

Agent-Based Simulation of N-Person Games with Crossing Payoff Functions

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We report on computer simulation experiments using our agent-based simulation tool to model uniform N -person games with crossing payoff functions. We study the case when agents are greedy simpletons who imitate the action of their neighbor that received the highest payoff for its previous action.

The individual agents may cooperate with each other for the collective interest or may defect, that is, pursue their selfish interests only. After a certain number of iterations the proportion of cooperators stabilizes to either a constant value or oscillates around such a value.

The payoff (reward/penalty) functions are given as two straight lines: one for the cooperators and another for the defectors. The payoff curves are functions of the ratio of cooperators to the total number of agents. Even if the payoff functions are linear, four free parameters determine them. In this investigation only crossing payoff functions are considered.

We have investigated the behavior of the agents systematically. The results show that the solutions are nontrivial and in some cases quite irregular. They show drastic changes for the Leader Game in the narrow parameter range of $1.72 \leq P \leq 1.75$. This behavior is similar to that observed in [1] for the N -Person Chicken Game. Irregular solutions were also found for the Reversed Stag Hunt Game.

1. Introduction

It is well known that there are 576 different two-person games of binary choice [2]. When two people play such a game, each has two choices: cooperate or defect. If both cooperate, they both receive a certain reward R . If both defect, they are both punished (P). If one of them cooperates but the other does not, then the cooperator receives a sucker's payoff S while the defector gets a temptation reward T . As there are $4! = 24$ possible preference orderings of these payoffs, 24 types of games are especially important to investigate.

These two-person games can readily be extended to an arbitrary number of players [3]. Agent-based simulation [4] is a convenient approach to investigate the arising N -person games. Simulation of the N -Person Prisoners' Dilemma Game has received considerable atten-

tion [5–8]. Very few papers are devoted to simulating other N -person games [9–11].

In this paper, agent-based simulation of uniform N -person games with crossing payoff functions is presented. The agents are considered to be greedy simpletons who imitate the action of their neighbor that received the highest payoff for its previous action. In this case only the relative payoffs count, so a thorough investigation is possible.

2. The model

The individual agents may cooperate with each other for the collective interest or may defect, that is, pursue their selfish interests. Their decisions to cooperate or defect will accumulate over time to produce a result that will determine the success or failure of the given artificial society.

Let us first consider the payoffs for both choices (Figure 1). The horizontal axis represents the number of cooperators related to the total number of agents. We assume that the payoffs are linear functions of this ratio x . We now extend the notations introduced in section 1. Point P corresponds to the punishment when all agents defect, point R is the reward when all agents cooperate, T is the temptation to defect when everybody else cooperates, and S is the sucker's payoff for cooperating when everyone else defects. $C(0)$ and $D(1)$ are impossible by definition, but we follow the generally accepted notation by extending both lines for the full range of $0 \leq x \leq 1$ and denoting $C(0) = S$ and $D(1) = T$, making it simpler to define the payoff functions.

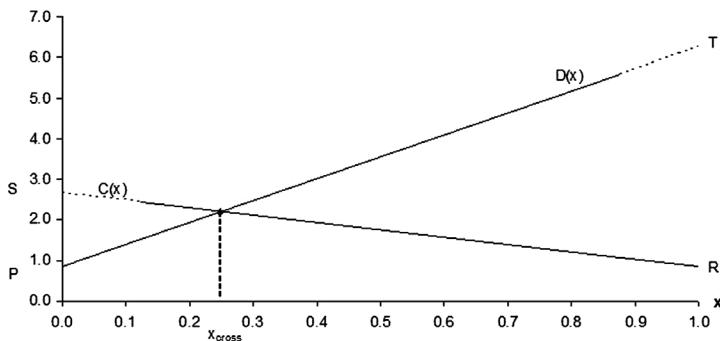


Figure 1. Payoff (reward/penalty) functions for defectors (D) and cooperators (C). The horizontal axis (x) represents the ratio of the number of cooperators to the total number of agents in the neighborhood. The vertical axis is the payoff provided by the environment. In this case, $D(x) = 0.85 + 5.4x$ and $C(x) = 2.65 - 1.8x$. The two lines cross at $x_{\text{cross}} = 0.25$. The payoff value corresponding to this point is $y_{\text{cross}} = C(x_{\text{cross}}) = D(x_{\text{cross}}) = 2.20$. The broken lines represent the fact that the values $C(0)$ and $D(1)$ are not possible by definition.

