Turing Patterns in Networks

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The first pattern formation model was proposed by the mathematician Alan M. Turing. This model consists of a system of reaction-diffusion equations that produces stationary patterns by means of the so-called "Turing instability." In this paper, we found the conditions that the network and the parameters need to fulfill in order to achieve the Turing instability in a particular reaction-diffusion system called the Mimura–Murray model on different network topologies, including some simulations on an innovative kind of network, based on the Wolfram model, that evolves over time, generating interesting topologies that exhibit lattice-like topology. In addition, the equations are solved and simulated in Wolfram Language, and some examples of applications in biology and sociology are presented.

Keywords: Turing Patterns; Turing Instability; Mimura–Murray Model; Network Topologies; Reaction-Diffusion System; Wolfram Model

1. Introduction

The English mathematician Alan Turing, in his 1952 article "The Chemical Basis of Morphogenesis," was the first to propose a mathematical model for the formation of spatial patterns in biological systems. Turing claimed that spatial patterns can arise as a result of instability in a reaction-diffusion-type mechanism for two chemicals (one acting as an activator and the other acting as an inhibitor), which he called morphogens. This mechanism of pattern generation is now called "Turing instability." Turing instability occurs when a steady state is stable without diffusion and unstable with diffusion, then under certain conditions, spatial inhomogeneities (stationary spatial patterns) will occur [1, 2]. The instability caused by the diffusion might appear counterintuitive at first, since diffusion is usually a process of stabilization and homogenization, but Turing's insight showed that from the interaction of two stabilizing processes, an instability can arise [3].

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After Turing's pioneering work, a general mathematical framework was developed to investigate the Turing instability with different reactions in continuous domains with distinct geometries and recently has been extended to discrete domains defined by networks [4, 5]. In such systems on networks, Turing instability leads to the spontaneous differentiation of the nodes of the network into groups of nodes where some have a high concentration of the activating substance and others a high concentration of the inhibiting substance.

In this paper, we present the Turing instability analysis of a particular reaction-diffusion system called the Mimura–Murray model for two species on different network topologies, but this analysis can be generalized and extended to more species, more reaction-diffusion models, and to other types of networks, for example, multilayer networks [6].

Additionally, we explore through various illustrative numerical simulations the Turing instability on dynamical networks given by Wolfram models that evolve over time using simple rules that converge into lattice structures [7] and can be used as a first approach to complement Turing's work on the morphogen equations for an assembly of cells [8]. This discrete case may be useful in many circumstances where the continuum limit is not adequate or applicable.

2. Background

The existence of pattern formation in discrete media was first introduced by Othmer and Scriven in the context of the diffusion of morphogens through a network of intercellular connections [9].

The patterns in this context correspond to different populations of nodes differentiated by their levels of concentrations of the different species and exhibit properties quite different from the continuum case. For example, multiple coexisting steady states can occur and hysteresis effects are present, indicating that the patterns are not particularly robust over time [10].

Since then, the theory has been extended to directed networks [11], multilayer networks [12] and time evolutionary networks [10]. And given the widespread prevalence of networks in a large number of socio-economic, biological, technological [13] and social systems, Turing's theory is increasingly being used to model self-organizing behaviors, among other complex phenomena. Although much has been studied regarding the application and the existence of Turing patterns in different types of complex networks, much remains to be understood about the precise role of network topology.

3. Reaction-Diffusion System

A reaction-diffusion system is a mathematical model that describes the dynamics of how the concentration of one or more substances varies in time and space, under the influence of two terms: one of reaction, in which the concentration increases or decreases by local interaction, and another by diffusion that causes substances to disperse in space [2, 3].

The general form of this system is:

$$\frac{\partial s}{\partial t} = D\nabla^2 s + f(s), \tag{1}$$

where s is the vector of the concentrations of the morphogens, f represents the reaction kinetics and D is the diagonal matrix of the diffusion coefficients.

The following work addresses the model for two chemical species, given by the equations:

$$\frac{\partial u}{\partial t} = d_1\nabla^2 u + f(u, v), \tag{2}$$

$$\frac{\partial v}{\partial t} = d_2\nabla^2 v + g(u, v), \tag{3}$$

where f and g are nonlinear functions [2]. Furthermore, we consider that u is an activator; this means that the production of u is autocatalytically stimulated by u, and v is an inhibitor. That means that this species inhibits the production of u.

