Field Theoretical Approach to the Conservation of Identity of a Complex Network System

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The concept of "identity" of a complex network system is proposed based on the method of the gauge field theory. The system is assumed to consist of many elements interacting with each other. The interaction weight is treated as a gauge potential. Change in the external environment surrounding the system is assumed to induce a unitary transformation of the state vector of the system, which is regarded as a gauge transformation. The identity is defined by the fact that the system has some invariant quantities under gauge transformation. Here the total hamiltonian of the whole system is assumed to be that of the invariant quantities representing the identity. The invariance of the identity is conserved by changing the gauge potential according to the given external environment. Some invariant functionals are obtained by introducing the noncommutative gauge field derived from the gauge potential of mutual interaction. Based on the concept of identity, new learning dynamics written in a covariant form are presented as a field equation of the gauge field, which is utilized to realize the state requisite for adaptation to a given new external environment. The learning dynamics are extended to the case of nonlinear interaction among the elements.

1. Introduction

A complex biological system is generally composed of many functional elements, like neurons in the brain. The interaction weight between the functional elements is organized flexibly so that elements can adapt to the given external environment. The invariant characteristics of the system in the adaptive process can be regarded as representing an "identity" of the system. In other words, the invariant characteristics guarantee that the system continues to conserve its identity while it adapts to changing external environments.

In the present paper, we propose a field theoretical approach to the conservation of the identity of a complex network system. The system is assumed to be composed of many discretely distributed elements with internal freedom. The elements interacting with each other play a sim-

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ilar role to neurons in the brain. An actual complex biological system is generally composed of many functional elements with a hierachical structure of their functional roles. However, in the present paper, we confine ourselves to the simplest case of a system composed of elements of only one kind interacting with each other. The interaction of connection weights between the elements is treated as a gauge field potential. Hence, we call it the "connection gauge field." The system is able to learn the input initially given from the external environment or to realize the requisite state necessary for adaptation to the external environment, by organizing the connection field among the elements [1, 2]. In other words, the connection field is organized in the system itself in order to adapt to changes of the external environment. Here the change is assumed to induce unitary transformation of the state vector of the system. The identity of the system is defined by the fact that the system conserves its characteristics while it interacts with changing external environments. Some invariant characteristic functionals are derived by taking into account the adaptive change of the gauge field.

Based on the concept of identity, new covariant learning dynamics of the system are presented as a field equation of the noncommutative gauge field. The learning process enables the systems to realize the state necessary for adaptation to changes of the external environment. The learning dynamics are extended to the case of nonlinear interaction among the elements [3].

2. Complex network system

Let us consider a complex network system that is composed of many elements. Each element is assumed to have internal freedom characterized by a state vector $\psi_i(t)$, $i = 1, 2, \ldots, N$. The units are assumed to interact with each other with the weight matrix $W \equiv \{W_{i,j}\}$, $i, j = 1, 2, \ldots, N$, and the external environment is assumed to be represented by the input force vector $S(t) = \{S_i(t), i = 1, 2, \ldots, N\}$, where $S_i(t)$ stands for an additive input force on element i.

In analogy with a spin system under an external magnetic field, we define the total hamiltonian $\mathcal{H}$ of the whole network system as

$$\mathcal{H} \equiv \psi^+ W \psi + \psi^+ W^+ \psi + \psi^+ S + S^+ \psi, \tag{1}$$

where $\psi(t)$ is a state vector whose ith component ψ_i, $i = 1, 2, \ldots, N$, represents the state vector of the ith element and $\psi^+(t)$ (or $S^+(t)$) is the conjugate transpose of $\psi(t)$ (or $S(t)$).

Let us assume that an input force $S'(t)$ representing a different external environment is given by the unitary transformation of $S(t)$,

$$S'(t) = U(t) S(t), \tag{2}$$

where $U(t)$ is a time-dependent unitary matrix satisfying $U^+(t) U(t) =$

$U(t)U(t)^+ = 1$ and equation (2) ensures a normalization condition for the intensity of the external force:

$$S'^+(t)S'(t) = S^+(t)S(t). \tag{3}$$

Equation (2) is interpreted to represent a change of the external environment.

