdots \end{array} \tag{1}$$

Assuming that the initial string is randomly generated, what is the probability that a given symbol has a dissenting right neighbor after t iterations of the procedure?

In order to answer this question, we will take advantage of the fact that the process described in the previous paragraph is actually cellular automaton Rule 142. It has a property which turns out to be crucial to the solution. If one counts the number of pairs "10" in equation (1) before and after the bit-flipping operation, it is easy to see that this

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number remains constant. We show that this is true for an arbitrary string, and take advantage of this fact to compute the probability of having a dissenting neighbor.

2. Definitions

Before proceeding, we will introduce several concepts of cellular automata theory. Let $\mathcal{G} = \{0, 1\}$ be called a *symbol set*, and $\mathcal{S} = \{0, 1\}^{\mathbb{Z}}$ be called the *configuration space*. A *block of radius* r is an ordered set $b_{-r}b_{-r+1} \dots b_r$ where $r \in \mathbb{N}$ and $b_i \in \mathcal{G}$. Let $r \in \mathbb{N}$ and let $\mathcal{B}_r$ denote the set of all blocks of radius r over $\mathcal{G}$. The number of elements of $\mathcal{B}_r$ (denoted by $\operatorname{card}\mathcal{B}_r$) equals 2^{2r+1}.

A mapping $f : \{0, 1\}^{2r+1} \mapsto \{0, 1\}$ will be called a *cellular automaton rule of radius* r. Alternatively, the function f can be considered a mapping of $\mathcal{B}_r$ into $\mathcal{B}_0 = \mathcal{G} = \{0, 1\}$.

Corresponding to f (also called a *local mapping*), we define a *global mapping* $F : S \to S$ such that $(F(s))_i = f(s_{i-r}, \dots, s_i, \dots, s_{i+r})$ for any $s \in S$. The *composition of two rules* $f, g \in \mathcal{F}$ can now be defined in terms of their corresponding global mappings F and G as $(F \circ G)(s) = F(G(s))$, where $s \in S$. We note that if $f \in \mathcal{F}_p$ and $g \in \mathcal{F}_q$, then $f \circ g \in \mathcal{F}_{p+q}$. For example, the composition of two radius-1 mappings is a radius-2 mapping:

$$
\begin{aligned}
&(f \circ g)(s_{-2}, s_{-1}, s_0, s_1, s_2) = \\
&\qquad f(g(s_{-2}, s_{-1}, s_0), g(s_{-1}, s_0, s_1), g(s_0, s_1, s_2)).
\end{aligned}
\tag{2}
$$

Multiple composition will be denoted by

$$
f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}. \tag{3}
$$

A *block evolution operator* corresponding to f is a mapping $\mathbf{f} : \mathcal{B} \mapsto \mathcal{B}$ defined as follows. Let $r \geq p > 0$, $a \in \mathcal{B}_r$, $f \in \mathcal{F}_p$, and let $b_i = f(a_{i-p}, a_{i-p+1}, \dots, a_{i+p})$ for $-r + p \leq i \leq r - p$. Then we define $\mathbf{f}(a) = b$, where $b \in \mathcal{B}_{r-p}$. Note that if $b \in B_1$ then $f(b) = \mathbf{f}(b)$.

In this paper, we are concerned with trajectories of a given configuration under consecutive iterations of F. Denoting the initial configuration by $s(0)$, the image of $s(0)$ after t iterations of F will be denoted by $s(t)$, that is,

$$
s(t) = F^t(s(0)), \tag{4}
$$

which implies that

$$
s(t + 1) = F(s(t)), \tag{5}
$$

and hence

$$
s_i(t + 1) = f(s_{i-r}(t), s_{i-r+1}(t), \dots, s_{i+r}(t)). \tag{6}
$$

Cellular automaton Rule 142, which is the subject of this paper, has the following local function

$$\begin{aligned}
&f(0,0,0)=0,\ f(0,0,1)=1,\ f(0,1,0)=1,\ f(0,1,1)=1,\\
&f(1,0,0)=0,\ f(1,0,1)=0,\ f(1,1,0)=0,\ f(1,1,1)=1,
\end{aligned} \tag{7}$$

which can also be written in an algebraic form

$$f(x_0,x_1,x_2)=x_1+(1-x_0)(1-x_1)x_2-x_0x_1(1-x_2). \tag{8}$$

3. Conservation

As shown in [1], Rule 142 is one of the few nontrivial elementary rules which posses the second order additive invariant. It conserves the number of blocks 10, and this fact can be formally described as follows. Let us first define a function $\xi(x_0,x_1)=x_0(1-x_1)$, which takes value 1 on block 10 and value 0 on all other blocks of length 2. We will call ξ the *density* of blocks 10. We will say that ξ is a density function of an additive invariant of f if

$$\sum_{i=0}^{L-1}\xi(f(s_i,s_{i+1},s_{i+2}),f(s_{i+1},s_{i+2},s_{i+3}))=\sum_{i=0}^{L-1}\xi(s_i,s_{i+1}) \tag{9}$$

for every positive integer L and for all $s_0,s_1,\ldots,s_{L-1}\in\{0,1\}$. In equation (9), and in all subsequent considerations, we assume that addition of all spatial indices is performed *modulo L*. That is, we will be concerned with periodic configurations, or, in other words, configurations with periodic boundary conditions where $s(i+L)=s(i)$ for all integers i.