3.1 Reaction-Diffusion System in a Discrete Domain

Since we are interested in the discrete form of this equation, we need to modify equations (2) and (3) to discrete form as follows.

If the system is embedded in a network instead of a continuous domain, we use a similar set of equations, by means of the analog of the Laplacian operator for networks, known as the Laplacian matrix [4, 5]:

$$L_{ij} = A_{ij} - k_i\delta_{ij}. \tag{4}$$

Here, A_{ij} is the symmetric adjacency matrix, whose elements are 1 if there is an edge between the nodes i and j and 0 otherwise, and $k_i = \Sigma_i A_{ij}$ is the degree of node i. Then the diffusion of a particular node is the sum of all incoming flows from its neighbors and proportional to their respective concentration differences. Substituting this modification, our initial system is transformed into a set of $2N$ differential equations (for $i = 1, 2, \ldots, N$):

$$\frac{\partial u_i}{\partial t} = \epsilon \sum_{j=1}^{N} L_{ij} u_j + f(u_i, v_i), \tag{5}$$

$$\frac{\partial v_i}{\partial t} = \epsilon\gamma \sum_{j=1}^{N} L_{ij} v_j + g(u_i, v_i), \tag{6}$$

where we introduce $\epsilon = d_1$ and $\gamma = d_2$ to align the notation with the continuous system.

4. Mimura–Murray Model

The reaction defined by the Mimura–Murray model that will be used in the following simulations specifies the functions f and g from equations (5) and (6) as:

$$f(u, v) = u\left(\frac{a + bu - u^2}{c} - v\right), \tag{7}$$

$$g(u, v) = v(u - (1 + dv)), \tag{8}$$

where a, b, c and d are real positive parameters.

This model has proven to be very versatile and has been applied to various phenomena of biological interest. Specifically, it has been used to model a predator-prey type system [14]. Our choice of this model is motivated by its extended use in the study of different reaction-diffusion dynamics in complex networks; however, our results are applicable to a wide range of dynamics.

5. Turing Instability on the Discrete Domain

Turing's crucial idea was that it is possible to have stable stationary states in the absence of diffusion, which become unstable in the presence of diffusion and form heterogeneous patterns, hence the name diffusion-driven instability. To obtain this diffusion-driven instability, we need to derive these conditions as follows.

5.1 Stability in the Absence of Diffusion

Similar to the more-studied classical continuous domain, we consider (u^*, v^*) an equilibrium point, which satisfies $f(u^*, v^*) = g(u^*, v^*) = 0$.We linearize the system in the absence of diffusion (with

$\epsilon = 0$ in equations (5) and (6)) around (u^*, v^*):

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{(u*,v*)} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{9}$$

where the matrix $\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{(u^*,v^*)}$ is the Jacobian, which we will denote by J.

In this way, to find the stability of the system, we will only have an eigenvalue problem:

$$|J - \lambda \mathrm{I}| = 0 \tag{10}$$

with the characteristic polynomial:

$$\lambda^2 - (f_u + g_v)\lambda + (f_u g_v - f_v g_u) = 0 \tag{11}$$

and solutions:

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{Tr}(J) \pm \sqrt{\mathrm{Tr}(J)^2 - 4\,\mathrm{Det}(J)}\right) \tag{12}$$

where $\mathrm{Tr}(J) = f_u + g_v$ and $\mathrm{Det}(J) = f_u g_v - f_v g_u$.

From this, the system in the absence of diffusion will be stable if $\mathrm{Re}[\lambda] < 0$. This holds if the following inequalities are satisfied:

$$\mathrm{Tr}[J] = f_u + g_v < 0 \tag{13}$$

$$\mathrm{Det}[J] = f_u g_v - f_v g_u > 0. \tag{14}$$

5.2 Diffusion-Driven Instability

Now we need to find the conditions for the instability in the presence of diffusion ($\epsilon > 0$). To achieve this, we introduce diffusion in the linear system at the equilibrium point:

$$\begin{pmatrix} \dfrac{du_i}{dt} \\ \dfrac{du_2}{dt} \end{pmatrix} = J\begin{pmatrix} u_i \\ v_i \end{pmatrix} + \sum_{j=1}^{N} L_{ij} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon\gamma \end{pmatrix} \nabla^2 \begin{pmatrix} u_i \\ v_i \end{pmatrix}. \tag{15}$$

To derive the necessary conditions for the steady state to become unstable, we consider an inhomogenous perturbation for the steady state, that is, $u = u^* + \eta_u$ and $v = v^* + \eta_v$ with $\|\eta_u\| \ll 1$ and $\|\eta_v\| \ll 1$.