Now we introduce a postulate that the total hamiltonian $\mathcal{H}$ is invariant to the change of the external environment:

$$\begin{aligned} &\psi'^+ W'\psi' + \psi'^+ W'^+\psi' + \psi'^+ S' + S^{+\prime}\psi' = \\ &\quad \psi^+ W\psi + \psi^+ W^+\psi + \psi^+ S + S^+\psi, \end{aligned} \tag{4}$$

where ψ' is a state vector under external force $S'(t)$. In other words, we confine ourselves to the change of external force under which the total hamiltonian $\mathcal{H}$ of the whole network system is conserved. Then, from equations (2) and (3), we have

$$\psi'(t) = U(t)\psi(t) \tag{5}$$

$$W' + W'^+ = U(W + W^+)U^+. \tag{6}$$

Equation (5) shows that the state vector ψ changes following the same transformation rule as equation (2) of the external force $S(t)$. Note that we can use equation (5) as the first starting assumption to define the change of the state of the system caused by the change of the external environment. Furthermore, equation (5) gives rise to the invariance of the norm of the state vector:

$$\psi'^+(t)\psi'(t) = \psi^+(t)\psi(t). \tag{7}$$

3. Identity of the network system

Let us introduce the concept of "identity" of the network system based on the fact that there exist invariant quantities in the process of the interaction with the external environment. From this consideration, the total hamiltonian $\mathcal{H}$ of the network system is already assumed to be an invariant quantity, as shown in equation (4).

Now we study other invariant quantities in the dynamical process of the network system. For this purpose, we regard equation (5) as a gauge transformation and also the weight matrix W as a gauge potential which follows the transformation rule:

$$W' = U\frac{\partial}{\partial t}U^+ + UWU^+. \tag{8}$$

Then it is easily shown from equation (8) that equation (6) is automatically satisfied by using the equality $UU^+ = U^+U = 1$.

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Furthermore, using equations (5) and (8), we can derive

$$D'\psi' \equiv \left(\frac{\partial}{\partial t} + W'\right)\psi' = U\left(\frac{\partial}{\partial t} + W\right)\psi = UD\psi, \tag{9}$$

where

$$D \equiv \left(\frac{\partial}{\partial t} + W\right)\psi \quad \text{or} \quad D' \equiv \left(\frac{\partial}{\partial t} + W'\right)\psi'$$

means a covariant derivative of ψ or ψ'. Iterating equation (9) n times will yield

$$D'^n\psi' = UD^n\psi. \tag{10}$$

A gauge field G is introduced from the gauge potential W. Using equation (8), we have

$$\begin{aligned} G' &\equiv \frac{\partial}{\partial t}(W' + W'^{+}) + [W', W'^{+}] \\ &= U\left(\frac{\partial}{\partial t}(W + W^{+}) + [W, W^{+}]\right)U^{+} \equiv UGU^{+}, \end{aligned} \tag{11}$$

where $[A, B]$ means a communication bracket, $[A, B] \equiv AB - BA$. Defining the covariant derivative D or D' of the potential A as

$$DA \equiv \frac{\partial}{\partial t}A + [W, A] \tag{12}$$

$$D'A' \equiv \frac{\partial}{\partial t}A' + [W', A'], \tag{13}$$

we can write G and G' as

$$G = D(W + W^{+}) \tag{14}$$

$$G' = D'(W' + W'^{+}) \tag{15}$$

and can derive (see appendix A)

$$D'^n(W' + W'^{+}) = UD^n(W + W^{+})U^{+}. \tag{16}$$

Now we can present some invariant functionals using the transformation rules given by equations (6), (10), and (14). As a typical one, we have

$$\begin{aligned} \mathcal{L} \equiv \mathcal{L}[&\psi^{+}\psi, \psi^{+}D\psi, (D\psi)^{+}D\psi, \psi^{+}(W + W^{+})^{j}\psi, \\ &T_r(D^k(W + W^{+})D^l(W + W^{+})\ldots D^m(W + W^{+})), \\ &\mathrm{Det}(I + D^n(W + W^{+})), \psi^{+}S, S^{+}\psi, S^{+}(W + W^{+})S], \end{aligned} \tag{17}$$

where j, l, k, m, and n are positive integers, 0, 1, 2, 3, …, and I is a unit matrix. $T_r(M)$ and $\mathrm{Det}(M)$ respectively mean the trace and determinant of matrix M.

In conclusion, identity is defined by the existence of the invariant functional written in terms of state vector ψ, gauge potential W, and their covariant derivatives $D\psi$ and DW. The invariant functional is conserved during adaptation to the change of the external environment, if the gauge field W is reorganized in the system itself as equation (8).

4. Learning dynamics of the network system

In this section we study a temporal evolution of the network system based on the idea that the evolution equations take invariant forms under transformation equations (2), (5), and (6). The evolution equations themselves can be regarded as the invariant characteristics that represent the identity of the system itself, because they take invariant forms in any environment given by unitary transformation. In other words, they are covariant equations in the sense that they have invariant forms even in different environments. The evolution equations are derived by taking the transformation rules summarized in equation (17) into account. A typical simple evolution equation is given by

$$\left(\frac{\partial}{\partial t} + W\right)\psi - \lambda S = 0 \tag{18}$$

$$\frac{\partial}{\partial t}(W + W^{+}) + [W, W^{+}] + \eta(-\psi \otimes \psi^{*} + M) = 0, \tag{19}$$

where λ and η are constants, $\otimes$ denotes a direct product, $*$ means the complex conjugate, and M is introduced as an external excitation source of the field W which is assumed to be transformed as $M' = UMU^{+}$. A simple example of M is given by $S \otimes S^{*}$, because $S \otimes S^{*}$ is shown to have the same transformation rule as that of $\psi \otimes \psi^{*}$ (appendix B)

$$\psi' \otimes \psi'^{*} = U\psi \otimes \psi^{*} U^{+} \tag{20}$$

$$S' \otimes S'^{*} = US \otimes S^{*} U^{+}, \tag{21}$$

which are the same as those of the remaining terms on the left-hand side of equation (19), given by equation (11).