We connect by straight lines point S with point R (cooperators' payoff function C) and point P with point T (defectors' payoff function D). Thus the payoff to each agent depends on its choice and on the distribution of other players among cooperators and defectors. The payoff function $C(x)$ is the same for all cooperators and $D(x)$ is the same for all defectors (uniform game). The two lines cross each other at the point x_{cross} .

There are 12 different orderings of the values of P , R , S , and T that lead to crossing payoff lines. Each of them represents a different type of game. For the payoff functions shown in Figure 1, for example, we have $T > S > P = R$. This is the borderline case in between $T > S > P > R$ (Battle of the Sexes Game) and $T > S > R > P$ (Benevolent Chicken Game).

We will use our agent-based model developed for simulated social and economic experiments with a large number of decision-makers operating in a stochastic environment [12]. The simulation environment is a two-dimensional array of the participating agents. Its size is limited only by the computer's virtual memory. The behavior of a few million interacting agents can easily be observed on the computer's screen.

There are two actions available to each agent, and each agent must choose between them (cooperation or defection). The cooperators and the defectors are initially distributed randomly over the array. In the iterative game the aggregate cooperation proportion changes in time, that is, over subsequent iterations. At each iteration, every agent chooses an action according to the rewards to its neighbors. The software tool draws the array of agents in a window on the computer's screen, with each agent in the array colored according to its most recent action. The updating occurs simultaneously for all agents.

After a certain number of iterations the proportion of cooperators stabilizes to either a constant value or oscillates around such a value. The experimenter can view and record the evolution of the society of agents as it changes in time. The outcome of the game strongly depends on the "personalities" of the agents, that is, on the type of their responses to their environment. The software tool allows for a number of different personalities and their arbitrary combinations. In this paper we assume that all agents are greedy simpletons.

3. Simulation

Throughout this paper the total number of agents is $500 \times 500 = 250,000$. The initial ratio of cooperators is 50% and the neighborhood of each agent is one layer deep, that is, each agent has exactly eight neighbors except those that are situated at the borders of the array.

Writing the equations of the payoff functions in the form of

$$C(x) = S + (R - S)x \quad (1)$$

and

$$D(x) = P + (T - P)x \quad (2)$$

our free parameters are the four constants P , R , S , and T . They are chosen in such a way that the two lines cross each other.

We have performed a systematic investigation of the game for hundreds of values of these parameters. The global ratio $X(t)$ of the total number of cooperators in the entire array as a function of time (iterations) was observed for all parameter values. (Note that X is different from x which refers to an agent's immediate neighbors only.) The final ratio of cooperators X_{final} around which $X(t)$ oscillates represents the solution of the game.

The initial random distribution of the 250,000 agents is shown in Figure 2. Black dots represent cooperators, white dots represent defectors. Figure 3 shows the distribution of the agents after 500 iterations for the payoff functions shown in Figure 1. We see that the ratio of black dots is well below 50% and they form linear clusters.

Figure 4 shows the $X(t)$ function for the payoff lines shown in Figure 1. The solution is reached after 100 iterations. $X(t)$ oscillates between $X_1 = 0.11$ and $X_2 = 0.32$, corresponding to a solution of $X_{\text{final}} = 0.22$ (indeed a low cooperation rate).