These perturbations then satisfy the following linearized ODEs:

$$\begin{pmatrix} \dfrac{d\eta_u}{dt} \\ \dfrac{d\eta_v}{dt} \end{pmatrix} = J\begin{pmatrix} \eta_u \\ \eta_v \end{pmatrix} + \sum_{j=1}^{N} L_{ij} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon\gamma \end{pmatrix} \nabla^2 \begin{pmatrix} \eta_u \\ \eta_v \end{pmatrix}. \tag{16}$$

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The perturbations η_u and η_v therefore satisfy a $2N$ linear ODE system, and their long-term behavior can be studied using the eigenvalues of the matrix

$$\begin{pmatrix} f_u\, I + \epsilon\, L & f_v\, I \\ g_u\, I & g_u\, I + \epsilon\gamma L \end{pmatrix}. \tag{17}$$

Here we can apply a similar technique as in Turing's work, expanding the dynamics on a well-chosen orthogonal basis. In the continuum case, we use the orthogonal basis of eigenfunctions of the Laplace operator, and in the discrete case, we can use the eigenvectors of the graph Laplacian L. Note that since L is a real positive semi-definite matrix, the eigenbasis for L is indeed orthogonal, and the corresponding eigenvalues are real and non-negative.

Denote by $\Phi_j{}^{(\alpha)}$ the set of N eigenvectors of L with accompanying eigenvalues Λ_a, which lie in the interval $[0, N]$. We assume for simplicity that the eigenvectors are ordered such that the eigenvalues Λ_a are in increasing order. Then, if we introduce the corresponding eigenvalue equation for the Laplacian matrix with eigenvalue and eigenvector decomposition, we will have:

$$\sum_{j=1}^{N} L_{ij} \Phi_j{}^{(\alpha)} = \Lambda_\alpha\, \Phi_j{}^{(\alpha)} \tag{18}$$

where $\alpha = 1, \ldots, N$. The disturbances can be expanded over the set of Laplacian eigenvectors as:

$$\eta_u = \sum_{a=1}^{N} c_a E^{(\lambda_a t)} \Phi_i{}^{(\alpha)} \tag{19}$$

$$\eta_v = \sum_{a=1}^{N} c_a\, B_a E^{(\lambda_a t)} \Phi_i{}^{(\alpha)}{}_i. \tag{20}$$

Substituting equations (19) and (20) in equations (16) and (17), we obtain the following eigenvalue equation:

$$\lambda_a \begin{pmatrix} 1 \\ B_a \end{pmatrix} = \begin{pmatrix} f_u + \epsilon\, \Lambda_a & f_v \Lambda_a \\ g_u \Lambda_a & g_u + \gamma \Lambda_a \end{pmatrix} \begin{pmatrix} 1 \\ B_a \end{pmatrix} \tag{21}$$

with characteristic equation:

$$\lambda_a{}^2 + b\,(\Lambda_a)\lambda_a + c(\Lambda_a) = 0 \tag{22}$$

and roots:

$$\lambda_{1,2} = \frac{1}{2}\left(-b(\Lambda_a) \pm \sqrt{-b(\Lambda_a)^2 - 4c(\Lambda_a)}\right) \tag{23}$$

with definitions:

$$b(\Lambda_a) = -(\mathrm{Tr}(J) + (\epsilon + \gamma)\Lambda_a) \tag{24}$$

$$c(\Lambda_a) = \epsilon\gamma{\Lambda_a}^2 + (\epsilon g_v + \gamma f_u)\Lambda_a + \mathrm{Det}(J). \tag{25}$$

From the conditions obtained for the system in the absence of diffusion, we can see that the only way to make the roots positive (unstable) is with the root of equation (20). Thus, to induce differentiation we need $\lambda\alpha(\Lambda\alpha) > 0$ for some α, which implies $c(\Lambda\alpha) < 0$, so:

$$c(\Lambda_a) = \epsilon\gamma{\Lambda_a}^2 + (\epsilon g_v + \gamma f_u)\Lambda_a + \mathrm{Det}(J) < 0. \tag{26}$$

The roots of $c(\Lambda_a)$ are:

$$\Lambda_{\alpha_1,\alpha_2} = \frac{1}{2\epsilon\gamma}\left(-(\epsilon g_v + \gamma f_u) \pm \sqrt{(\epsilon g_v + \gamma f_u)^2 - 4\,\epsilon\gamma\,\mathrm{Det}(J)}\right). \tag{27}$$