A simple stationary solution of equations (18) and (19) for the case when the source M is given by $M = S \otimes S$ becomes

$$\psi = S \tag{22}$$

$$W = \lambda S \otimes S, \tag{23}$$

where S is assumed to be a real constant vector satisfying $|S|^2 = 1$. This is easily shown as follows. $W_{i,j} = \lambda S_i S_j$, so that $(W\psi) = \lambda(S \otimes S)S = \lambda S$, and furthermore, we note that $W = W^{+}$ and $\psi \otimes \psi^{*} = S \otimes S$. Then equations (18) and (19) are automatically satisfied.

Noting the covariant property of equations (22) and (23), we can derive a time-dependent solution of equations (18) and (19) under the

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external force $S'(t) \equiv U(t)S$ and $M'(t) = S'(t) \otimes S'(t)$

$$\psi'(t) = U(t)\psi(t) = S'(t) \tag{24}$$

$$W'(t) = U(t)\frac{\partial}{\partial t}U^+(t) + \lambda US \otimes SU^+. \tag{25}$$

They are easily shown to satisfy

$$\left(\frac{\partial}{\partial t} + W'\right)\psi' - \lambda S' = 0 \tag{26}$$

and

$$\frac{\partial}{\partial t}\left(W' + W'^+\right) + [W', W'^+] = 0, \tag{27}$$

where we used equation (24) with

$$\psi' \otimes \psi'^* = M' = S'(t) \otimes S^{*\prime}(t). \tag{28}$$

Equation (27) means the condition of zero field, $G' = 0$, that is, pure gauge potential. Equation (24) shows that the external force $S'(t)$ is reflected directly in the state vector $\psi'(t)$. In other words, the state vector ψ' forms a mirror image of the external force $S'(t)$.

Equations (18) and (19) are written in the covariant forms in the sense that they are invariant under gauge transformation equations (5) and (8). We may regard equation (18) as the Langevin equation with a stochastic force λS; then equations (18) and (19) compose learning dynamics having an extended form from that of the Synergetic computer [3]. Here the excitation force M may be interpreted to represent the final condition of the direct product $\psi \otimes \psi$ which must be realized for the adaptation to a new external environment. In other words, the learning dynamics are interpreted to describe an adaptation process of the system to a new external environment. Furthermore, we note that various nonlinear terms with respect to ψ and $W + W^+$ can be included on the left-hand sides of equations (18) and (19). For example,

$$\left(\frac{\partial}{\partial t} + W\right)\psi - \lambda S + F(|\psi|^2)\psi = 0 \tag{29}$$

$$\frac{\partial}{\partial t}(W + W^+) + [W, W^+] + \eta(-\psi \otimes \psi^* + M) + \sum_{n=1}^{\infty} a_n (W + W^+)^n = 0, \tag{30}$$

where a_n, $n = 1, 2, 3, \ldots$ are arbitrary constants and $F(|\psi|^2)$ means an arbitrary function of $|\psi|^2$. The nonlinear terms of $W + W'$ in equation (30) will generally cause a phase change in the connection gauge field W.

5. Electromagnetic-like connection field

In this section we consider that each network element is subject to the local unitary transformation, $\psi'^j = U_j \psi_j$, $j = 1, 2, \ldots, N$, where U_j is a unitary transformation acting on the state of the element at position j, and ψ_j represents the internal state of the element situated at position j. Then the global unitary transformation U takes the diagonal form:

$$U = \begin{pmatrix} U_1, & 0, & 0, & \ldots, & 0, \\ 0, & U_2, & 0, & \ldots, & 0, \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0, & 0, & 0, & \ldots, & U_n, \end{pmatrix}. \tag{31}$$

Transformation rule equation (8) of the weight matrix W is decoupled as

$$W'_{ij} = U_i W_{ij} U_j^+ \tag{32}$$

for $i \neq j$ and

$$V'_i = U_i \frac{\partial}{\partial t} U_i^+ + U_i V_i U_i^+ \tag{33}$$

for $W_{ij} \equiv V_i$.