If at $x < x_{\text{cross}}$ the cooperators get a higher payoff than the defectors and at $x > x_{\text{cross}}$ the defectors get a higher payoff than the cooperators,

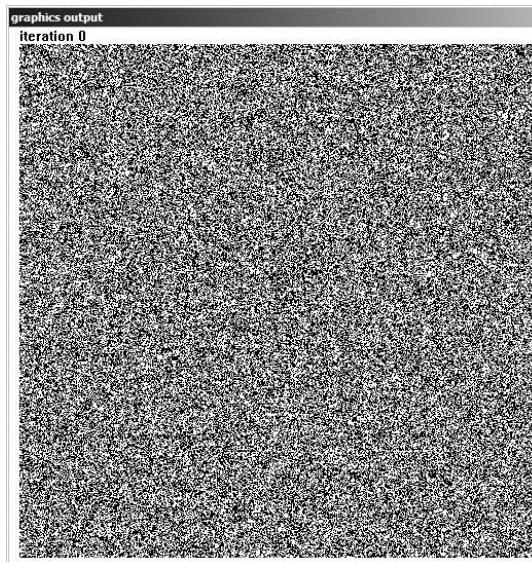


Figure 2. The initial random distribution of the 250,000 agents. Black dots represent cooperators, white dots represent defectors.

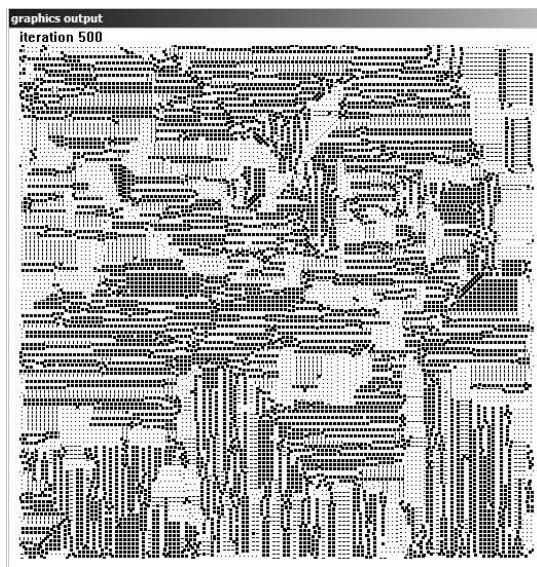


Figure 3. The distribution of the 250,000 agents after 500 iterations for the payoff functions shown in Figure 1. Black dots represent cooperators, white dots represent defectors.

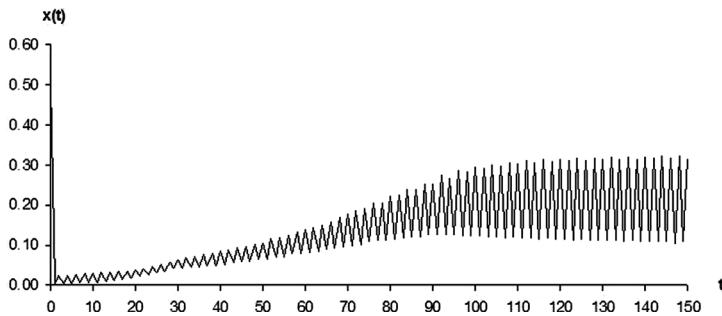


Figure 4. The $X(t)$ function for the payoff lines shown in Figure 1. The solution is $X_{\text{final}} = 0.22$.

then the intersection point x_{cross} of the two payoff functions is a Nash equilibrium for rational players. Indeed, under these conditions at $x < x_{\text{cross}}$ the number of cooperators increases and this number decreases when $x > x_{\text{cross}}$. This is, however, not true for the greedy simpletons because x refers to the immediate neighbors only while the final ratio of cooperators X_{final} represents the ratio of the total number of cooperators in the entire array. In the case of the payoff functions presented in Figure 1, for example, the solution is $X_{\text{final}} = 0.22$ while $x_{\text{cross}} = 0.25$.