Therefore, $c(\Lambda_a)$ is negative if some of the Laplacian eigenvalues are in the range $[\Lambda_{a_1}, \Lambda_{a_2}]$, which guarantees the existence of positive growth factors λ_a. In this way, we obtain the conditions for instability in the presence of diffusion:

$$\epsilon\, g_v + \gamma f_u > 0 \tag{28}$$

$$(\epsilon g_v + \gamma f_u)^2 - 4\,\epsilon\gamma(f_u g_v - f_v g_u) > 0. \tag{29}$$

So the conditions to achieve Turing instability and differentiation of the nodes of our network are:

$$f_u + g_v < 0 \tag{30}$$

$$f_u g_v - f_v g_u > 0 \tag{31}$$

$$\epsilon\, g_v + \gamma f_u > 0 \tag{32}$$

$$(\epsilon g_v + \gamma f_u)^2 - 4\,\epsilon\gamma(f_u g_v - f_v g_u) > 0, \tag{33}$$

with f_u, f_v, g_u and g_v as the coefficients of the Jacobian evaluated at the equilibrium point.

If we comply with these conditions and start from an almost homogeneous distribution of concentrations, the species are gradually regulated until a spontaneous differentiation of the nodes of the network arises, and as time increases, the dynamics of the system do not undergo any further changes [2].

Interestingly, the appearance of positive growth rates λ_a corresponds to the superposition of the Laplacian eigenvalues with the instability regime and since

$$\mathrm{Tr}(-L) = \Sigma_i k_i = \Sigma_a \Lambda_a,$$

we have:

$$\langle k \rangle = \langle \Lambda_a \rangle$$

where $\langle k \rangle = \frac{2m}{N}$ is the average degree of the network.

Thus, for a fixed set of diffusion coefficients and dynamic parameters, Turing instabilities can be triggered in any type of network topology by adjusting the average connectivity of the system into a regime with positive growth rates. In contrast, for fixed network topology, the diffusion coefficients can be adjusted (within significant limits) to generate instabilities [5].

6. Simulations

Now that we have identified the conditions for the appearance of Turing patterns, we can observe the Turing space; this will allow us to see when spontaneous differentiation occurs. For example, Figure 1 shows the parameter space for γ and ϵ, holding the other parameter values fixed. In this image, the green area shows the value of the parameters ϵ and γ that will allow us to obtain the Turing pattern based on previous analysis.

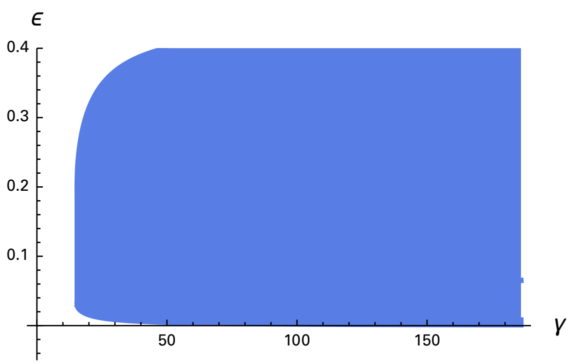

Figure 1. Parameter space γ versus ϵ fixing $a = 35$, $b = 16$, $c = 9$ and $d = 2/5$.

From the parameter space, we can select $\epsilon = 0.015$ and $\gamma = 2$ to observe the instability parameter range determined by the diffusion coefficients. In this range, the dynamics at the stable point become unstable with a positive eigenvalue. In Figure 2, we can see the characteristic curve $c(\Lambda_a)$, which relates the Laplacian eigenvalues with the dynamical eigenvalues and allows us to choose the average degree of our network nodes to be able to generate a pattern. For example, for a Watts–Strogatz-type network with $\Lambda_a = \langle k \rangle = 200$, shown in

Figure 3, spontaneous differentiation occurs, as we can see in the network graph and the graph of the index versus the concentration of u. This graph of the index versus the concentration of u indicates that there are two types of nodes, one type with a large quantity of the inhibitor species and the other type with a small quantity of the inhibitor, meaning that it has a large quantity of the activator.