The field transformations follow from equations (32) and (33):

$$\frac{\partial}{\partial t} W'_{ij} + V'_i W'_{ij} - W'_{ij} V'_j = U_i \left(\frac{\partial}{\partial t} W_{ij} + V_i W_{ij} - W_{ij} V_j \right) U_j^+ \tag{34}$$

$$V'_i + V'^+_i = U_i (V_i + V_i^+) U_i^+ \tag{35}$$

and

$$\frac{\partial}{\partial t}(V'_i + V'^+_i) + [V'_i, V'^+_i] = U_i \left(\frac{\partial}{\partial t}(V_i + V_i^+) + [V_i, V_i^+] \right) U_i^+. \tag{36}$$

Using these transformation rules, we can express gauge invariant dynamics of ψ_i, W_{ij}, and V_i.

As a typical case, we have

$$\left(\frac{\partial}{\partial t} + V_i \right) \psi_i + \sum_{i \neq j} W_{ij} \psi_j = F(|\psi|^2)\psi_i + H(|\psi_i|^2)\psi_i \tag{37}$$

$$\frac{\partial}{\partial t} W_{ij} + V_i W_{ij} - W_{ij} V_j + a W_{ij} = b \psi_i \otimes \psi_j^* \tag{38}$$

$$\frac{\partial}{\partial t}(V_i + V_i^+) + [V_i, V_i^+] + p(V_i + V_i^+) = q \psi_i \otimes \psi_i^*, \tag{39}$$

where $|\psi|^2 = \psi\psi^+$, $|\psi_i|^2 = \psi \otimes \psi^*$, F, and H are arbitrary functions of $|\psi|^2$ and $|\psi_i|^2$, respectively, and a, b, p, and q are constant coefficients.

Furthermore, if the local unitary transformation is given by a phase transformation,

$$U_j = e^{-i\theta_j}, \tag{40}$$

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V_k and W_{ik} become scalar complex quantities and equations (33) and (34) reduce to

$$W'_{jk} = W_{jk} e^{-i(\theta_j - \theta_k)} \tag{41}$$

$$V'_k = i\frac{\partial}{\partial t}\theta_k + V_k \tag{42}$$

from which the following invariant equations are derived:

$$-\frac{\partial}{\partial t}\ln W'_{jk} - V'_j + V'_k = -\frac{\partial}{\partial t}\ln W_{jk} - V_j + V_k \tag{43}$$

and

$$\ln W'_{jk} - \ln W'^{*}_{kj} = \ln W_{jk} - \ln W^{*}_{kj}. \tag{44}$$

Analogous to the gauge transformation of electromagnetic potential with a transformation function $i\theta_k$, we can regard V_k in equation (42) as a scalar potential at position k and $\ln W_{kj}$ in equation (43) as the line integral of a vector potential from position k to j.

This idea gives rise to the interpretation that equations (43) and (44) show the invariant property of an "electric field" $-\partial/\partial t \ln W'_{jk} - (V'(j) - V'(k))$ and a "magnetic flux" $\ln W_{jk} - \ln W^{*}_{kj}$, respectively. It is worth noting that the magnetic flux vanishes when the connection field W becomes hermitian, $W = W^{+}$.

In the case of phase transformation, the dynamics presented by equations (37), (38), and (39) reduce to

$$\left(\frac{\partial}{\partial t} + V_i\right)\psi_i + \sum_{i \neq j} W_{ij}\psi_j = F(|\psi|^2)\psi_i + H(|\psi_i|^2)\psi_i \tag{37$'$}$$

$$\frac{\partial}{\partial t}W_{ij} + (V_i - V_j)W_{ij} + W_{ij} = \psi_i\psi_j^{*} \tag{38$'$}$$

$$\frac{\partial}{\partial t}(V_i + V_i^{*}) + \frac{1}{2}(V_i + V_i^{*}) = \psi_i\psi_i^{*}, \tag{39$'$}$$

where we used the fact that W_{ij} and V_i became complex functions and assumed $a = b = q = 1$ and $p = 1/2$ in equations (37), (38), and (39).

A steady-state solution of equations (37′), (38′), and (39′) under the assumption of uniform potential $V_i = V_j$ is given by

$$W_{ij} = \psi_i\psi_j^{*} \tag{45}$$

$$V_i = W_{ii} = |\psi_i|^2 \tag{46}$$

$$|\psi|^2 - F(|\psi|^2) = H(|\psi_i|^2) \tag{47}$$

for a nonzero solution of ψ. Here, the first assumption of uniform potential is satisfied because equation (47) shows that the nonzero solutions of ψ are all equal,

$$|\psi_i|^2 \equiv |\psi_0|^2, \tag{48}$$

so that we have the uniform scalar potential,

$$V_i = |\psi_0|^2. \tag{49}$$

The quantity $|\psi_0|^2$ is determined by

$$m|\psi_0|^2 - F(m|\psi_0|^2) = H(|\psi_0|^2), \tag{47$'$}$$

where m is the number of nonzero solutions of ψ among $\psi_1, \psi_2, \ldots$, and ψ_N. The line integral of the "vector potential" becomes

$$\ln W_{ij} = \ln |\psi_0|^2 + i(\theta_i - \theta_j), \tag{50}$$

where we write $\psi_i = \psi_0 e^{i\theta_i}$. The magnetic flux defined by $\ln W_{ij} - \ln W_{ji}^*$ is found, from equation (44), to become zero, together with the electric field defined by equation (43), so that $V(i)$ and W_{ij} compose a pure gauge field.

