Classification of Elementary Cellular Automata Based on Their Limit Cycle Lengths in $\mathbb{Z} / k$

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In this paper we introduce a classification of elementary cellular automata based solely on numerical properties of the lengths of their limit cycles on finite lattices $\mathbb{Z} / k$. The classification has a formal definition, and it could in principle be proved whether a given cellular automaton belongs to a given class. It will remain open if this is generally possible, that is, if the question is decidable.

Keywords: cellular automata; limit sets; limit cycles; cycle length spectra; decidability problems; classification

1. Introduction

We assume that the concepts of cellular automata are known and will use the terms "rule" and "cellular automaton" synonymously. The naming of rules by decimal numbers is according to Wolfram's notation.

The paper is organized as follows: First we introduce limit cycles and cycle length spectra. We further introduce a software tool called "Another kind of atlas" that we developed to display and investigate cellular automata, their limit cycles and cycle length spectra. Then we summarize some numerical observations on cycle lengths related to lattice size k. We then give a short overview of classifications of cellular automata and specify the announced classification of elementary cellular automata based on the numerical observations. Finally we compare it with other classifications and end with a short discussion of open questions.

2. Limit Cycles on Finite Lattices

Limit sets and especially limit cycles of cellular automata on finite lattices have already been studied [1].

A lattice of size k with two cell states has a state space of size 2^k. Each configuration $\alpha \in \left\{2^k\right\}$ necessarily evolves into a limit cycle, 2^k

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being the natural upper bound for limit cycle lengths. In the following, we will use the term "cycle" instead of "limit cycle" for conciseness.

Wuensche/Lesser's [2] as well as Wolfram's atlas [3] depicts many state transition diagrams including limit cycles and gives numbers on limit cycle lengths and multiplicities but only for rather small lattice sizes $k \lesssim 20$. Wolfram's atlas gives cycle length plots of all elementary automata but only for the single-black-cell initial configuration.

Wolfram et al. in [4] investigated limit cycles more carefully and calculated their lengths by algebraical means (namely generating functions) but only for linear or additive rules, mainly rule 90.

Limit cycle lengths play a role in some classifications (see below). In Li and Packard's classification (see [5]) cellular automata are classified as "chaotic" when they have "exponentially divergent cycle lengths as lattice length is increased."

The cycle length spectrum $\Gamma_R(k)$ is the mapping from lattice size k to the set of cycle lengths that rule R gives rise to. Cycles with periodic spatial configurations (and their lengths) can be ignored because they already appeared for smaller k. Of special interest is the maximal cycle length $L_R(k)$ and the envelope $\Lambda_R(k)$ of this function.

For the sake of our investigations, we calculated the complete cycle length spectrum for all elementary cellular automata—of which there are 88 up to equivalence—and all lattice sizes up to $k = 22$. For rule 45 we had to stop at $k = 18$ due to limited processing power. Beyond these numbers, only partial cycle length spectra were calculated by starting from 10 000 random initial configurations. The largest k that could be achieved even for complex rules like 18, 73 or 110 was $k = 56$, due to limitations of processing power and time. For some other rules, we went up to $k = 29$ and $k = 36$; for rule 45, only up to $k = 22$. For details, see the Appendix.

Features of complexity, chaoticity and exponentiality can be found not only in spacetime diagrams of cellular automata but also in the plots of their cycle length spectra. Cycle length spectra have the advantage that there is only one diagram to look at, not overwhelmingly many as for spacetime diagrams. Nevertheless, we can literally see order and chaos, linear, exponential and—surprise!—quadratic growth. Last but not least, cycle length spectra are identically the same for equivalent rules.

We developed and used an interactive software tool (called Another kind of atlas) to visually inspect the cycle length spectra of different rules and find regularities, especially in the form of more or less densely populated "lines" $L(k)$ in the L-k plots. We also implemented filters to find such lines, be they linear, quadratic or exponential:

- $L(k) = k + n$
- $L(k) = k \cdot n$
- $L(k) = k \cdot n / m$
- $L(k) = k \cdot (k + n)$
- $L(k) = n \cdot k^2$
- $L(k) = 2^n - 2^m$ for $m = 0, 1, 2, 3, 4$

The table views in Figure 1 gave us a first hint to a quadratic dependence of L on k for rules 41, 54 and 110.

For more details on the software tool, see the Appendix.

There is a cycle with	
L=k·(k+0)	for rule = 54, 106, 110
L=k·(k+1)	for rule = 41, 45, 73, 106
L=k·(k+2)	for rule = 54, 73, 106, 110
L=k·(k+4)	for rule = 54, 106, 110
L=k·(k+5)	for rule = 106, 110
L=k·(k+6)	for rule = 41, 110
L=k·(k+7)	for rule = 45, 54, 110
L=k·(k+8)	for rule = 45, 54, 106, 110
L=k·(k+9)	for rule = 41, 54, 73, 106, 110
L=k·(k+10)	for rule = 73
L=k·(k+11)	for rule = 45, 54, 106, 110

There is a cycle with	
L=1·k²	for rule = 54, 106, 110
L=2·k²	for rule = 45, 54, 106, 110
L=3·k²	for rule = 45
L=4·k²	for rule = 45
L=6·k²	for rule = 106
L=11·k²	for rule = 45
L=41·k²	for rule = 106
L=67·k²	for rule = 106

Figure 1. Two table views in Another kind of atlas, giving hints to quadratic cycle length growth for rules 41, 54 and 110. Rules 45, 73 and 106 that also pop up here will turn out to exhibit exponential cycle length growth.

3. Numerical Properties of Cycle Lengths

We made a great many observations that relate cycle lengths L to lattice size k.

- There are rules with only constant cycle lengths $L = 1, 2$, for example, rules 0 (FALSE), 51 (NOT) and 204 (IDENTITY).
- There are rules where each cycle length is a multiple of k, for example, rules 170 (LEFT-SHIFT) and 184.
- There are rules with a pseudo-envelope $\Lambda_R(k)$ that is quadratic in k, namely rules 54 and 110.
- There are rules with an envelope $\Lambda_R(k)$ depending exponentially on k. We found either $\Lambda_R(k) = a \cdot \left(2^{f(k)} - 1\right)$ with $a = 1, 2$ or $\Lambda_R(k) = k \cdot \left(2^{f(k)} - 1\right)$ with some linear function $f(k)$.