For $\langle \Lambda_a \rangle = \langle k \rangle = 150$ (Figure 4), $\langle \Lambda_a \rangle = \langle k \rangle = 100$ (Figure 5), $\langle \Lambda_a \rangle = \langle k \rangle = 50$ (Figure 6) and $\langle \Lambda_a \rangle = \langle k \rangle = 10$ (Figure 7), we can also observe the differentiation on the nodes for these cases. So $\langle k \rangle$ can be tuned to trigger Turing instability.

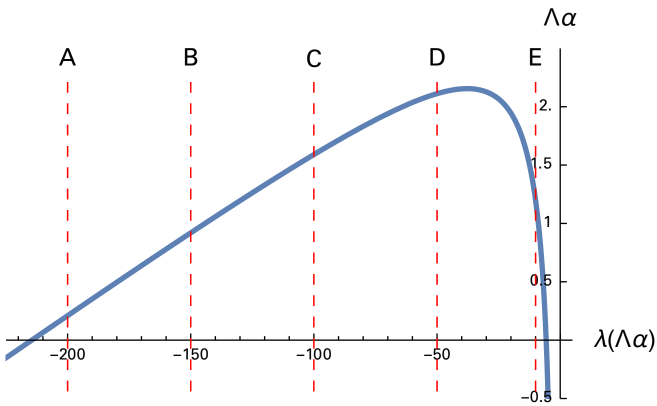

Figure 2. Characteristic curve.

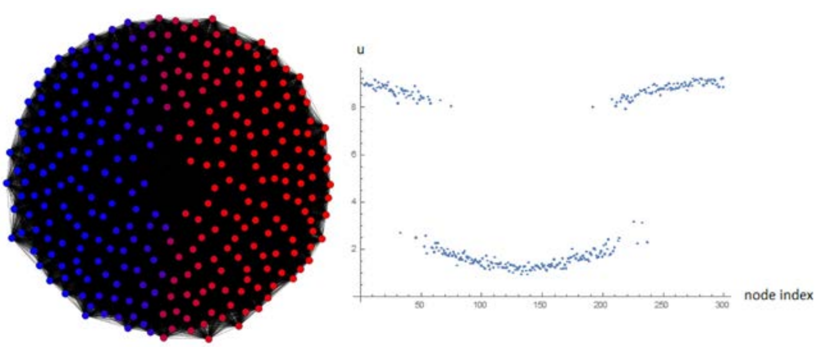

Figure 3. Differentiation in the network for $\langle k \rangle = 200$.

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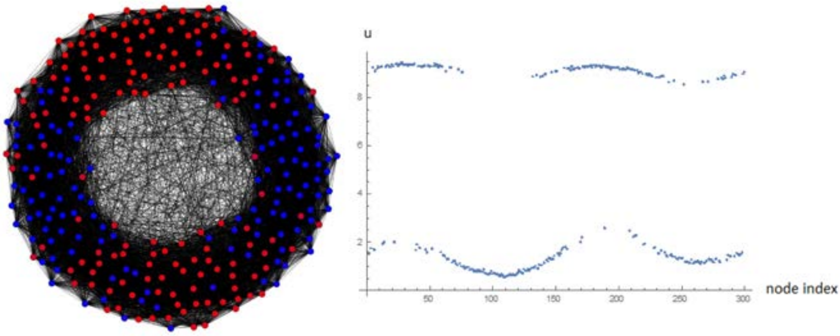

Figure 4. Differentiation in the network for $\langle k \rangle = 150$.

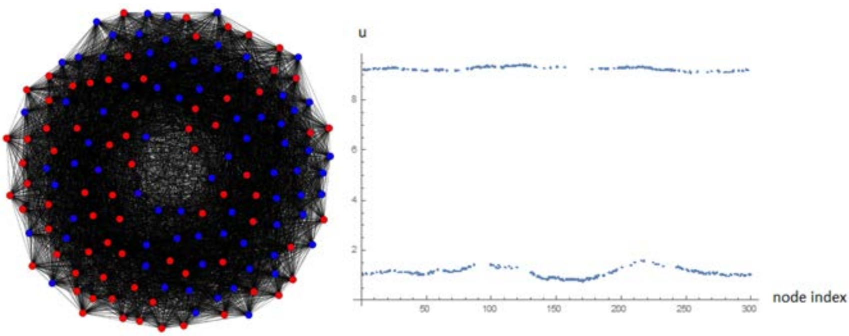

Figure 5. Differentiation in the network for $\langle k \rangle = 100$.