6. Nonlinear network system

We consider learning dynamics with *a* nonlinear interaction terms of ψ on the right-hand side of equation (18). Here we confine ourselves to a typical nonlinear term $\Gamma\psi^* \otimes \psi \otimes \psi$, which is equivalent to taking into account the fourth-order interaction energy,

$$\psi^+ \otimes \psi\Gamma\psi^+ \otimes \psi \equiv \psi_i^*\psi_j\Gamma_{(j,i)(l,k)}\psi_k^*\psi_l, \tag{51}$$

where the four-dimensional matrix Γ has the components $\Gamma_{(i,j)(l,k)}$, $i, j, k, l = 1, 3, \ldots, N$ and is assumed, for simplicity, to be hermitian, $\Gamma_{(i,j)(l,k)} = \Gamma^*_{(j,i)(k,l)}$.

Assuming also the invariant property of the fourth-order energy term,

$$\psi'^* \otimes \psi'\Gamma' \otimes \psi^{*\prime} \otimes \psi' = \psi^* \otimes \psi\Gamma \otimes \psi^* \otimes \psi, \tag{52}$$

we derive a transformation rule of Γ (see appendix D):

$$\Gamma'_{(i,j)(k,l)} = U_{ix}U_{k\mu}\Gamma_{(\lambda,\mu)(\nu,\theta)}U^+_{\mu j}U^+_{\theta l}. \tag{53}$$

Furthermore, using equation (53), we derive the transformation rule (see appendix D)

$$\frac{\partial}{\partial t}\Gamma' + [W', \Gamma'] = U \otimes U\left(\frac{\partial}{\partial t} + [W, \Gamma]\right)U^+ \otimes U^+, \tag{54}$$

where $[W, \Gamma]$ is an abbreviated notation in which the $(i, j)(k, l)$ component is defined by

$$[W, \Gamma]_{(i,j)(k,l)} \equiv W_{i\alpha}\Gamma_{(\alpha,j)(k,l)} - \Gamma_{(i,\alpha)(k,l)}W_{\alpha j} + W_{k\alpha}\Gamma_{(i,j)(\alpha,l)} - \Gamma_{(i,j)(k,\alpha)}W_{\alpha l}. \tag{55}$$

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Equations (53) and (54) are comparable to equations (6) and (11).

Taking equation (B.3) into account, we have a symbolically written set of covariant learning dynamics including a nonlinear interaction of ψ:

$$\left(\frac{\partial}{\partial t} + W\right)\psi + \Gamma\psi \otimes \psi^* \otimes \psi - \lambda S = 0 \tag{56}$$

$$\frac{\partial}{\partial t}(W + W^+) + [W, W^+] + \eta(-\psi \otimes \psi^* + M) = 0 \tag{57}$$

$$\dot{\Gamma} + [W, \Gamma] + \zeta(-\psi \otimes \psi^* \otimes \psi \otimes \psi^* + Q) = 0 \tag{58}$$

where ζ is a constant and Q is introduced as an excitation source of field Γ, having the same transformation rule as equation (54). A simple example for Q is given by $Q = S \otimes S^* \otimes S \otimes S^*$. Equations (56), (57), and (58) are also written in the covariant forms under the unitary transformation of equation (5), as in the case of equations (18) and (19), and are also understood to be the learning process of the state to be realized for adaptation to the new external environment. Here the field sources M and Q represent the requisite final states of the direct products $\psi \otimes \psi^*$ and $\psi \otimes \psi^* \otimes \psi \otimes \psi^*$ for adaptation to the given external environment, and the source S plays the role of stochastic force [3].

7. Composite adaptive system

We discuss here a composite adaptive system composed of the two kinds of elements, $\psi^{(1)}$ and $\psi^{(2)}$, which are assumed to be disconnected in the initial state. That is, the initial connection field W is assumed to take the block diagonal form:

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}, \tag{59}$$

where $W = W^+$ is assumed for simplicity.