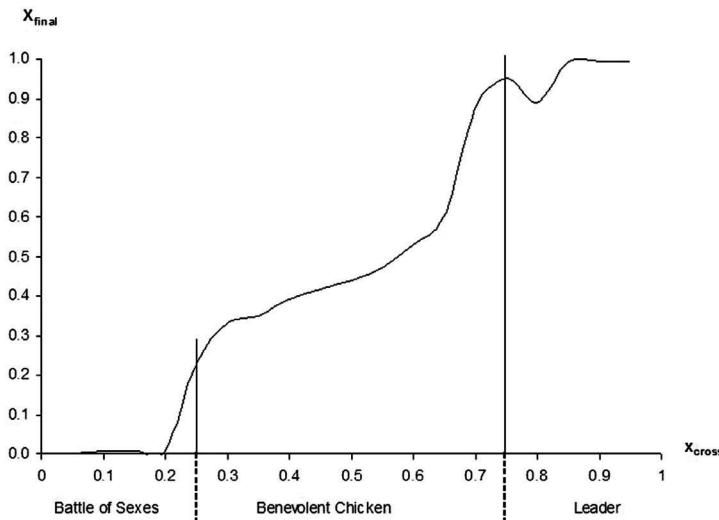


Figure 5. The solution of the game X_{final} as a function of x_{cross} . The values of $(R - S) = -1$, $(T - P) = 3$, and $y_{\text{cross}} = 2.2$ are fixed.

In view of this fact, we were first interested to find out how the value of X_{final} changes as a function of x_{cross} . To investigate this dependence, we fixed the values of the payoff lines' slopes at $R - S = -1$ and $T - P = 3$ and moved the two lines so that x_{cross} changed but the payoff value $y_{\text{cross}} = 2.2$ corresponding to x_{cross} remained constant. We expected a monotonically increasing $X_{\text{final}}(x_{\text{cross}})$ dependence. The result is shown in Figure 5. Indeed, for $x_{\text{cross}} < 0.2$ the solution is total defection and it is overwhelming cooperation for $x_{\text{cross}} > 0.85$. The transition in between these two limiting values, however, is quite interesting. As can be seen, in the region $0.75 < x_{\text{cross}} < 0.80$ the value of X_{final} even decreases as x_{cross} grows.

It is interesting to note the transitions between different games as the value of x_{cross} changes. At these values of the fixed parameters we have

$$P = 2.2 - 3x_{\text{cross}}$$

$$R = 1.2 + x_{\text{cross}}$$

$$S = 2.2 + x_{\text{cross}}$$

$$T = 5.2 - 3x_{\text{cross}}.$$

Using these expressions, we can easily determine the games to which different regions of x_{cross} correspond. For $0 \leq x_{\text{cross}} \leq 0.2$ we have the Battle of the Sexes Game, for $0.3 \leq x_{\text{cross}} \leq 0.7$ it is the Benevolent Chicken Game, and for $0.8 \leq x_{\text{cross}} \leq 1.0$ we have the Leader Game.

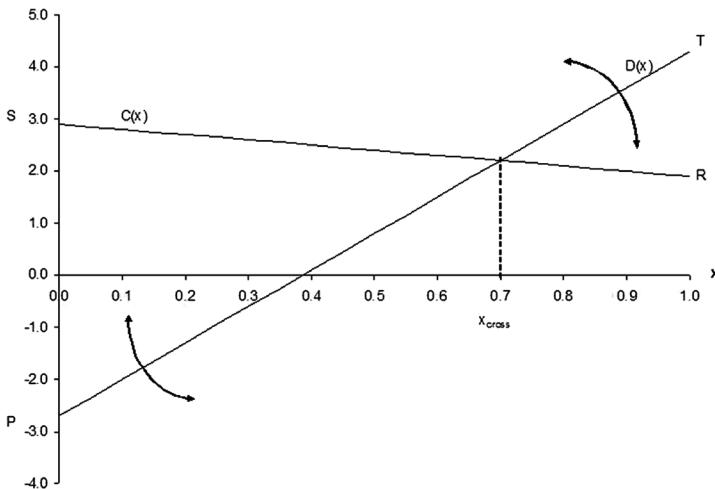


Figure 6. Rotating the $D(x)$ function around the intersection point of the two lines $x_{\text{cross}} = 0.7$, $y_{\text{cross}} = 2.2$. $C(x) = 2.9 - x$.

$x_{\text{cross}} = 0.25$ and $x_{\text{cross}} = 0.75$ are borderline cases in between two games. The transitions between games are relatively smooth.