Other numerical findings, not immediately related to our classification task:

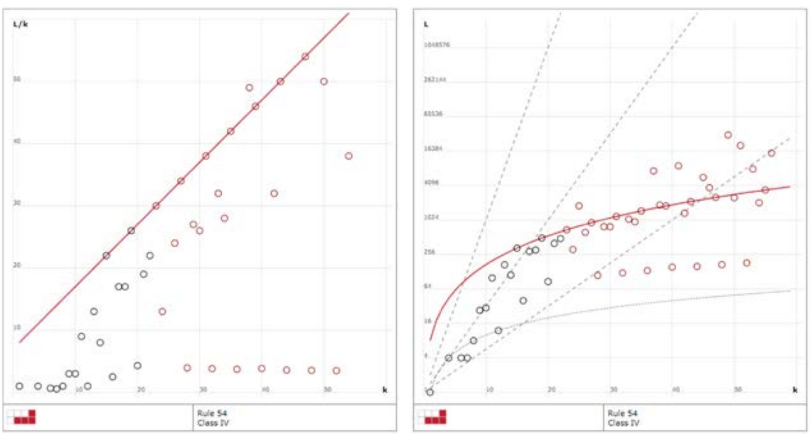
- Rules 26 and 154 are the only rules with linear growth that have cycle lengths $L = 4 \cdot k$.
- Rule 54 is the only rule with abundant cycles of length $L = k \cdot (k + 7)$.
- Rule 110 is the only rule with abundant cycles of length $L = 7$.

We now take a closer look at the nontrivial cases, the constant and linear cases considered trivial. The plots show the length $L_R(k)$ of the longest cycle for each rule R and k. Black circles indicate cycles that are guaranteed to be the longest; red circles indicate the longest cycle found in partial cycle length spectra, which is not necessarily the absolutely longest. The dashed lines are the lines $L(k) = 2^k, 2^{k/2}, 2^{k/4}$; the dotted line is $L(k) = k$.

3.1 The Quadratic Cases

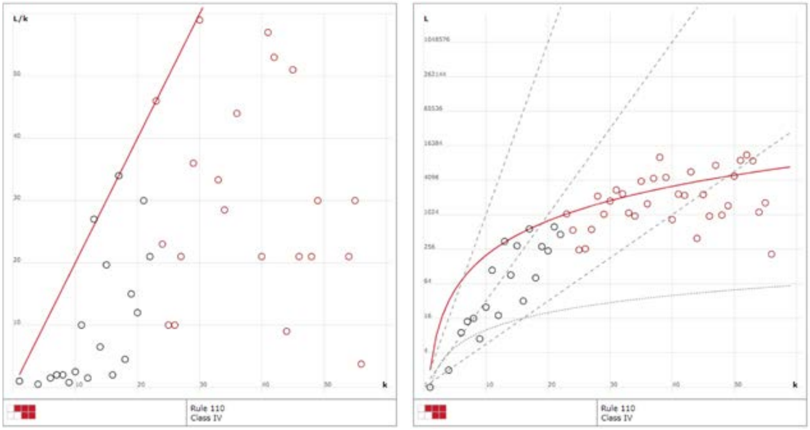
3.1.1 Rule 54

For rule 54, there is a pseudo-envelope $\Lambda_{54}(k)$ depending quadratically on k, best to be seen when plotting L / k over k. The exact dependence is $\Lambda_{54}(k) = k \cdot (k + 7)$. Logarithmic scaling reveals that $\Lambda_{54}(k)$ is in fact not the true envelope.



3.1.2 Rule 110

For rule 110, there is a pseudo-envelope $\Lambda_{110}(k) = 2 \cdot k^2$, but it does not match perfectly: for larger k, the lengths fall short of $2 \cdot k^2$. This is comparable to the case of rule 45 (see below).



This is how true values of $L_{110}(k)$ deviate from $2 \cdot k^2$.

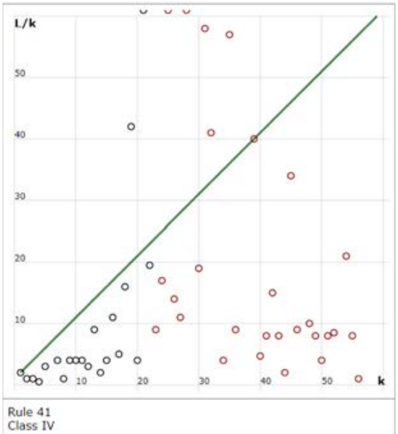
k	L	$L(k)$
13	351	$2 \cdot k^2 + 1 \cdot k$
17	578	$2 \cdot k^2 + 0 \cdot k$
23	1058	$2 \cdot k^2 + 0 \cdot k$
30	1770	$2 \cdot k^2 - 1 \cdot k$
50	4825	$2 \cdot k^2 - \frac{7}{2} \cdot k$

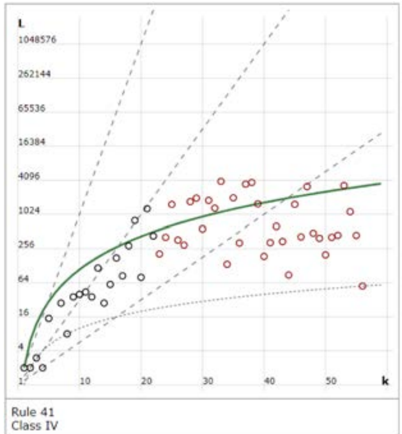
Taking all cycles into account—not only the longest—we find other lengths lying near $\Lambda_{110}(k) = 2 \cdot k^2$.

k	L	$L(k)$
28	1568	$2 \cdot k^2 + 0 \cdot k$
28	1652	$2 \cdot k^2 + 3 \cdot k$
30	1770	$2 \cdot k^2 - 1 \cdot k$
43	3999	$2 \cdot k^2 + 7 \cdot k$
61	6954	$2 \cdot k^2 - 8 \cdot k$
73	10 366	$2 \cdot k^2 - 4 \cdot k$
87	15 486	$2 \cdot k^2 + 4 \cdot k$

3.1.3 Rule 41

For rule 41, a pseudo-envelope $\Lambda_{41}(k)$ depending quadratically on k can hardly be seen, even when plotting L / k over k. Rule 41 is a notorious borderline case that is classified quite differently by different classifications.