The right-hand side of equation (9) simply denotes the number of blocks 10 in the configuration $s=(s_0,s_1,\ldots,s_{L-1})$, and the left-hand side denotes the number of these blocks in the image of s under f. Figure 1 shows an example of a configuration s consisting of 11 sites, and its two consecutive images under Rule 142, that is, $F_{142}(s)$ and $F_{142}^2(s)$, where F_{142} denotes the global function of Rule 142. Periodic boundary conditions are assumed. The initial configuration s contains three blocks 10 labeled a, b, and c, and one can clearly see that the number of blocks 10 remains constant after each application of F_{142}. Moreover, since the number of blocks 10 remains constant, we can label them with distinctive labels, allowing us to keep track of individual blocks. For example, looking again at Figure 1, we could say that block a remains in the same position after the first iteration, but moves to the left by one site in the second iteration. Similarly, block b moves by one site to the left in both iterations shown in Figure 1.

To formalize the concept of the motion of blocks, we will first prove that Rule 142 conserves the number of blocks 10 in an arbitrary periodic

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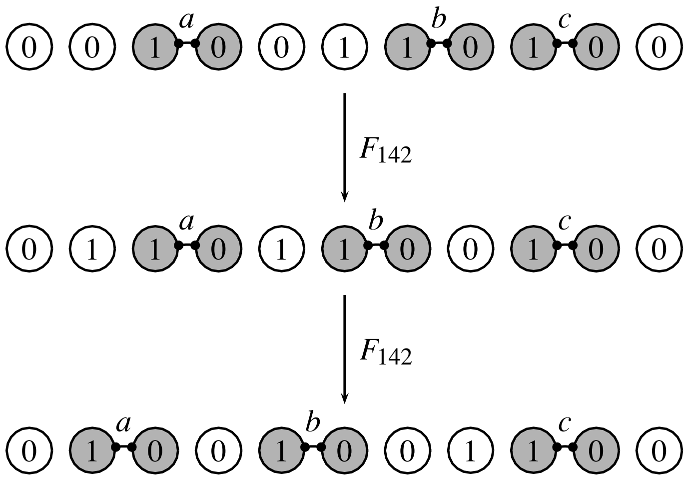

Figure 1. Two consecutive images of a sample configuration under Rule 142, assuming periodic boundary conditions. Blocks 10 are labeled with letters a, b, and c.

configuration. Let us note that

$$\xi(f(s_i, s_{i+1}, s_{i+2}), f(s_{i+1}, s_{i+2}, s_{i+3})) = \\ f(s_i, s_{i+1}, s_{i+2})(1 - f(s_{i+1}, s_{i+2}, s_{i+3})). \tag{10}$$

Using equation (8), the right-hand side of equation (10) becomes somewhat complicated, but it drastically simplifies if one notes that all variables s_i in this equation are boolean, and $x^n = x$ for all positive n if $x \in \{0, 1\}$. After this simplification, one obtains

$$\xi(f(s_i, s_{i+1}, s_{i+2}), f(s_{i+1}, s_{i+2}, s_{i+3})) = \\ - s_i s_{i+1} + s_{i+1} + s_i s_{i+1} s_{i+2} - s_{i+1} s_{i+2} s_{i+3}. \tag{11}$$

We now regroup terms on the right-hand side

$$\xi(f(s_i, s_{i+1}, s_{i+2}), f(s_{i+1}, s_{i+2}, s_{i+3})) = \\ s_{i+1}(1 - s_{i+2}) + s_{i+1} s_{i+2}(1 - s_{i+3}) - s_i s_{i+1}(1 - s_{i+2}), \tag{12}$$

and finally write the last equation as

$$\xi(f(s_i, s_{i+1}, s_{i+2}), f(s_{i+1}, s_{i+2}, s_{i+3})) = \\ \xi(s_{i+1}, s_{i+2}) - J(s_{i+1}, s_{i+2}, s_{i+3}) + J(s_i, s_{i+1}, s_{i+2}), \tag{13}$$