Now we consider a composite environment represented by the unitary transformation

$$U(t) = \frac{1}{\sqrt{1 + |\varepsilon(t)|^2}} \begin{pmatrix} I & \epsilon(t)I \\ -\epsilon^*(t)I & I \end{pmatrix}, \tag{60}$$

where I is the unit matrix and $\varepsilon(t)$ is an interaction parameter with the initial value $\varepsilon(0) = 0$. The connection field potential under the composite environment takes the form

$$\begin{aligned} W' + W'^+ &= U(W + W^+)U^+ \\ &= \frac{2}{1 + |\varepsilon(t)|^2} \begin{pmatrix} W^{(1)} + |\varepsilon|^2 W^{(2)} & \epsilon(W^{(2)} - W^{(1)}) \\ \epsilon^*(W^{(2)} - W^{(1)}) & W^{(2)} + |\varepsilon|^2 W^{(1)} \end{pmatrix}, \end{aligned} \tag{61}$$

where we used $W^+ = W$ in equation (6). The two kinds of elements begin to interact at time $t = 0$ through the connection gauge field equation (61). Here, an additive invariant quantity is shown to be given by

$$\begin{aligned} t_r(I + W' + W'^+) &= t_r(I + W + W^+) \\ &= t_r(I + W_1 + W_1^+) + t_r(I + W_2 + W_2^+), \end{aligned} \tag{62}$$

and a multiplicative invariant quantity by

$$\mathrm{Det}(I + W' + W'^+) = \mathrm{Det}(I + W_1 + W_1^+)\mathrm{Det}(I + W_2 + W_2^+). \tag{63}$$

Note that these invariant quantities are written in the forms without mutual interaction. In other words, these quantities are conserved in the interacting evolution process from noninteracting initial states.

8. An adaptive network system

In this section, we propose a simple model of an adaptive network system whose state is changeable depending upon external requirements.

The state vector ψ of the system is assumed to obey

$$\frac{\partial}{\partial t}\psi(t) + W(t)\psi(t) = 0 \tag{64}$$

where $W(t)$ is, in general, a time-dependent connection weight matrix.

The system is assumed to be in a state ψ_0 at initial time t_0.

Now we suppose that a state transition of the system is required for some reason. In other words, a new state ψ_s at time t_1 is required to be achieved, which is in general different from the state $\psi(t_1)$ determined from equation (64).

The problem we must solve is how to realize the state transition from the original state ψ_0 to the new state ψ_s under the same connection weight $W(t)$.

For this purpose let us introduce an external force term $-\varphi(t)$ on the right-hand side of equation (64):

$$\frac{\partial}{\partial t}\psi(t) + W(t)\psi = -\varphi(t). \tag{65}$$

Here $\varphi(t)$ is a state vector of the adjoint system which obeys

$$-\frac{\partial}{\partial t}\varphi(t) + W^+(t)\varphi(t) = 0. \tag{66}$$

The adjoint equation (66) is derived from equation (64) through the time-inversion ($t \to -t$) followed by the transpose of the weight matrix $W(t)$. The time-reversed property of equation (67) enables us to incorporate the final desired condition ψ_s as an initial condition of the external force $-\varphi(t)$ of equation (65).

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Introducing the fundamental matrix $K(t, t_0)$ which satisfies

$$\frac{\partial}{\partial t}K(t, t_0) + W(t)K(t, t_0) = 0 \tag{67}$$

with the initial condition,

$$K(t_0, t_0) = I \tag{68}$$

we have the solution of equation (64) written as

$$\psi(t) = K(t, t_0)\psi(t_0). \tag{69}$$

Here are the properties of the fundamental matrix:

$$K(t, t_0) = K(t, \tau)K(\tau, t_0) \tag{70}$$

and

$$K(\tau, t) = K(t, \tau)^{-1}. \tag{71}$$

Further, the solution of the adjoint equation (66) is also written as

$$\varphi(t) = K^+(t_0, t)\varphi(t_0). \tag{72}$$

The innerproduct $\varphi^+\psi$ is shown from equations (65) and (66) to satisfy

$$\frac{\partial}{\partial t}(\varphi^+\psi) = -\varphi^+\varphi. \tag{73}$$

Integrating equation (73) and substituting equation (72) into the resultant equation, we have

$$\varphi^+(t_0)K(t_0, t)\psi(t) - \varphi^+(t_0)\psi_0 = -\varphi^+(t_0)L(t)\varphi(t_0) \tag{74}$$

where $L(t, t_0)$ represents the Gram matrix:

$$L(t, t_0) = \int_{t_0}^{t} d\tau K(t_0, \tau)K^+(t_0, \tau). \tag{75}$$

Further, if we choose the initial condition of φ_0 as

$$\varphi(t_0) = L^{-1}(t_1, t_0)(\psi_0 - K(t_0, t_1)\psi_s) \tag{76}$$

then we have

$$\varphi^+(t_0)K(t_0, t_1)\psi(t)(t_1) = \varphi^+(t_0)K(t_0, t_1)\psi_s \tag{77}$$

which yields the desired result:

$$\psi(t_1) = \psi_s. \tag{78}$$

Note that the initial condition of equation (76) coincides with that derived from the controllability problem in control theory [4]. However, the point to be stressed here is that the external force on the right-hand side of equation (66) is given by the adjoint state vector $\varphi(t)$ to the original one $\psi(t)$.