Questions:

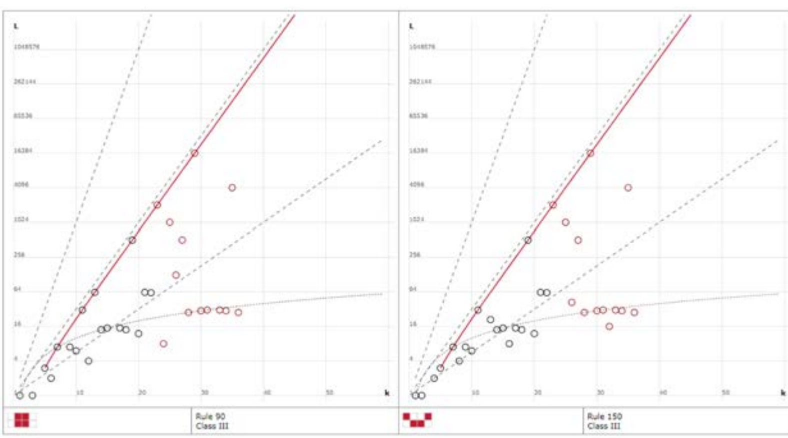
- Is there a closed form of the pseudo-envelope $\Lambda_{110}(k)$, especially of the error term $\epsilon(k) = a(k) \cdot k$ in $L_{110}(k) = 2 \cdot k^2 + \epsilon(k)$?
- If not, is there an order $O(f(k))$ and what is the least upper bound for $|a(k)|$? The least we can say is that $a(k) < k$, and presumably $a(k)$ is of order $O(\sqrt{k})$.

3.2 The Case $\Lambda_R(k) = 2^{f(k)} - 1$

Here we found true envelopes $\Lambda_R(k)$.

3.2.1 Rules 90 and 150

Rules 90 and 150 have identical maximal cycle lengths and thus identical envelopes $\Lambda_R(k)$. We find $f(k) = (k-1)/2$.

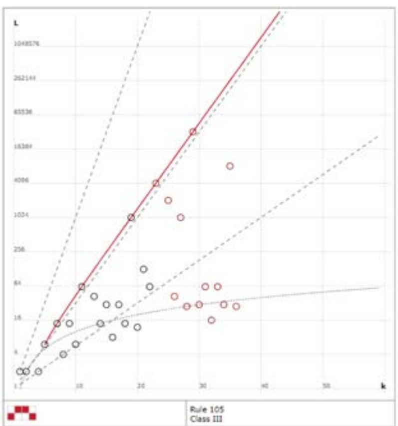


k	L	$f(k)$
5	$3 = 2^2 - 1$	2
7	$7 = 2^3 - 1$	3
11	$31 = 2^5 - 1$	5
13	$63 = 2^6 - 1$	6
19	$511 = 2^9 - 1$	9
23	$2047 = 2^{11} - 1$	11
29	$16\,383 = 2^{14} - 1$	14

3.3 The Case $\Lambda_R(k) = 2 \cdot \left(2^{f(k)} - 1\right)$

3.3.1 Rule 105

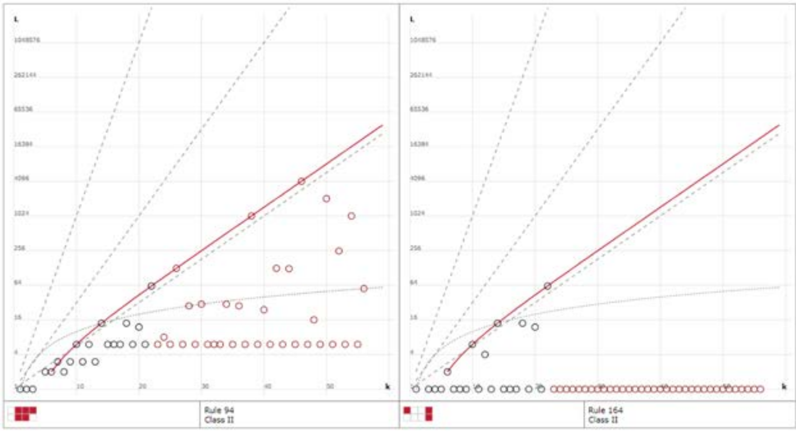
For rule 105, we find $f(k) = (k - 1) / 2$.



k	L	$f(k)$
5	$6 = 2 \cdot \left(2^2 - 1\right)$	2
7	$14 = 2 \cdot \left(2^3 - 1\right)$	3
11	$62 = 2 \cdot \left(2^5 - 1\right)$	5
19	$1022 = 2 \cdot \left(2^9 - 1\right)$	9
23	$4094 = 2 \cdot \left(2^{11} - 1\right)$	11
39	$32\,766 = 2 \cdot \left(2^{14} - 1\right)$	14

3.3.2 Rules 94 and 164

For rule 94 and similarly for rule 164, we find $f(k) = (k - 2) / 4$. Note that these rules are class II rules according to Wolfram's classification, exhibiting a kind of exponential growth nevertheless.

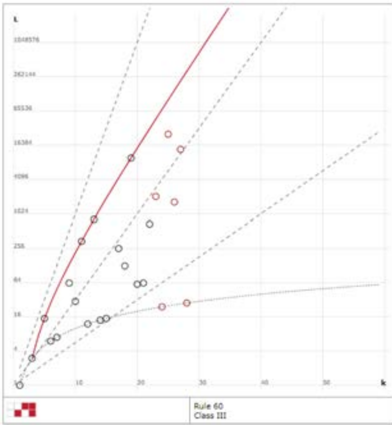


k	L	$f(k)$
6	$2 = 2 \cdot \left(2^1 - 1\right)$	1
10	$6 = 2 \cdot \left(2^2 - 1\right)$	2
14	$14 = 2 \cdot \left(2^3 - 1\right)$	3
22	$62 = 2 \cdot \left(2^5 - 1\right)$	5
26	$126 = 2 \cdot \left(2^6 - 1\right)$	6
38	$1022 = 2 \cdot \left(2^9 - 1\right)$	9
46	$4094 = 2 \cdot \left(2^{11} - 1\right)$	11