where

$$J(x_0, x_1, x_2) = x_0 x_1 (x_2 - 1). \tag{14}$$

This leads to

$$\sum_{i=0}^{L-1} \xi(f(s_i, s_{i+1}, s_{i+2}), f(s_{i+1}, s_{i+2}, s_{i+3})) =$$
$$\sum_{i=0}^{L-1} \xi(s_{i+1}, s_{i+2}) - \sum_{i=0}^{L-1} J(s_{i+1}, s_{i+2}, s_{i+3}) + \sum_{i=0}^{L-1} J(s_i, s_{i+1}, s_{i+2}), \quad (15)$$

and, since $\sum_{i=0}^{L-1} J(s_{i+1}, s_{i+2}, s_{i+3}) = \sum_{i=0}^{L-1} J(s_i, s_{i+1}, s_{i+2})$, the conservation condition of equation (9) follows.

Equation (13) resembles the continuity equation, with J playing a role of current, or flow of blocks 10. To see this, let us note that $J(s_i(t), s_{i+1}(t), s_{i+2}(t))$ takes a nonzero value only when $s_i(t)$,$s_{i+1}(t)$, $s_{i+2}(t) = 1, 1, 0$. Consider now a configuration containing block 110, surrounded by sites of undetermined state. Using equation (7) to define the local function for Rule 142, we can construct a partial state of the configuration at the next time step. Denoting by $*$ an arbitrary value in the set $\{0, 1\}$, we have $f(*, 1, 1) = 1$, $f(1, 1, 0) = 0$, and $f(1, 0, *) = 0$, hence

$$\begin{aligned} s(t) &= \ldots 110 \ldots \\ s(t+1) &= \ldots 100 \ldots \end{aligned} \quad (16)$$

We can clearly see that the block 10, when preceded by 1, moves by one site to the left in a single iteration. Similar arguments could be used to demonstrate that the block 10 preceded by 0 does not move:

$$\begin{aligned} s(t) &= \ldots 010 \ldots \\ s(t+1) &= \ldots ?10 \ldots \end{aligned} \quad (17)$$

where "?" denotes an undetermined value. We can see that indeed only blocks 110 can contribute to the current, in agreement with equation (14).

4. Initial distribution

Let us now go back to the problem stated in the introduction. In order to make the problem well posed, we need to define the probability distribution μ from which the initial string is drawn. Since we know that Rule 142 conserves the number of blocks 10, it is natural to consider an initial distribution parameterized by the density of blocks 10. Let us define the expected value of ξ at site i as

$$\rho(i, t) = E_\mu\left[\xi(s_i(t), s_{i+1}(t))\right] = E_\mu\left[s_i(t)(1 - s_{i+1}(t))\right]. \quad (18)$$

Assuming that the initial distribution μ is translation-invariant, $\rho(i, t)$ will not depend on i, and we will therefore define $\rho(t) = \rho(i, t)$. Furthermore, since ξ is the density function of a conserved quantity, $\rho(t)$ is t-independent, so we define $\rho = \rho(t)$.

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The desired distribution parameterized by ρ can be obtained as follows. Let $\rho \in [0, 1/2]$ be the target density of blocks 10, and let $\{X_i\}_{i=0}^{L-1}$ be a collection of independently and identically distributed Bernoulli random variables such that

$$Pr(X_i = 1) = 2\rho, \tag{19}$$

$$Pr(X_i = 0) = 1 - 2\rho, \tag{20}$$

for all $i \in \{0, 1\}$. The initial configuration will be given by

$$s_i(0) = \left(\sum_{j=0}^{i} X_j \right) \mod 2. \tag{21}$$

Let $P_t(b)$ denote the probability for the occurrence of block b in configuration $s(t)$. If the density of blocks 10 in the initial configuration is ρ, then the probability of having a dissenting neighbor at time t will be denoted by $P_{\text{dis}}(\rho, t)$. A site s_i has a dissenting right neighbor if $s_{i-1}s_is_{i+1} = 110$ or $s_{i-1}s_is_{i+1} = 001$. $P_{\text{dis}}(\rho, t)$ is therefore given by

$$P_{\text{dis}}(\rho, t) = P_t(110) + P_t(001). \tag{22}$$

Although two block probabilities appear on the right-hand side of equation (22), we will show that $P_{\text{dis}}(\rho, t)$ can be expressed in terms of a single block probability.

As a first step, we note that the following properties are direct consequences of equation (21).

Proposition 1. Let $P_o(b)$ denote the probability for the occurrence of block b in the configuration drawn from the distribution given by equation (21). Then we have:

(*i*) $P_0(1) = P_0(1) = 1/2$.

(*ii*) $P_0(10) = P(01) = \rho$.