Finally, we note the fact that the covariant property of the system of equations (65) and (66) under the unitary transformation $U(t)$ is also retained:

$$\psi'(t) = U(t)\psi(t) \tag{5}$$

$$W'(t) = U(t)\frac{\partial}{\partial t}U^{+}(t) + U(t)W(t)U^{+}(t) \tag{8}$$

$$\varphi'(t) = U(t)\varphi(t) \tag{79}$$

$$K'(t, t_0) = U(t)K(t, t_0)U^{+}(t) \tag{80}$$

These covariant properties assure an identity of the adaptive system following equations (65) and (66).

9. Conclusion

A field theory of the identity of a complex network system was developed. The change of the external environment was assumed to induce unitary gauge transformation of the state of the system. The identity was defined as the invariance of the system under gauge transformation. The interaction weight between the elements of the complex system was treated as a gauge field and the invariance of the identity was maintained by reorganizing the gauge field according to the given external environment.

Covariant learning dynamics were presented as a process of adaptation to a given external environment. Finally, we note that the fundamental idea developed here can be extended to a more general change of the system represented by

$$\psi' = A\psi, \tag{81}$$

where A is a general transformation matrix having a generalized inverse A^{-1}. Equation (81) plays the role of equation (5), where the unitary matrix U is replaced by matrix A and the inverse of A is made to correspond to the hermitian conjugate U^{+} of U.

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Appendix

A. Transformation of the gauge field

Let the transformation of matrix Z be

$$Z' = UZU^+. \tag{A.1}$$

Then the covariant derivation $D'Z'$ defined by equation (12) becomes

$$\begin{aligned} D'Z' &= \frac{\partial}{\partial t}Z' + [W', Z'] \\ &= \frac{\partial}{\partial t}(UZU^+) + \left[U\frac{\partial U^+}{\partial t} + UWU^+, UZU^+\right] \\ &= \frac{\partial}{\partial t}UZU^+ + U\frac{\partial Z}{\partial t}U^+ + UZ\frac{\partial}{\partial t}U^+ + U\frac{\partial U^+}{\partial t}UZU^+ \\ &\quad - UZU^+U\frac{\partial U^+}{\partial t} + UWU^+UZU^+ - UZU^+UWU^+ \\ &= U\frac{\partial Z}{\partial t}U^+ + U[W, Z]U^+ + \left(\frac{\partial}{\partial t}U\right)ZU^+ \\ &\quad - \frac{\partial U}{\partial t}U^+UZU^+ + UZ\frac{\partial}{\partial t}U^+ - UZ\frac{\partial}{\partial t}U^+ \\ &= U\left(\frac{\partial Z}{\partial t} + [W, Z]\right)U^+ = U(DZ)U^+. \end{aligned} \tag{A.2}$$

The repeated use of equations (A.1) and (A.2) gives rise to equation (16):

$$D'^nZ' = U(D^nZ)U^+. \tag{A.3}$$

B. Transformation of the direct product of ψ

Using equation (5), we have

$$\begin{aligned} (\psi' \otimes \psi'^*)_{i,j} &= \psi'_i\psi'^*_j = (U\psi)_i(U\psi)^*_j = U_{i\alpha}\psi_\alpha U^*_{j\beta}\psi^*_\beta \\ &= U_{i\alpha}\psi_\alpha\psi^*_\beta U^*_{j\beta} = U_{i\alpha}\psi_\alpha\psi^*_\beta U^+_{\beta j} = (U(\psi' \otimes \psi'^*)U^+)_{i,j}, \end{aligned} \tag{B.1}$$

where the repeated indices are assumed to be summed. Equation (B.1) gives rise to equation (21).

Similarly, we have

$$\begin{aligned} \psi'_i\psi'^*_j\psi'_h\psi'^*_l &= (U\psi)_i(U\psi)^*_j(U\psi)_k(U\psi)^*_l \\ &= U_{i\lambda}U_{kv}\psi_\lambda\psi_v U^*_{j\mu}\psi^*_\mu U^*_{l\theta}\psi^*_\theta \\ &= U_{i\lambda}U_{kv}\psi_\lambda\psi_v\psi^*_\mu\psi^*_\theta U^+_{\mu j}U^+_{\theta l}, \end{aligned} \tag{B.2}$$

which can be written symbolically as

$$\psi' \otimes \psi^{*\prime} \otimes \psi' \otimes \psi'^* = U \otimes U(\psi \otimes \psi^* \otimes \psi \otimes \psi)U^+ \otimes U^+. \tag{B.3}$$