3.4 The Case $\Lambda_R(k) = k \cdot \left(2^{f(k)} - 1\right)$

3.4.1 Rule 60

For rule 60, we find:

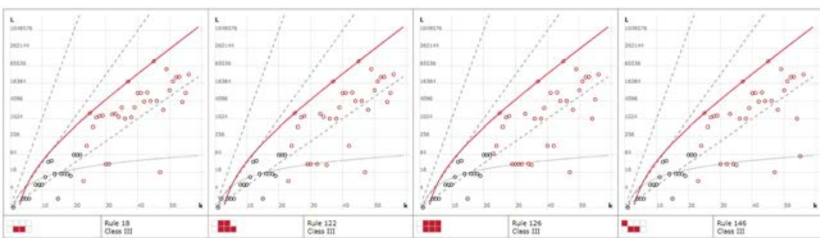


k	L	L/k	$f(k)$
3	3	$1 = 2^1 - 1$	1
5	15	$3 = 2^2 - 1$	2
11	341	$31 = 2^5 - 1$	5
13	819	$63 = 2^6 - 1$	6
19	9709	$511 = 2^9 - 1$	9
25	25 575	$1023 = 2^{10} - 1$	10

Except for $k = 25$, we find $f(k) = (k - 1) / 2$. The reason for the exception at $k = 25$ may be that the longest cycle was not found for $k = 25$; that is, that the cycle with length 25 575 is not the longest.

3.4.2 Rules 18, 122, 126, 146

Rules 18, 122, 126, 146 show identical maximal cycle lengths for $k = 12, 25, 37, 45$ and thus have identical envelopes $\Lambda_R(k)$.



k	L	L/k	$f(k)$
12	36	$3 = 2^2 - 1$	2
25	1575	$63 = 2^6 - 1$	6
37	18 907	$511 = 2^9 - 1$	9
45	92 115	$2047 = 2^{11} - 1$	11

Except for $k = 12$, we find $f(k) = (k - 1) / 4$.

For all the rules given, the same questions have to be asked and answered:

- Are there infinitely many $L \in \Lambda_R(k)$?
- Is the asymptotic density of $L \in \Lambda_R(k)$ finite or does it vanish?
- Is the set of k with $L_R(k) \in \Lambda_R(k)$ decidable?

With respect to the quadratic rules 41, 54 and 110, these questions need to be investigated:

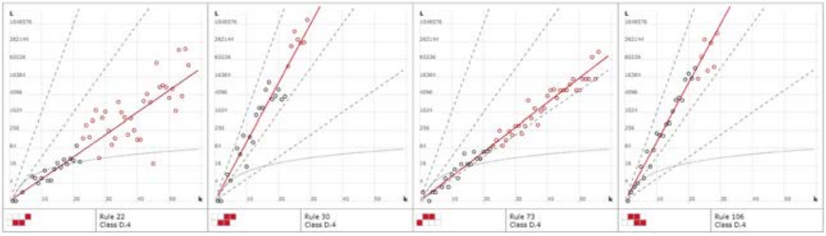
- Are there possibly pseudo-envelopes of polynomial degree greater than 2?
- Are there rules with quadratic growth of cycle lengths other than rules 41, 54 and 110?

3.5 Mean or Bounded Exponential Growth

There are rules that did not allow us to make such exact numerical observations, namely rules 22, 30, 45, 73 and 106, for all of which no closed functional form for any envelope could be found. For these rules, the maximal cycle lengths are either scattered around or bounded by an exponential function.

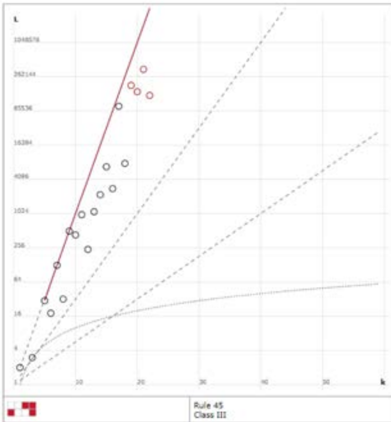
3.5.1 Rules 22, 30, 73, 106

For these rules, some sort of exponential "regression" line seems to exist around which the maximal cycle lengths are scattered. Here are rough, hand-drawn approximations.



3.5.2 Rule 45

The maximal cycle lengths for rule 45 are bounded from above by $L(k) = 2^k - 2$, which is essentially the size of the state space. For $k = 5, 7$, the value $L(k) = 2^k - 2$ is actually taken; beyond this, the lengths fall short quickly.



k	L
5	$30 = 2^5 - 2$
7	$126 = 2^7 - 2$
9	$504 = 2^9 - 2^3$
17	$78,812 < 2^{17} - 2^{15}$
19	$18,3920 < 2^{19} - 2^{18}$
21	$352,884 < 2^{21} - 2^{20}$

Question:

- Is there a closed functional form of the envelope $\Lambda_{45}(k)$, especially of the error term ϵ in $L_{45}(k) = 2^k - \epsilon(k)$?

4. Classifications of Cellular Automata

Classification of cellular automata consists of:

1. the definition of a set of mutually exclusive properties (which define disjoint classes)
2. the classifying itself, that is, the estimation, calculation or in general determination of the class to which any cellular automaton under consideration belongs

The properties are typically global or emergent in nature, not immediately tellable from the local rules. Classifications may be restricted to elementary cellular automata, but most are generalizable to higher dimensions, larger radii and larger alphabets.

There are two dichotomies of classifications: formal versus informal and phenotypic versus genotypic (see Gutowitz [6]). Typical informal classifications are Wolfram's qualitative classifications (see [1, 7, 8]) without formal definitions and proof of mutual exclusiveness. Classifying is done more or less at discretion and is more of a class *estimation*. Informal classifications typically have borderline cases where it is not clear to which class some cellular automata belong. Many formal classifications turn out to be undecidable (see [9]), so may have (undecidable) borderline cases as well.

Phenotypic classifications are often informal and require the long-term observation of the evolution of a cellular automaton with its local rule applied to a large number of simple or random initial configurations. Genotypic classifications are immediately based on properties of the local rules and typically have formal definitions that allow us in principle to prove whether a given cellular automaton belongs to a given class.