(*iii*) $P_0(b) = P_0(\bar{b})$, where $\bar{b}$ denotes boolean conjugation of block b, that is, $\bar{b}_i = 1 - b_i$.

Rule 142 exhibits boolean self-conjugacy, that is, replacing all zeros by ones and *vice versa* in equation (7) does not change the definition. This fact together with Proposition 1(*iii*) implies that $P_t(110) = P_t(001)$, hence

$$P_{\text{dis}}(\rho, t) = 2P_t(110). \tag{23}$$

Kolgomorov consistency conditions for block probabilities require that

$$P_t(110) + P_t(111) = P_t(11),$$
$$P_t(10) + P_t(11) = P_t(1),$$

hence

$$P_t(110) = P_t(1) - P_t(10) - P_t(111). \tag{24}$$

Using the fact that $P_t(10) = \rho$, we obtain

$$P_{\mathrm{dis}}(\rho, t) = 2P_t(1) - 2\rho - 2P_t(111). \tag{25}$$

Proposition 2 will lead to the elimination of $P_t(1)$ from equation (25).

Proposition 2. Let the initial configuration $s(0)$ be drawn from the distribution given by equation (21), and let $s(t)$ be obtained from $s(0)$ by iterating Rule 142 t times, so that $s(t) = F_{142}^t(s(0))$. Then we have

$$P_t(1) = P_t(0) = 1/2. \tag{26}$$

Proof. We will prove by induction. Obviously, $P_0(1) = P_0(0) = 1/2$ by Proposition 1. Let us assume that $P_t(1) = 1/2$ for some t. Block 1 has four preimages under $\mathbf{f}_{142}$: 001, 010, 011, and 111. This leads to

$$P_{t+1}(1) = P_t(001) + P_t(010) + P_t(011) + P_t(111). \tag{27}$$

Kolgomorov consistency conditions require that $P_t(011) + P_t(111) = P_t(11)$, and, as remarked before, boolean self-conjugacy of Rule 142 implies $P_t(001) = P_t(110)$. This yields

$$P_{t+1}(1) = P_t(110) + P_t(010) + P_t(11). \tag{28}$$

Using consistency conditions again we get

$$P_{t+1}(1) = P_t(10) + P_t(11) = P_t(1), \tag{29}$$

and this, by the induction hypothesis, yields $P_{t+1}(1) = 1/2$, concluding the proof. ■

Proposition 2 simplifies equation (25) to

$$P_{\mathrm{dis}}(\rho, t) = 1 - 2\rho - 2P_t(111). \tag{30}$$

Now the only thing left is to compute the probability for the occurrence of block 111 in configuration $s(t)$.

5. Preimages

In order to compute $P_t(111)$, we will use some properties of preimages of the block 111. Let $\mathbf{f}_{142}^{-1}(111)$ be a set of preimages of 111 under $\mathbf{f}_{142}$. Then we have

$$P_t(111) = \sum_{b \in \mathbf{f}_{142}^{-1}(111)} P_{t-1}(b), \tag{31}$$

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generalizing, we can write

$$P_t(111) = \sum_{b \in \mathbf{f}_{142}^{-t}(111)} P_0(b), \tag{32}$$

where again $\mathbf{f}_{142}^{-t}(111)$ is a set of preimages of 111 under $\mathbf{f}_{142}^{t}$, that is, under t iterations of $\mathbf{f}_{142}$. To find $P_t(111)$ using this property, two steps are needed: first, we have to find the set of preimages of 111, and then find probabilities of their occurrences in the initial distribution. Figure 2 shows three levels of preimages of 111. Upon inspection of this figure, two properties become apparent.

Proposition 3. Let b be a t-step preimage of 111, that is, $b \in \mathbf{f}_{142}^{-t}(111)$. Then we have that

(i) the length of b is $3 + 2t$, and

(ii) b ends with 111.

The first property is an obvious consequence of the definition of $\mathbf{f}_{142}^{t}$, and the second one can be easily proved by induction (omitted here).

Further inspection of Figure 2 leads to the necessary and sufficient condition for a block b to be a t-step preimage of 111. Before stating this condition formally, we will explain it using an example. Consider the block $b = 011100111$, which is a preimage of 111 in three steps since $\mathbf{f}_{142}(011100111) = 1100111$, $\mathbf{f}_{142}(1100111) = 00111$, and $\mathbf{f}_{142}(00111) = 111$. Let us now assume that we start with "capital" of 1. We move along the string $b = b_0 b_1 \ldots b_8$ starting from $i = 6$ in the direction of decreasing i. Every time we see that b_{i-1} is different from b_i, we decrease our capital by 1. If $b_{i-1} = b_i$, we increase our capital by 1. We stop at $i = 1$.