C. Transformation of the electromagnetic-like connection field

Using equations (32) and (33), we have

$$\frac{\partial}{\partial t} W'_{ij} = \frac{\partial}{\partial t} U_i W_{ij} U_j^+ + U_i \frac{\partial}{\partial t} W_{ij} U_j^+ + U_i W_{ij} \frac{\partial}{\partial t} U_j^+ \tag{C.1}$$

$$\begin{aligned} V'_i W'_{ij} &= U_i \frac{\partial}{\partial t} U_i^+ W'_{ij} + U_i V_i U_i^+ W'_{ij} \\ &= U_i \frac{\partial}{\partial t} U_i^+ U_i^+ W_{ij} U_j^+ + U_i V_i U_i^+ U_i W_{ij} U_j^+ \\ &= -\frac{\partial}{\partial t} U_i W_{ij} U_i^+ + U_i V_i W_{ij} U_j^+, \end{aligned} \tag{C.2}$$

where we used $\partial/\partial t(U_i U_i^+) = 0$.

Similarly, we have

$$W'_{ij} V'_j = U_i W_{ij} \dot{U}^+{}_j + U_i W_{ij} V(j) U_j^+. \tag{C.3}$$

Combining equations (C.1), (C.2), and (C.3), we obtain equation (34):

$$\frac{\partial}{\partial t} W'_{ij} + V'_i W'_{ij} - W'_{ij} V'_j = U_i \left(\frac{\partial}{\partial t} W_{ij} + V_i W_{ij} - W_{ij} V_j \right) U_j^+. \tag{C.4}$$

In a similar way, we have equation (35):

$$\begin{aligned} V'_i + V'^+_i &= U_i \frac{\partial}{\partial t} U_i^+ + \frac{\partial}{\partial t} U_i U_i^+ + U_i (V_i + V_i^+) U_j^+ \\ &= U_i (V_i + V_i^+) U_i^+. \end{aligned} \tag{C.5}$$

D. Nonlinear interaction among the elements

Starting from equation (52) with equation (5), we have

$$\Gamma'_{(i,j)(k,l)} (U\psi)_i^* (U\psi)_j (U\psi)_k^* (U\psi)_l = \Gamma_{(\lambda,\mu)(\nu,\theta)} \psi_\lambda^* \psi_\mu \psi_\nu^* \psi_\theta. \tag{D.1}$$

Expanding the left-hand side of equation (D.1), we obtain

$$\Gamma'_{(i,j)(k,l)} U_{\lambda i}^+ \psi_\lambda^* U_{j\mu} \psi_\mu U_{\nu k}^+ \psi_\nu^* U_{l\theta} \psi_\theta = \Gamma_{(\lambda,\mu)(\nu,\theta)} \psi_\lambda^* \psi_\mu \psi_\nu^* \psi_\theta. \tag{D.2}$$

Noting that $UU^+ = U^+U = 1$, we have

$$\left. \begin{aligned} &U_{j\mu} U_{\mu j'}^+ = \delta_{jj'}, \quad U_{k'\nu} U_{\nu k}^+ = \delta_{k'k} \\ &U_{l\theta} U_{\theta l'}^+ = \delta_{ll'}. \end{aligned} \right\} \tag{D.3}$$

Furthermore, using equation (D.3), we can derive equation (53):

$$\Gamma'_{(i,j)(k,l)} = U_{i\lambda} U_{k\nu} \Gamma_{(\lambda,\mu)(\nu,\theta)} U_{\mu j}^+ U_{\theta l}^+ \tag{D.4}$$

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Differentiating both sides of equation (D.4) with respect to time and using equations derived from equation (8),

$$\frac{\partial U}{\partial t} = UW - W'U \tag{D.5}$$

$$\frac{\partial U^{+}}{\partial t} = U^{+}W' - WU^{+}. \tag{D.6}$$

We have equation (54) with equation (55)

$$\frac{\partial}{\partial t}\Gamma'_{(i,j)(k,l)} + [W', \Gamma']_{(i,j)(k,l)} = \\ U_{i\lambda}U_{k\nu}\left(\frac{\partial}{\partial t}\Gamma_{(\lambda,\mu)(\nu,\theta)} + [W, \Gamma]_{(\lambda,\mu)(\nu,\theta)}\right)U^{+}_{\mu,j}U^{+}_{\theta l} \tag{D.7}$$

after straightforward calculation, where $[W, \Gamma]_{(\lambda,\mu)(\nu,\theta)}$ or $[W', \Gamma']_{(i,j)(k,l)}$ is an abbreviated notation defined as

$$[W, \Gamma]_{(\lambda,\mu)(\nu,\theta)} \equiv \\ W_{\lambda\rho}\Gamma_{(\rho,\mu)(\nu,\theta)} - \Gamma_{(\lambda,S)(\nu,\theta)}W_{S\mu} + W_{\nu q}\Gamma_{(\lambda,\mu)(q,\theta)} - \Gamma_{(\lambda,\mu)(\nu,\gamma)}W_{\gamma\theta}. \tag{D.8}$$

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