The classification we will propose will be a formal phenotypic one.

Many classifications—for a good overview see [10]—aim in one way or another at complexity features like chaoticity, entropy and (exponential) growth. In most—if not all—classifications, the classes can be ordered from the simplest to the most complex. This allows classifications to be compared quantitatively. (Historical side note: It was a bit misleading that Wolfram named his nontrivial classes "class III" and "class IV," while in fact class III comprises the more complex, that is, more chaotic cellular automata compared to class IV.)

5. Classification Based on the Numerical Findings

We start by writing down defining formulas $\phi_i(R)$ for classes of cellular automata, but they would probably not pass the exclusiveness test. This will come in the next step. Remember that $\Gamma_R(k)$ is the set of cycle lengths that rule R gives rise to on $\mathbb{Z}/k$.

We give the formulas in two versions, a weaker and a stronger one:

- $(\exists_\infty\, k)\varphi(k)$ ("there are infinitely many k with $\varphi(k)$") defined by $(\exists\, k)\varphi(k) \wedge (\forall\, k)\varphi(k) \to (\exists\, k' > k)\varphi(k')$.
- $p(\varphi) > 0$ ("the asymptotic probability of being φ does not vanish") with $p(\varphi)$ defined by
$$p(\varphi) :\, = \lim_{K\to\infty} \frac{|\{k < K \mid \varphi(k)\}|}{K}.$$

It will have to be investigated which version is more appropriate.

Definition 1.

$$\begin{array}{rrrrcl}
\phi_0(R) = & (\forall\, k > 1) & & & & \Gamma_R(k) = \{\} \\
\phi_1(R) = & & (\exists_\infty\, k) & 1 & \in & \Gamma_R(k) \\
\phi_2(R) = & & (\exists_\infty\, k) & 2 & \in & \Gamma_R(k) \\
\phi_3(R) = & & (\exists_\infty\, k) & k \cdot 1 & \in & \Gamma_R(k) \\
\phi_4(R) = & & (\exists_\infty\, k) & k \cdot 2 & \in & \Gamma_R(k) \\
\phi_5(R) = & (\exists\, a > 2) & (\exists_\infty\, k) & k \cdot a & \in & \Gamma_R(k) \\
\phi_6(R) = & (\exists\, a, b \geq 0) & (\exists_\infty\, k) & k^2 \cdot b + k \cdot a(k) & \in & \Gamma_R(k) \\
\phi_7(R) = & & (\exists_\infty\, k) & \left(2^{(k-2)/4} - 1\right) \cdot 2 & \in & \Gamma_R(k)
\end{array}$$

$$\phi_8(R) = \quad (\exists_\infty\, k)\ \left(2^{(k-1)/4} - 1\right) \cdot k \ \in\ \Gamma_R(k)$$
$$\phi_9(R) = \quad (\exists_\infty\, k)\ \left(2^{(k-1)/2} - 1\right) \cdot 1 \ \in\ \Gamma_R(k)$$
$$\phi_{10}(R) = \quad (\exists_\infty\, k)\ \left(2^{(k-1)/2} - 1\right) \cdot 2 \ \in\ \Gamma_R(k)$$
$$\phi_{11}(R) = \quad (\exists_\infty\, k)\ \left(2^{(k-1)/2} - 1\right) \cdot k \ \in\ \Gamma_R(k)$$

With asymptotic probability $p(\phi)$:

Definition 2.

$$\phi'_0(R) = \quad (\forall\, k > 1) \quad \Gamma_R(k) = \{\}$$
$$\phi'_1(R) = \quad p(\ 1 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_2(R) = \quad p(\ 2 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_3(R) = \quad p(\ k \cdot 1 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_4(R) = \quad p(\ k \cdot 2 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_5(R) = \quad (\exists\, a > 2)\ p(\ k \cdot a \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_6(R) = \quad (\exists\, a, b \geq 0)\ p(\ k^2 \cdot b + k \cdot a(k) \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_7(R) = \quad p(\ \left(2^{(k-1)/4} - 1\right) \cdot 2 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_8(R) = \quad p(\ \left(2^{(k-2)/4} - 1\right) \cdot k \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_9(R) = \quad p(\ \left(2^{(k-1)/2} - 1\right) \cdot 1 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_{10}(R) = \quad p(\ \left(2^{(k-1)/2} - 1\right) \cdot 2 \ \in\ \Gamma_R(k)) > 0$$
$$\phi'_{11}(R) = \quad p(\ \left(2^{(k-1)/2} - 1\right) \cdot k \ \in\ \Gamma_R(k)) > 0$$

It may be observed that a regularity in the sequences of properties ϕ_{1-3}, ϕ_{3-5}, ϕ_{9-11} is not to be found for the properties ϕ_{7-8}.

In any case, the term $a(k)$ for ϕ_6 needs more careful consideration.

We now define mutually exclusive classes 0 to 12:

Definition 3.

- Class 11: Rule R is in class 11 if and only if $\phi_{11}(R)$ holds.
- Class 10: Rule R is in class 10 if and only if it is not in class 11 and $\phi_{10}(R)$ holds.
- Class 9: Rule R is in class 9 if and only if it is not in classes ≤ 10 and $\phi_9(R)$ holds.
- …
- Class 2: Rule R is in class 2 if and only if it is not in classes ≤ 3 and $\phi_2(R)$ holds.
- Class 1: Rule R is in class 1 if and only if it is not in classes ≤ 2 and $\phi_1(R)$ holds.

- Class 0: Rule R is in class 0 if and only if $\phi_0(R)$ holds.
- Class 12: Rule R is in class 12 if and only if it is not in any of the classes ≤ 11.

We assume that classes 0 to 5 are provably inhabited by Wolfram's class I and II rules. For classes 6 to 11, it must be proved rule by rule. They might be empty, so all the other rules would be in class 12, which is just the unspecific "other rules" class.

Assuming that our observations for small $k \approx 4$ can be generalized to arbitrarily large k—the transitions from $(\exists_4\ k)$ to $(\exists_\infty\ k)$ can be made—these would be the rules by class shown in Table 1.