Clearly, it is possible to traverse $b = 011100111$ following this procedure without making the capital negative. It turns out that this is a general property of preimages of 111. If b is a preimage of 111, then it is possible to traverse it keeping the capital nonnegative. If b is not a preimage of 111, the capital will become negative at some point. A more formal statement of this property follows.

Proposition 4. Let t be a nonnegative integer, and let $b = b_0 b_1 \ldots b_{2t+2}$ be a binary string of length $3 + 2t$ ending with 111. Define χ to be a function of two variables such that $\chi(u, v) = 1$ if $u = v$, and $\chi(u, v) = -1$ otherwise. The string b is a preimage of 111 under $\mathbf{f}_{142}^{t}$ if and only if the inequality

$$1 + \sum_{i=0}^{k} \chi(b_{2t-i-1}, b_{2t-i}) \geq 0 \tag{33}$$

is satisfied for all $k = 0, 1, \ldots, 2t - 1$.

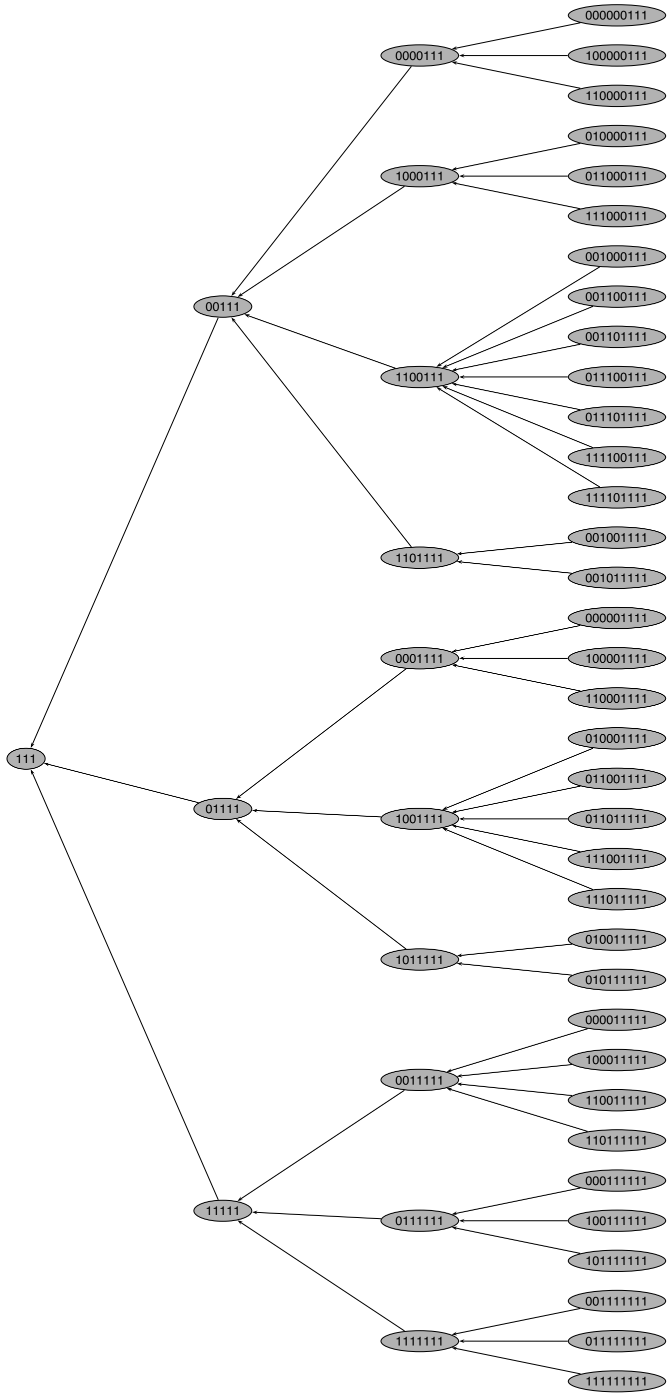

Figure 2. Tree of preimages of the block 111 under Rule 142.

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Instead of proving this proposition directly, we will show that it can be derived from a similar result previously obtained for a related cellular automaton rule.

6. Rule 226

In [2] it has been observed that

$$f_{60} \circ f_{142} = f_{226} \circ f_{60}, \tag{34}$$

where

$$f_{60}(x_0, x_1, x_2) = x_0 + x_1 \mod 2 \tag{35}$$

$$f_{226}(x_0, x_1, x_2) = x_0 x_1 - x_1 x_2 + x_2. \tag{36}$$

This means that there exists a local mapping (Rule 60) which transforms Rule 142 into Rule 226. Rule 226 and its image under spatial reflection, Rule 184, are the only nontrivial elementary number-conserving rules, and many results regarding their dynamics have been established [3–10]. For our purpose, one such result will be particularly useful.