Next, to investigate the role of the relative angle between the two payoff lines, we fixed the $C(x)$ function as $C(x) = 2.9 - x$, the intersection point of the two lines at $x_{\text{cross}} = 0.7$, $y_{\text{cross}} = 2.2$, and rotated the $D(x)$ function around this point (Figure 6). As we rotate $D(x)$, its end points change:

$$-10.0 \leq P \leq 2.9$$

$$R = 1.9$$

$$S = 2.9$$

$$T = P + \frac{2.2 - P}{0.7} \quad (7.43 \geq T \geq 1.9).$$

(The upper limit of P is chosen as S to maintain the condition that at $x < x_{\text{cross}}$ the cooperators get a higher payoff than the defectors and at $x > x_{\text{cross}}$ the defectors get a higher payoff than the cooperators.) Under these conditions for $-10.0 \leq P \leq 0.5$ we have the Benevolent Chicken Game, for $0.6 \leq P \leq 1.8$ it is the Leader Game, for $2.0 \leq P \leq 2.1$ we have the Reversed Benevolent Chicken Game, and for $2.3 \leq P \leq 2.9$ it is the Reversed Stag Hunt Game. $P = 0.56$, $P = 1.9$, and $P = 2.2$ are borderline cases in between two games. The transitions between games are quite smooth.

We present the result of the simulation X_{final} as a function of P in Figure 7. For the region $-10.0 \leq P \leq -0.6$ the result is nearly constant around the value of $X_{\text{final}} = 0.82$. Therefore, we only show the result

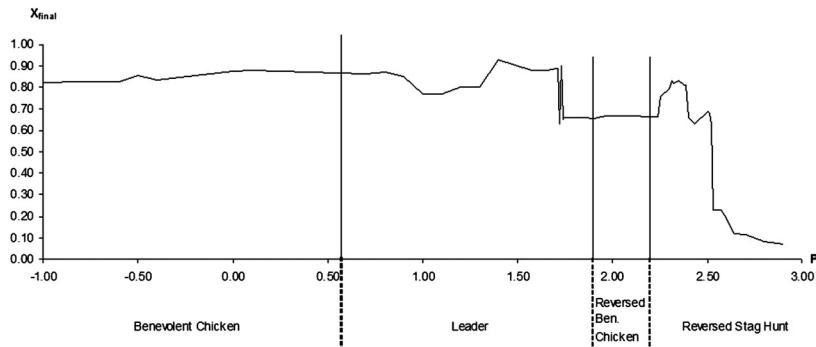


Figure 7. The solution of the game X_{final} as a function of P . The $D(x)$ function is rotated around the fixed intersection point.

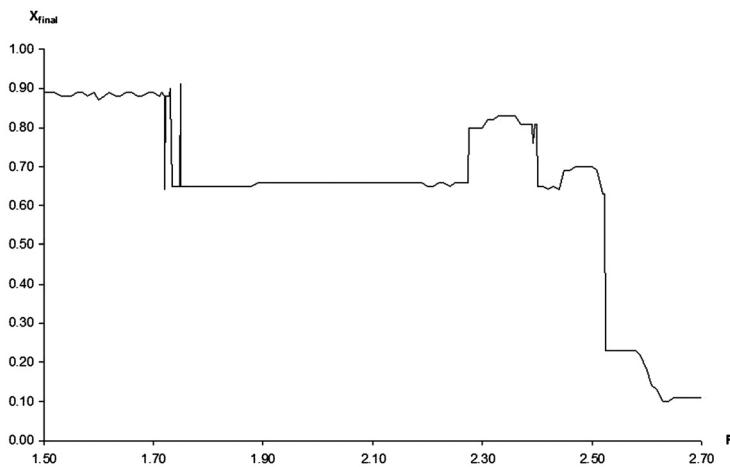


Figure 8. The solution of the game X_{final} as a function of P in the range of $1.5 \leq P \leq 2.7$. The $D(x)$ function is rotated around the fixed intersection point.

for $-1.0 \leq P \leq 2.9$. Several remarkable features can be immediately noticed. First, neither the Benevolent Chicken nor the Reversed Benevolent Chicken games are sensitive to the rotation of the $D(x)$ function. However, the Leader and the Reversed Stag Hunt games behave quite strangely. In order to show this behavior in more detail, X_{final} is given as a function of P for the region $1.5 \leq P \leq 2.7$ in Figure 8.