Class 0	0, 8, 32, 128, 136, 160
Class 1	4, 12, 13, 36, 44, 72, 76, 77, 78, 104, 132, 140, 200, 204, 232
Class 2	1, 5, 19, 23, 28, 29, 33, 50, 51, 108, 156, 178
Class 3	2, 10, 24, 34, 40, 42, 46, 56, 57, 130, 138, 152, 162, 168, 170, 172, 184
Class 4	3, 6, 7, 9, 11, 14, 15, 27, 35, 38, 43, 58, 62, 134, 142
Class 5	25, 26, 37, 74, 154
Class 6	41, 54, 110
Class 7	94, 164
Class 8	18, 122, 126, 146
Class 9	90, 150
Class 10	105
Class 11	60
Class 12	22, 30, 45, 73, 106

Table 1. Assignment of rules to the classes defined in Definition 3.

The main conjecture of this paper is:

Conjecture 1. The assignment of rules to classes as given in Table 1 is correct.

It must be noted that no proof is given—and known to the author—for any of the complex rules starting from class 6.

To avoid too-small classes, it makes sense to group them into larger classes A (constant cycle length, classes 0 to 2), B (linear growth, classes 3 to 5), C (quadratic growth, class 6), D (deterministic exponential growth, classes 7 to 11) and X (stochastic exponential growth, class 12). We then have Table 2.

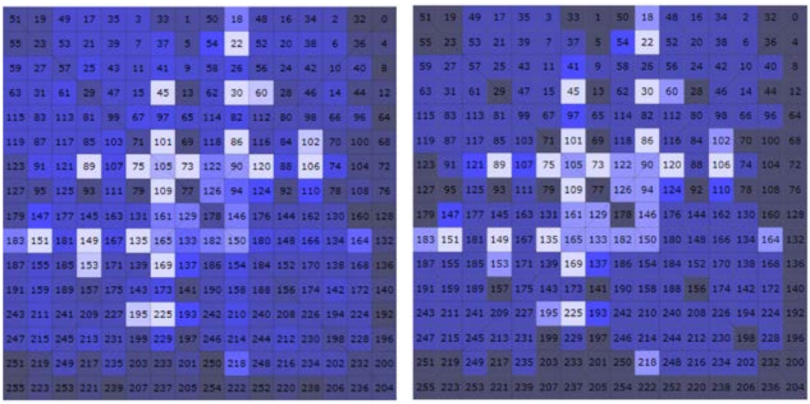
Class A	0, 8, 32, 128, 136, 160, 4, 12, 13, 36, 44, 72, 76, 77, 78, 104, 132, 140, 200, 204, 232, 1, 5, 19, 23, 28, 29, 33, 50, 51, 108, 156, 178
Class B	2, 10, 24, 34, 40, 42, 46, 56, 57, 130, 138, 152, 162, 168, 170, 172, 184, 3, 6, 7, 9, 11, 14, 15, 27, 35, 38, 43, 58, 62, 134, 142 , 25, 26, 37, 74, 154
Class C	41, 54, 110
Class D	94, 164, 18, 122, 126, 146, 90, 150, 105, 60
Class X	22, 30, 45, 73, 106

Table 2. Assignment of rules to coarser classes.

6. Comparison with Other Classifications

In the beginning, our proposed classification was made completely without regard to other classifications (see [10]) with respect to their criteria and their outcome.

Nevertheless, our classification shows strong similarities with many of the others, but since they have quite different grainedness (from only 2 classes up to 13), they are hard to compare. So we decided to reduce them to dichotomic classifications with only two classes—"simple" and "complex"—by merging the lower and the higher classes, respectively. See Figures 2–4.



(a) (b)

Figure 2. Classifications by cycle length with (a) 13 classes and with (b) five classes. We depict classifications by arranging the 256 rules in a 16×16 grid according to their numerical value in the so-called "canonical encoding scheme" (see Figure 3) and coloring them by shades of blue, from dark blue for simple rules ($C(R) = 0$) to light blue for the most complex ones ($C(R) = 1$). Here we consider normalized class numbers $C(R)$, assuming that

the classes of a classification can be naturally ordered from "simple" to "chaotic." When a classification C defines n ordered classes $C_0, \ldots, C_{n-1}$, these correspond to normalized class numbers $c_i = i/(n-1)$.

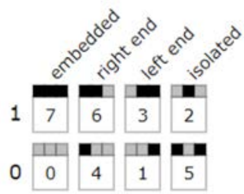


Figure 3. The canonical encoding scheme of a rule arranges the local configurations from $0 = 000$ to $7 = 111$ in a systematic and meaningful way and encodes by 1 that the color of the center cell is kept and by 0 that it is inverted. The canonical encoding gives rule 110 position (12,7) in the 16×16 rule matrix. In the canonical encoding, swapping the two rows and swapping the two inner columns yield equivalent rules. The number of 1s is the same for equivalent rules.

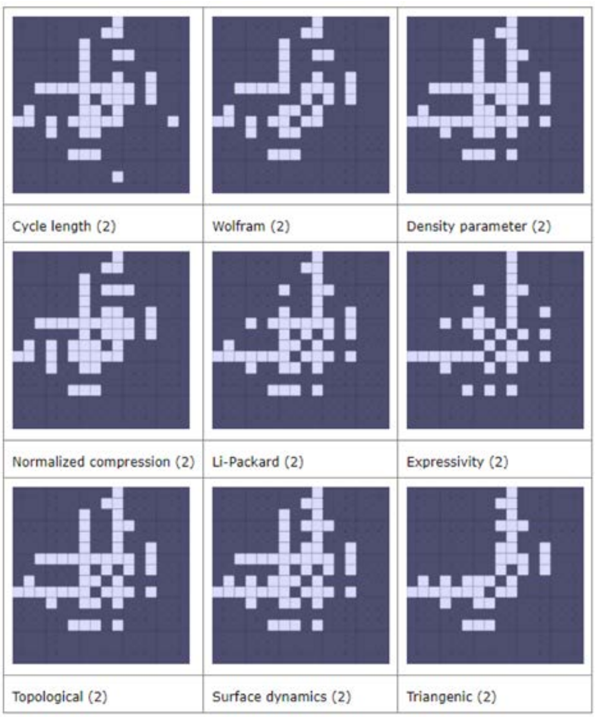


Figure 4. Different dichotomic classifications of elementary cellular automata. Rules are arranged the same as earlier: dark blue indicates "simple" rules; light blue indicates "complex" rules.