Proposition 5. Under Rule 226, t-step preimages of 00 have the following properties.

(i) In each preimage, the number of zeros exceeds the number of ones.

(ii) The block $a_0 a_1 \ldots a_{2t+1}$ is a t-step preimage of 00 if and only if it ends with two zeros and

$$1 + \sum_{i=2}^{k} \xi(s_{2t+1-i}) \geq 0 \tag{37}$$

for every $2 \leq k \leq 2t + 1$, where $\xi(0) = 1$, $\xi(1) = -1$.

(iii) The number of t-step preimages of 00 containing exactly n_0 zeros and n_1 ones is equal to

$$\frac{n_0 - n_1}{n_0 + n_1} \binom{n_0 + n_1}{n_1}, \tag{38}$$

where $n_0 + n_1 = 2t + 2$.

Proof of this result can be found in [11], with further generalization in [8]. The proof is based on the fact that the enumeration of preimages of 00 under Rule 226 (or 184) is equivalent to the problem of enumerating planar lattice paths between two points, subject to some constraining conditions. This path enumeration problem can then be solved using combinatorial methods.

We now relate preimages of Rules 226 and 142.

Proposition 6. The number of t-step preimages of 111 under Rule 142 is equal to the number of t-step preimages of 00 under Rule 226.

Proof. We will explicitly construct a bijection T between $\mathbf{f}_{142}^{-t}(111)$ and $\mathbf{f}_{226}^{-t}(00)$. Let x be a block of length m, $b_0 b_1, \ldots, b_{m-1}$. We define

$$[T(x)]_i = x_i + x_{i+1} \mod 2 \tag{39}$$

for $i = 0, 1, \ldots, m-2$. $T(x)$ is therefore a block of length $m-1$. Since T is a block evolution operator of Rule 60, a relationship similar to equation (34) must hold, that is,

$$T \circ \mathbf{f}_{142} = \mathbf{f}_{226} \circ T. \tag{40}$$

Let us now assume that b is a t-step preimage of 111 under $\mathbf{f}_{142}$. This means that $\mathbf{f}_{142}^{t}(b) = 111$. Since $T(111) = 00$, we have $T(\mathbf{f}_{142}^{t}(b)) = 00$. Using equation (40) we obtain $\mathbf{f}_{226}^{t}(T(b)) = 00$. This means that if b is a t-step preimage of 111 under $\mathbf{f}_{142}$, then $T(b)$ is a t-step preimage of 00 under Rule 226.

Now let us consider a transformation inverse to T. In general, T is not invertible, but if restricted to the set of preimages of 111 under $\mathbf{f}_{142}$, it becomes invertible.

For an arbitrary block y, there exist two different blocks x such that $T(x) = y$, and one can show that these two blocks are related by boolean conjugacy. For example, we have $T(111) = 00$ and $T(000) = 00$. We have to define T^{-1} such that this ambiguity is removed. This can be done as follows. Let a be a t-step preimage of 00 under Rule 226, $a = a_0 a_1 \ldots a_{2t+1}$. We define

$$\begin{aligned}
[T^{-1}(a)]_{2t+2} &= 1 \\
[T^{-1}(a)]_{2t+1} &= 1 + a_{2t+1} \mod 2 \\
[T^{-1}(a)]_{2t} &= 1 + a_{2t} + a_{2t+1} \mod 2 \\
&\ldots \\
[T^{-1}(a)]_{0} &= 1 + a_0 + a_1 + \cdots + a_{2t+1} \mod 2,
\end{aligned}$$

or in a general form

$$[T^{-1}(a)]_i = \begin{cases} 1 + \sum_{j=i}^{2t+1} a_j \mod 2 & \text{if } i = 0, 1, \ldots, 2t+1, \\ 1 & \text{if } i = 2t+2. \end{cases} \tag{41}$$

One can easily show that the transformation of equation (41) is indeed an inverse of T, and in addition we guarantee that when a ends with two zeros, $T^{-1}(a)$ ends with three ones, as required for a t-step preimage of 111 under Rule 142.

Now, if a is a t-step preimage of 00 under Rule 226, we have $\mathbf{f}_{226}^{t}(a) = 00$, hence $\mathbf{f}_{226}^{t}(T(T^{-1}(a))) = 00$, and by equation (40) we obtain

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$T(\mathbf{f}_{142}^t(T^{-1}(a))) = 00$. The last equation implies that $\mathbf{f}_{142}^t(T^{-1}(a)) = 111$, which means that $T^{-1}(a)$ is a t-step preimage of 111 under Rule 142, as required. ■

Proposition 4 follows from the above result and Proposition 5(*ii*).