The behavior of the Reversed Stag Hunt Game is quite remarkable. At $P = 2.275$ X_{final} suddenly jumps from 0.66 to 0.80, then rises to 0.83, then at $P = 2.394$ has a dip down to 0.76, rises again to 0.81, then at $P = 2.4$ jumps down to 0.65, rises again and at $P = 0.525$ suddenly changes from 0.63 to 0.23, and at $P = 2.65$ reaches its final value of

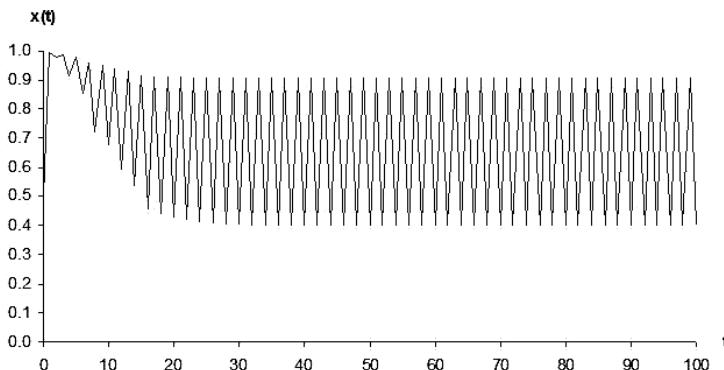


Figure 9. The $X(t)$ function for $P = 2.24$.

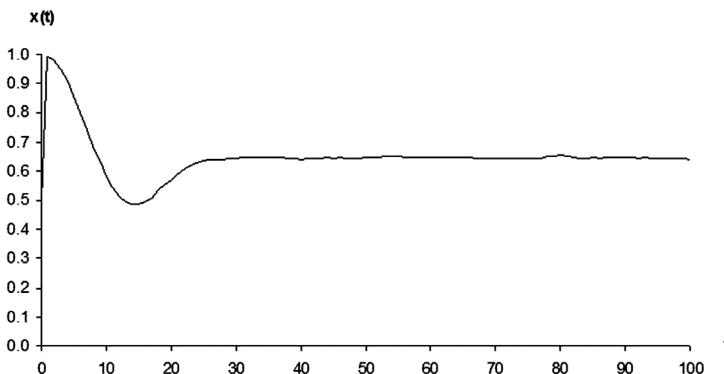


Figure 10. The $X(t)$ function for $P = 2.41$.

0.11. The character of the $X(t)$ function also drastically changes at these points. The $X(t)$ function for $P = 2.24$ wildly fluctuates in between 0.40 and 0.91 so that $X_{\text{final}} = 0.65$ (Figure 9). Figure 10 shows the $X(t)$ function for $P = 2.41$. In this case $X_{\text{final}} = 0.65$ again, but there are practically no fluctuations. The graphics outputs are also different. We show them for $P = 2.24$ and $P = 2.41$ in Figures 11 and 12, respectively.

The Leader Game behaves even more strangely. There are several wild fluctuations in between $X_{\text{final}} = 0.64$ and $X_{\text{final}} = 0.90$ in the narrow region of $1.72 \leq P \leq 1.75$. We found similar behavior in the N-Person Chicken Game [1]. These fluctuations are shown in more detail in Figure 13.

As shown earlier, the amplitude of the $X(t)$ function's fluctuation varies substantially. Figure 14 shows the amplitude as a function of the P parameter. We can see that the amplitude is very large (between 0.40 and 0.50) in the region $1.751 \leq P \leq 2.274$ and it is less than 0.1 outside of this region. The change is very abrupt at both ends of this region.

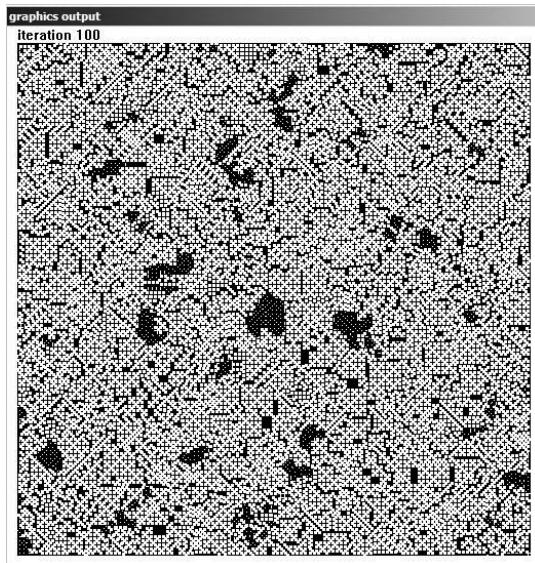


Figure 11. The distribution of the 250,000 agents after 100 iterations for $P = 2.24$. Black dots represent cooperators, white dots represent defectors.