We have chosen classifications that look quite similar at first sight. Others from [10] are neglected, for example, Wuensche's [2].

- by Wolfram [1]
- by density parameter (Fatés, [11])
- by normalized compression (Zenil et al., [12]) (This classification is dichotomic by nature.)
- by Li and Packard [5]
- by expressivity (Redeker et al., [13])
- by topology (Chua et al., [14])
- by surface dynamics (Seck et al., [15])

As a dichotomic benchmark classification, we introduced the "triangenic" rules, that is, the rules that generate triangles as shown in Figure 5.

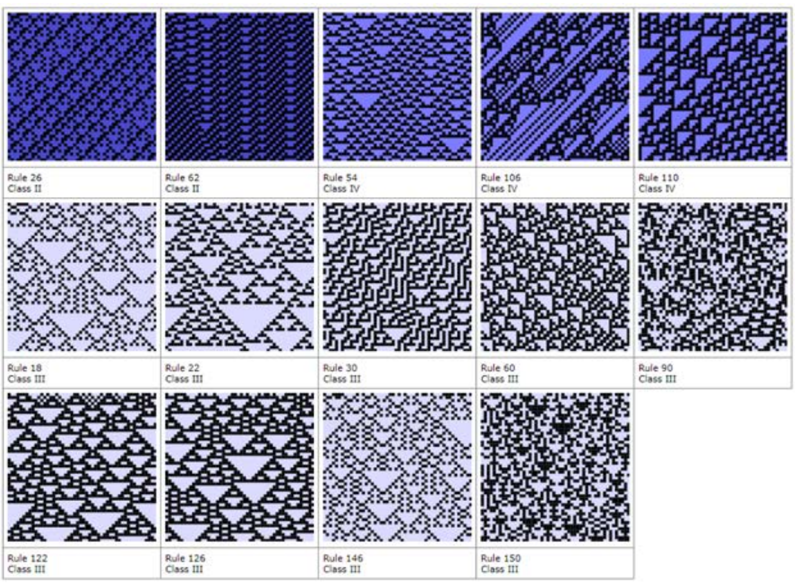


Figure 5. The rules whose evolution exhibits characteristic triangles, symmetric and asymmetric ones and even distorted (rule 106). Some of the triangles generate from a continuous sequence of black cells; others allow isolated white cells in their upper border. There are no other rules of this kind.

Figure 6 shows the rules that are not classified the same in all of these classifications.

Rule	Cycle length (2)	Wolfram (2)	Density parameter (2)	Normalized compression (2)	Li-Packard (2)	Expressivity (2)	Topological (2)	Surface dynamics (2)	Triangenic (2)
26	■	■	□	■	□	□	□	□	□
41	□	□	□	□	■	■	□	□	■
54	□	□	□	□	□	■	□	□	□
62	■	■	■	□	■	■	■	□	□
73	□	■	□	□	□	□	□	□	■
94	□	■	□	□	■	■	■	■	■
106	□	□	□	□	□	■	□	□	□
122	□	□	□	□	□	■	□	□	□
154	■	■	□	■	□	□	□	□	■
164	□	■	■	■	■	■	■	■	■

Figure 6. The classification of rules 54, 106, 122 dissents only for the classification by expressivity. The classification of rule 164 dissents only for the classification by cycle length. Neglecting the triangenic classification, the only dissenting classification of rule 73 is by Wolfram.

The closest classifications are Li and Packard's and the topological classification, which only differ for rule 41, and the classifications by cycle length and by normalized compression, which only differ for rules 62 and 164.

The cycle length classification differs from Wolfram's for the rules 73, 94 and 164, which are "simple" (class II) for Wolfram but "complex" for us.

In the Appendix, we show the set of complex rules according to different classifications as induced subgraphs of the folded rule space.

7. Open Questions

As mentioned earlier, no partial proof of Conjecture 1 is given—and known to the author—for any of the complex rules starting from class 6. It is assumed that the proofs have to be performed rule by rule and need not follow a common proof scheme [16].

The error term $\epsilon(k) = a(k) \cdot k$ in $L_{110}(k) = 2 \cdot k^2 + \epsilon(k)$ (see the formula for $\phi_6(R)$) needs more careful consideration. It is this term that makes the formula unspecific.

But even when just stating that $a(k) < O(k)$, it remains to be investigated and decided which of the two versions—the one with $(\exists_\infty k)\varphi(k)$ or the one with $p(\varphi) > 0$—is the more appropriate one. And of course, Theorem 1 needs to be proved, presumably rule by rule.

On a more general level, it needs to be investigated if the cycle length classification will stay meaningful when one generalizes to larger dimensions, neighborhoods and alphabets. Possibly, the number of classes based on cycle length growth increases unpredictably [16].

Appendix

A. Appendix

A.1 Calculating Cycle Length Spectra

We found all limit cycles and their lengths for all rules and $k \leq 18$ by the following brute force method:

1. We looped over all numbers from 0 to $2^k - 1$, interpreting them as initial configurations of an ECA over $\mathbb{Z} / k$.
2. We evolved the configurations by the local function of each rule until either:
   - a configuration was reached that was already attained in one of the loop steps ("runs") before
   - a configuration γ was reached that was already attained in the current run
3. In the latter case, γ was stored as the representative of a limit cycle, and the length of the run minus the index of γ in the current run was stored as the length L of the cycle.
4. We tracked all indices of a cycle (starting from γ with index 0) at which the cycle was met by transients.
5. We determined the subperiod l of a cycle being the number of steps until the initial configuration γ reappeared shifted by some number $\sigma < k$.

For other k, we stopped looping after some rule- and k-dependent number of runs.

A.2 Induced Subgraphs of Complex Rules in Folded Rule Space

The concept of a folded rule space was introduced in [5]. It is essentially the graph with equivalence classes of ECAs as nodes and an edge between two nodes when the minimal Hamming distance between their corresponding rules is 1. It has 88 nodes and 288 edges, and its automorphism group has order 128. The nodes can be arranged as shown in Figure A.1.