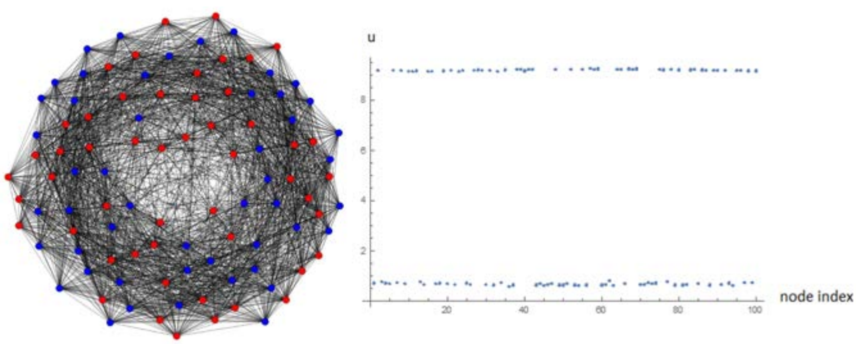

Figure 6. Differentiation in the network for $\langle k \rangle = 50$.

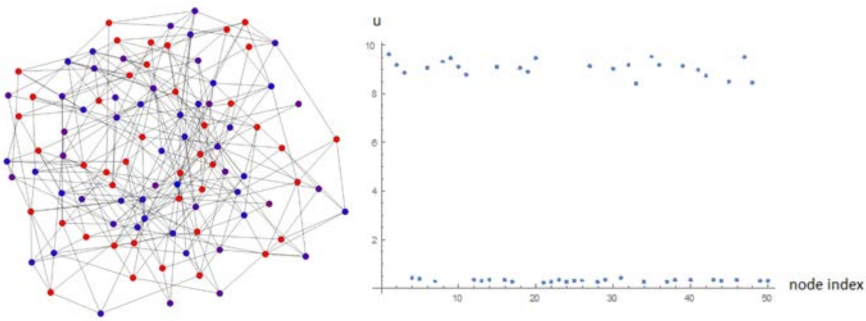

Figure 7. Differentiation in the network for $\langle k \rangle = 10$.

7. Evolution of the Network Using the Wolfram Model

So far, it has been seen that reaction-diffusion models have unavoidable limitations in describing many phenomena in the biological and chemical sciences. A good way to generalize the systems and give them more applications is to take into account the effect of a growing domain. We believe that by taking this effect into account we will be able to better describe some phenomena in the biological and chemical sciences. To simulate this growth in the domain, we will use a special kind of evolving network based on the Wolfram model (WM) [7].

Evolving networks are networks that change as a function of time. They are a natural extension of network science, as almost all real-world networks evolve, either adding or removing nodes or links over time [15, 16]. Usually, all of these processes occur simultaneously, as in social media, where people make and lose friends over time, creating and destroying edges.

In the same way, the WM generates networks that evolve over time. It begins with a small graph and a simple rule that is applied recursively over the network. Surprisingly, self-organization into macroscopic structures appears, including two-dimensional lattices, among other complex structures. We use the WM and some of its "notable universes" (sets of rules) to perform the simulation on some interesting network topologies. We are especially interested in rules defined by two ternary relations, specially the 2^3→3^3 case. Extensive search in all 79 million or so 2^3 → 3^3 possible rules has shown that rules with slow growth are quite rare and are strongly localized to about 10 broad regions in the space of possible rules [7]. From the rules with slow growth, only a small subset form nontrivial globular structures, and of these, perhaps 10% exhibit obvious lattice-like patterns [7]. We are interested in these rules for their simple mechanism of dynamic update yet complex self-organizing behavior into lattices, with striking visual connections to the two-dimensional continuous case commonly linked to stripe animal patterning, among other phenomena. Consider the case in Figure 8 as an example of a WM with Turing patterning: notice the stripe patterning on the last network.