7. Probability for the occurrence of 111

The bijective transformation T constructed in the proof of Proposition 6 has a property which will be useful in computing $P_t(111)$. Let us call the block $x = x_0x_1$ a *matching pair* if $x_0 = x_1$, and a *mismatched pair* if $x_0 \neq x_1$. If $a = T(b)$, then the number of matching pairs in b is equal to the number of zeros in a, while the number of mismatched pairs in b is equal to the number of ones in a. This fact, together with Proposition 5 immediately leads to the conclusion that under Rule 142, the number of t-step preimages of 111 with exactly n_0 matching pairs and n_1 mismatched pairs is equal to

$$\frac{n_0 - n_1}{n_0 + n_1}\binom{n_0 + n_1}{n_1}, \tag{42}$$

where $n_0 + n_1 = 2t + 2$. The probability for the occurrence of a matching pair in the initial configuration drawn from the initial distribution is 2ρ, and the mismatched pair is $1 - 2\rho$. Therefore, the probability for the occurrence of a block with a prescribed sequence of matching and mismatched pairs such that it has exactly n_0 matching pairs and n_1 mismatched pairs is equal to $(2\rho)^{n_1}(1 - 2\rho)^{n_0}$. This implies that the probability that a block of length $2t + 3$, randomly selected from the distribution of equation (21), is a t-step preimage of 111 with exactly n_0 matching pairs is equal to

$$\begin{aligned}\left(\frac{1}{2}\right)\frac{n_0 - n_1}{n_0 + n_1}\binom{n_0 + n_1}{n_1}(2\rho)^{n_1}(1 - 2\rho)^{n_0} = \\ \frac{n_0 - n_1}{4t + 4}\binom{2t + 2}{n_1}(2\rho)^{n_1}(1 - 2\rho)^{n_0}.\end{aligned} \tag{43}$$

The factor 1/2 in front comes from the fact that there are always two strings with a given sequence of pairs (related by boolean conjugacy), but only one of them is a preimage of 111.

The smallest possible number of matching pairs in a t-step preimage of 111 is $t + 2$ (recall that the number of matching pairs must exceed the number of mismatched pairs), while the maximum possible number is $2t + 2$ (all zeros). Summing equation (43) over n_0 we obtain

$$P_t(111) = \sum_{n_0=t+2}^{2t+2} \frac{n_0 - (2t + 2 - n_0)}{4t + 4}\binom{2t + 2}{2t + 2 - n_0}(2\rho)^{2t+2-n_0}(1 - 2\rho)^{n_0}. \tag{44}$$

Introducing a new summation index $j = n_0 - (t+1)$ we get

$$P_t(111) = \sum_{j=1}^{t+1} \frac{j}{2t+2} \binom{2t+2}{t+1-j} (2\rho)^{t+1-j} (1-2\rho)^{t+1+j}, \tag{45}$$

and as a result, equation (30) becomes

$$P_{\text{dis}}(\rho, t) = 1 - 2\rho - \sum_{j=1}^{t+1} \frac{j}{t+1} \binom{2t+2}{t+1-j} (2\rho)^{t+1-j} (1-2\rho)^{t+1+j}, \tag{46}$$

where $\rho \in [0, 1/2]$.

8. Equilibrium probability

We will now show how to obtain the equilibrium probability, that is, $\lim_{t\to\infty} P_{\text{dis}}(\rho, t)$. In order to find the limit $\lim_{t\to\infty} P_t(111)$ we can write equation (45) in the form

$$P_t(111) = \sum_{j=1}^{t+1} \frac{j}{2t+2} b(t+1-j, 2(t+1), 2\rho), \tag{47}$$

where

$$b(k, n, p) = \binom{n}{k} p^k (1-p)^{n-k} \tag{48}$$

is the binomial distribution function. Using the de Moivre–Laplace limit theorem, binomial distribution for large n can be approximated by the normal distribution

$$b(k, n, p) \sim \frac{1}{\sqrt{2\pi n p(1-p)}} \exp \frac{-(k-np)^2}{2np(1-p)}. \tag{49}$$

To simplify notation, let us define $T = t + 1$. Now, using equation (49) to approximate $b(T-j, 2T, 2\rho)$ in equation (47), and approximating the sum by an integral, we obtain

$$P_t(111) = \int_1^T \frac{x}{2T} \frac{1}{\sqrt{8\pi T\rho(1-2\rho)}} \exp \frac{-(T-x-4T\rho)^2}{8T\rho(1-2\rho)} dx. \tag{50}$$