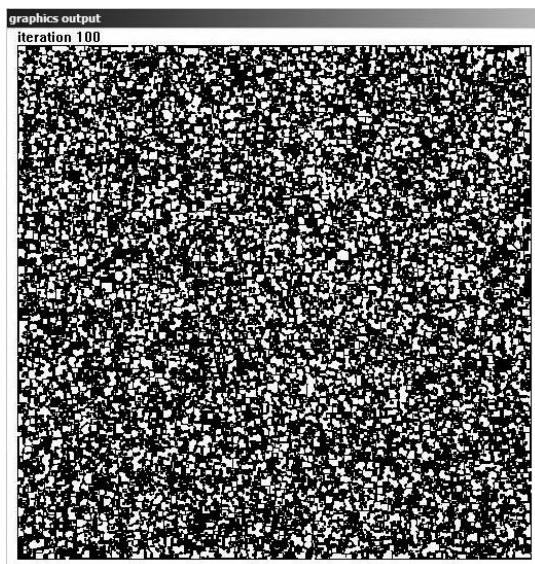


Figure 12. The distribution of the 250,000 agents after 100 iterations for $P = 2.41$. Black dots represent cooperators, white dots represent defectors.

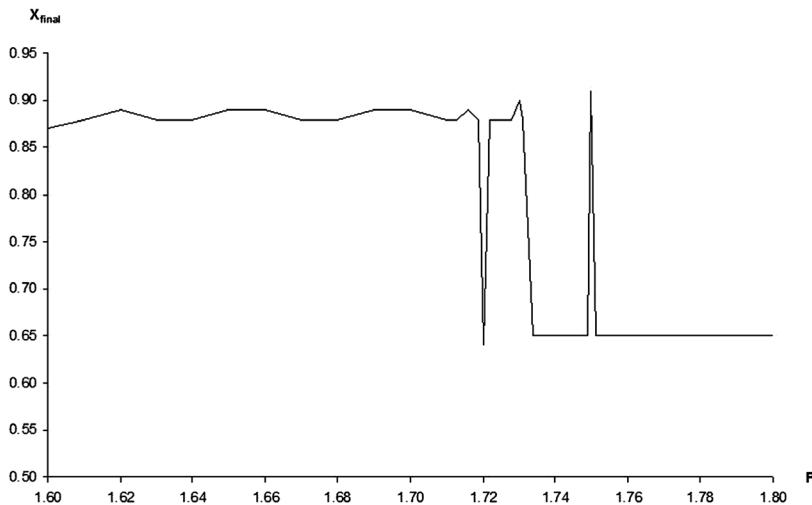


Figure 13. The solution of the game X_{final} as a function of P in the narrow region of $1.6 \leq P \leq 1.8$. The $D(x)$ function is rotated around the fixed intersection point.

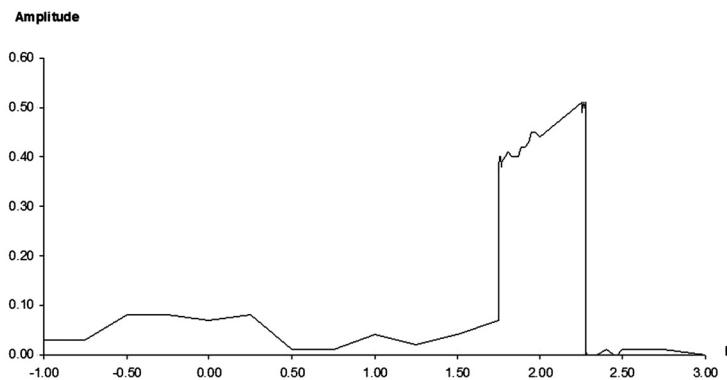


Figure 14. The amplitude of the $X(t)$ function's fluctuation as a function of the P parameter.