Some interesting differences between the structures of the subgraphs induced by the sets of complex rules (according to the different classifications) can best be seen in force-directed graphs like those in Figure A.2.

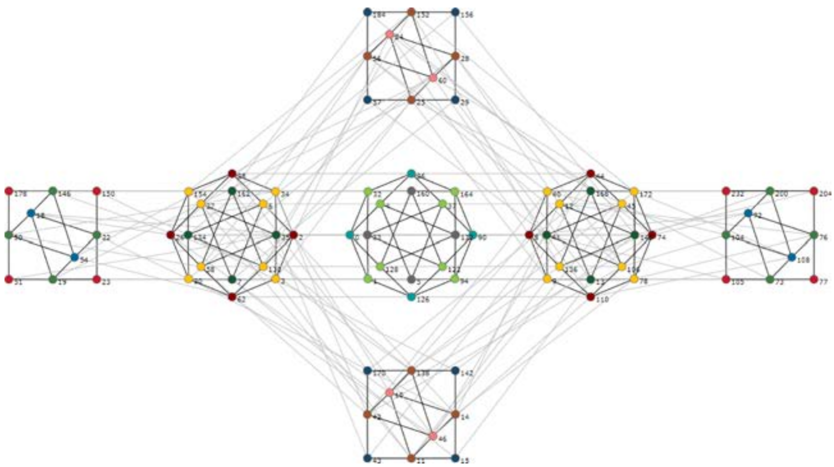
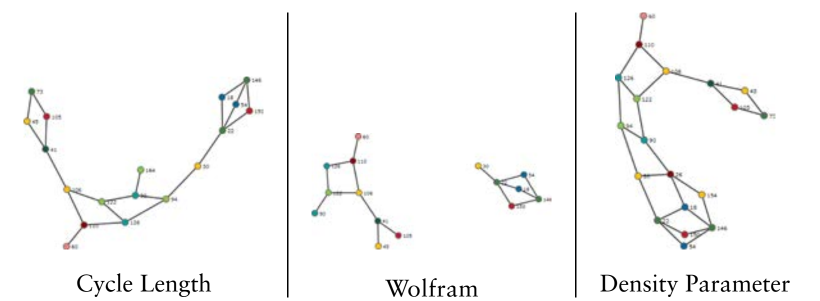


Figure A.1. Node colors represent conjugate nodes: when there is an automorphism α with $\alpha(R_1) = R_2$ then rules R_1 and R_2 have the same color. The black lines are just for highlighting some symmetric subgraphs, three of them being isomorphic to the four-dimensional hypercube.



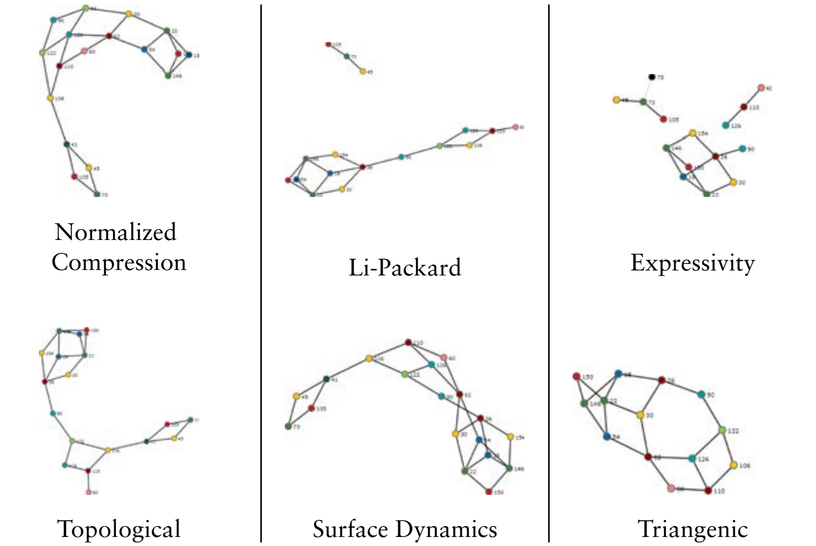


Figure A.2. Subgraphs induced by the sets of complex rules according to different classifications. Most of them are connected, and some have minimal node degree 2.

A.3 Another Kind of Atlas

Initially, we implemented Another kind of atlas for the visualization of limit cycles and cycle length spectra. Later on, other features were added, for example:

- visualization and comparison of classifications
- tables of properties
- state transition diagrams
- interactive spacetime diagrams
- investigation of tilings and defects
- investigation of rule 110's ether and particles
- colorings of spacetime diagrams
- regexp search for limit cycles
- stochastification
- sonification (experimental)
- graph drawings of the folded rule space

The atlas is a work in progress. The tool can be made available on request to the author.

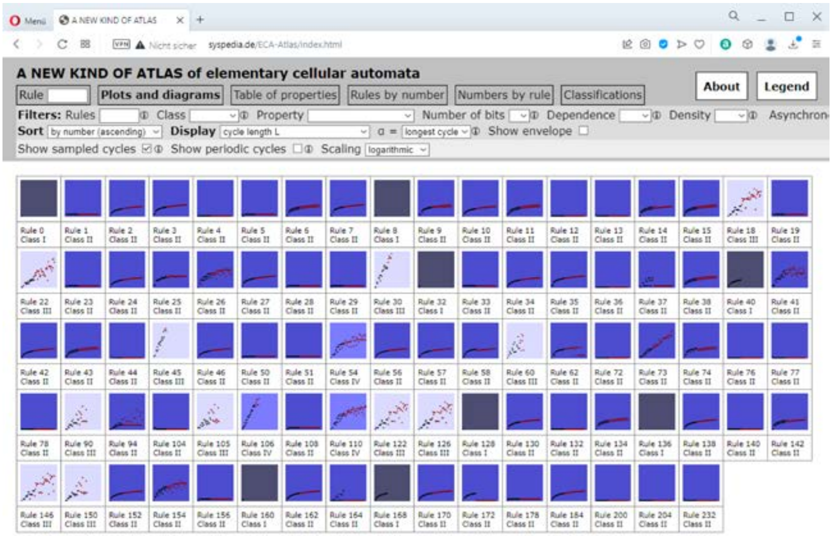


Figure A.3. Start page of the atlas, displaying the cycle spectra of all elementary cellular automata.

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