Integration yields

$$P_t(111) =$$
$$\sqrt{\frac{\rho(1-2\rho)}{2\pi T}} \left\{ \exp\left(\frac{-(1-T+4\rho T)^2}{8T\rho(1-2\rho)} \right) - \exp\left(\frac{-2\rho T}{2(1-2\rho)} \right) \right\}$$
$$+ \frac{1}{4}(1-4\rho) \left\{ \operatorname{erf}\left(\frac{4\rho T}{\sqrt{8\rho(1-2\rho)T}} \right) - \operatorname{erf}\left(\frac{1-T+4\rho T}{\sqrt{8\rho(1-2\rho)T}} \right) \right\},$$

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where $\mathrm{erf}(x)$ denotes the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt. \tag{51}$$

The first term (involving two exponentials) in equation (51) tends to 0 with $T \to \infty$. Moreover, since $\lim_{x\to\infty} \mathrm{erf}(x) = 1$, we obtain

$$\lim_{t\to\infty} P_t(111) = \frac{1}{4}(1-4\rho)\left\{1 - \lim_{T\to\infty} \mathrm{erf}\left(\frac{1-T+4\rho T}{\sqrt{8\rho(1-2\rho)T}}\right)\right\}.$$

Now, noting that

$$\lim_{T\to\infty} \mathrm{erf}\left(\frac{1-T+4\rho T}{\sqrt{8\rho(1-2\rho)T}}\right) = \begin{cases} 1 & \text{if } 4\rho \geq 1, \\ -1 & \text{otherwise,} \end{cases} \tag{52}$$

we obtain

$$\lim_{t\to\infty} P_t(111) = \begin{cases} 1/2 - 2\rho & \text{if } \rho < 1/4, \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

The final expression for the equilibrium probability becomes

$$\lim_{t\to\infty} P_{\mathrm{dis}}(\rho, t) = \begin{cases} 2\rho & \text{if } \rho < 1/4, \\ 1 - 2\rho & \text{otherwise.} \end{cases} \tag{54}$$

9. Current

The equilibrium probability calculated in section 8 exhibits a singularity at $\rho = 1/4$. This singularity is of a similar nature as the jamming transition observed in cellular automaton Rules 184, 226, and related models.

Recall that in section 3 we defined the current J (equation (14)). The expected value of the current is i-independent, so we can define the expected current as

$$\begin{aligned} j(\rho, t) &= E_\mu\big[J(s_i(t), s_{i+1}(t), s_{i+2}(t))\big] \\ &= E_\mu\left[s_i(t)s_{i+1}(t)(s_{i+2}(t) - 1)\right]. \end{aligned} \tag{55}$$

The graph of $j(\rho, \infty)$ as a function of ρ is known as a *fundamental diagram*. Using the notion of block probabilities we can rewrite equation (55) in an alternative form as

$$j(\rho, t) = -P_t(110), \tag{56}$$

and using equation (23)

$$j(\rho, t) = -\frac{1}{2} P_{\mathrm{dis}}(\rho, t). \tag{57}$$

The probability of having a dissenting neighbor, as we can see, is proportional to the expected current.

Since the current J represents the flow of blocks 10, the expected current must be equal to

$$j(\rho, t) = \rho v(\rho, t), \tag{58}$$

where $v(\rho, t)$ is the expected velocity of a block 10 at time t. Using equation (54) this velocity is given by

$$\lim_{t\to\infty} v(\rho, t) = \begin{cases} -1 & \text{if } \rho < 1/4, \\ 1 - (1/2\rho) & \text{otherwise.} \end{cases} \tag{59}$$

We can see that for densities of blocks 10 smaller than 1/4, the average velocity remains constant and equal to -1, which means that all blocks are moving to the left. At $\rho = 1/4$ a *jamming transition* occurs, and when ρ increases beyond 1/4, more and more blocks are stopped. This phenomenon is very similar to jamming transitions in discrete models of traffic flow, which have been extensively studied in recent years (see [12] and references therein).

10. Conclusions

We investigated dynamics of the cellular automaton Rule 142. It can be transformed into Rule 226 by a surjective transformation, which turns out to be invertible if restricted to preimages of 111. This transformation allows computing the probability of having a dissenting neighbor, which, in turn, allows computing the expected current of blocks 10. Rule 142 exhibits a jamming transition similar to transitions occurring in discrete models of traffic flow.

It is worth mentioning that there are other cellular automaton rules conserving the number of blocks 10 which also exhibit singularities of fundamental diagrams; for example, Rules 35 and 14, as reported in [1]. For these rules, however, no transformations exist relating them to other rules with singularities, thus the method presented in this paper cannot be easily applied. Nevertheless, the nature of singularities in these rules appears to be the same, thus some relationship between them and Rules 184/226 may exist. This problem is currently under investigation and will be reported elsewhere.

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