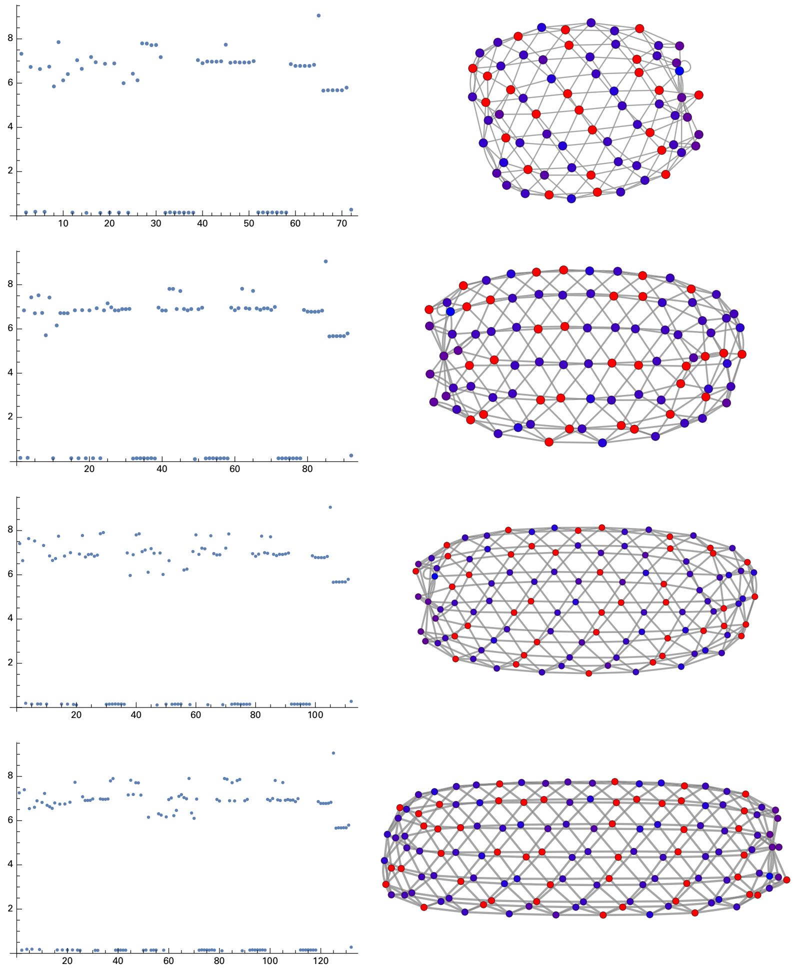

Figure 8. Growth through rules for WM 1268.

In this evolving networks case, when some new nodes are added into the graph, we are assuming the new nodes have a random value of u and v at the beginning, the dynamics of the reaction-diffusion system work as shown previously and as time goes by, the nodes arrive at stable u and v values.

Other examples of Turing patterns in Wolfram's remarkable universes are Figures 9–13 for the notable universes 3255, 1517, 3639, 1167 and 1986, respectively.

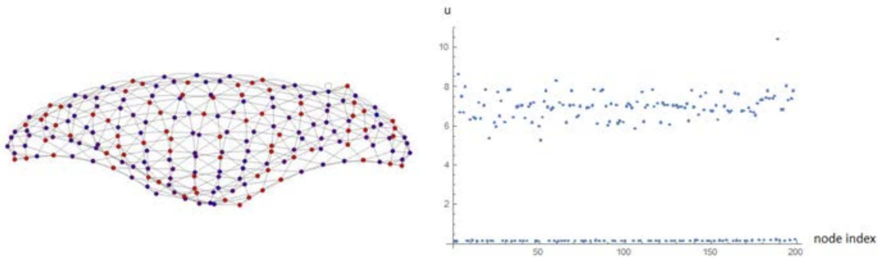

Figure 9. Turing pattern in the WM 3255.

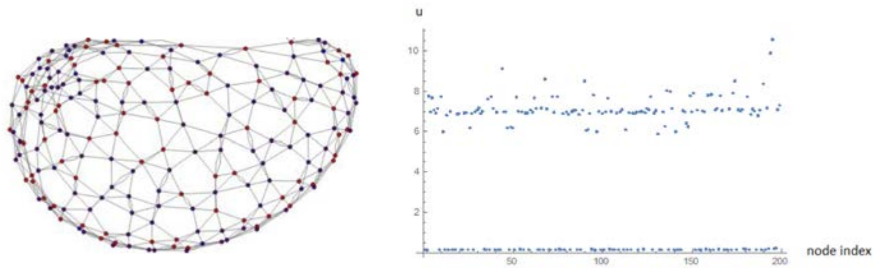

Figure 10. Turing pattern in the WM 1517.