Finally, we fixed the $D(x)$ function as $D(x) = 2.9 - x$, the intersection point of the two lines at $x_{\text{cross}} = 0.7$, $y_{\text{cross}} = 2.2$, and rotated the $C(x)$ function around this point (Figure 15). As we rotate $D(x)$, its end points change:

$$3.2 \leq S \leq 28.1$$

$$P = 2.9$$

$$T = 1.9$$

$$R = S + \frac{2.2 - S}{0.7} \quad (1.77 \geq R \geq -8.9).$$

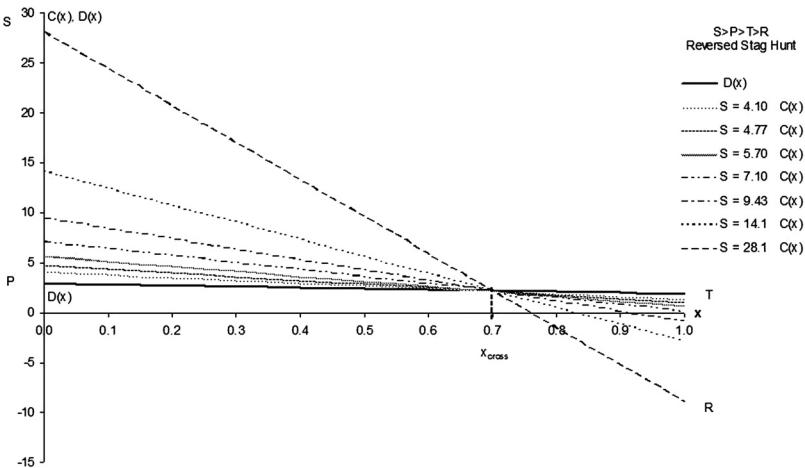


Figure 15. Rotating the $C(x)$ function around the intersection point of the two lines $x_{\text{cross}} = 0.7$, $y_{\text{cross}} = 2.2$. $D(x) = 2.9 - x$.

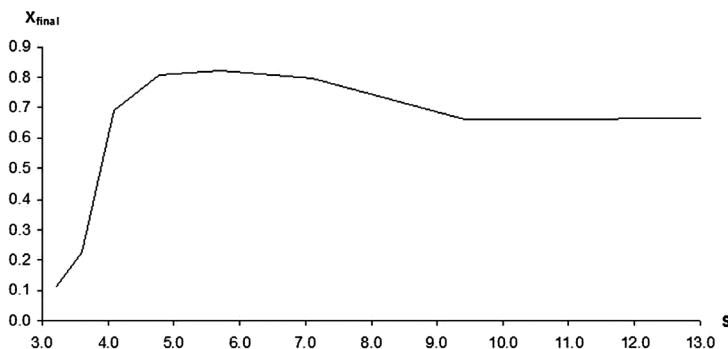


Figure 16. The solution of the game X_{final} as a function of S . The $C(x)$ function is rotated around the fixed intersection point.

These conditions correspond to the Reversed Stag Hunt Game. The result of the simulation X_{final} as a function of S is shown in Figure 16. This function is smooth as opposed to $X_{\text{final}}(P)$ (see Figure 8). The $X_{\text{final}}(S)$ function starts at a very low cooperation ratio, then begins to rise and reaches its maximum of $X_{\text{final}} = 0.82$ at $S = 5.7$. From here, it goes down smoothly. After $S \geq 9.4$ it is practically steady at $X_{\text{final}} = 0.66$.

4. Conclusion

The solutions of N -person games with crossing payoff functions are nontrivial and in some cases quite irregular. They show drastic changes

in the Leader Game in the narrow parameter range of $1.72 \leq P \leq 1.75$. This behavior is similar to that observed in [1] for the N-Person Chicken Game. Irregular solutions were also found for the Reversed Stag Hunt Game. The explanation of these phenomena would require a thorough mathematical analysis of N -person games, which is not yet available in the literature. Further simulations of N -person games may lead us to a better understanding of these results.

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