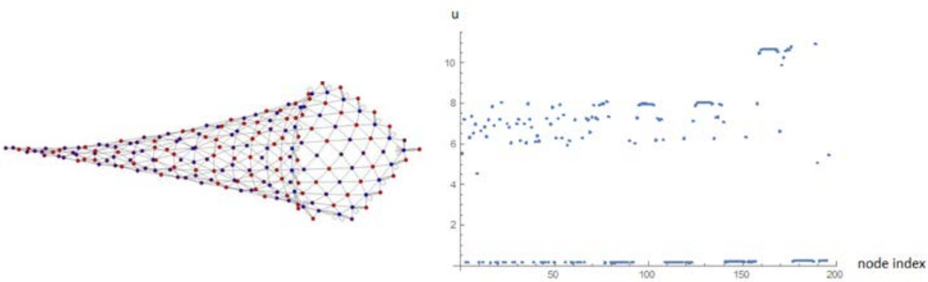

Figure 11. Turing pattern in the WM 3639.

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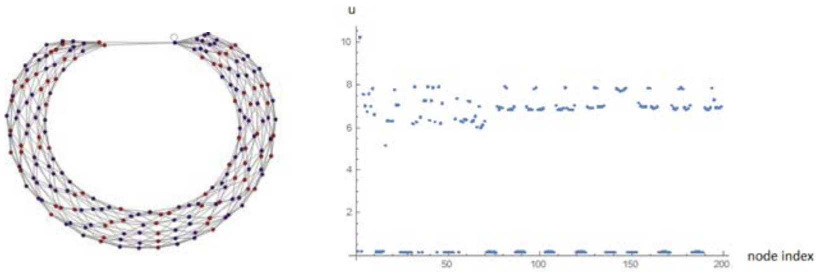
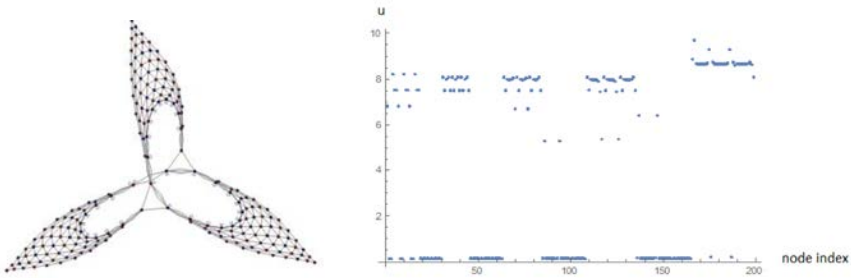


Figure 12. Turing pattern in the WM 1167.



Infinity

Figure 13. Turing pattern in the WM 1986.

The novel part of Turing patterns on these types of networks that evolve over time, generating interesting topologies that exhibit lattice-like networks based on the WM, is that this type of topology in the form of a lattice can show Turing patterns in networks in a visible way as we would see it in the continuous domain; such visible patterns can be hard to find on networks.

8. Discussion

In this paper, we analyzed the extension of the classical Turing pattern formation analysis to different complex network topologies. In great resemblance to the two-dimensional analysis, the Turing conditions are obtained and consequently allowed to fix parameters that produce self-organizing patterns. However, there are important differences between the classical framework and the network framework, due to the effect of the network topology in the diffusion. We observe that the Turing space is completely determined by the graph of the Laplacian spectrum and therefore intimately linked to network connectivity.

We also observe that, in principle, the differentiation of nodes can be induced in any type of network topology, adjusting the average connectivity of the system, allowing a degree of tuning and control of the degree of overlap between the graph spectrum and the instability regime; conversely, for a fixed network topology, dynamic parameters can be adjusted within significant limits to generate Turing patterns.

Additionally, the novel part of Turing patterns on these types of networks that evolve over time, generating interesting topologies that exhibit lattice-like networks based on the WM, is that this type of topology in the form of a lattice can show Turing patterns in networks in a visible way as we would see it in the continuous domain, so it could be of interest when compared with some simulation methods in continuous domains that use meshes of discretization, such as the finite element method.

9. Conclusion

In this paper, the theory of pattern formation in the Mimura–Murray reaction-diffusion system on complex networks was discussed in a way that can be generalized and applied to other spacetime dynamics. Likewise, the conditions necessary for the formation of the patterns were determined in detail by means of the so-called "Turing instability." These equations were implemented and solved to be able to observe the resulting pattern on different network topologies, including some simulations on an innovative kind of network that evolves over time, generating interesting topologies that exhibit lattice-like networks based on the Wolfram model.

We conclude that the results of this paper can enrich the investigation of pattern formation, since it presents a great variety of possible applications in different socioeconomic, biological, technological and social systems. For example, it could be useful to model polarization and opinion formation, and the socio-economic segregation in many social systems, similar to how the predator-prey model is used.

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