Realizability of Parity Logic in Finite Cellular Automata

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We investigate a range-4 cellular automaton that was designed and (falsely) proven to realize the parity logic for any odd input size N. We find that it fails to perform at $N = 13$ and provide a Boolean bottleneck conjecture to redirect research efforts.

Keywords: finite cellular automata; Boolean parity; Boolean bottleneck; compression; state transition diagrams; functional graphs

1. Introduction

Despite more than a half-century of investigations in cellular automata (CAs) [1, 2] and their ruliology [3], several questions of foundational relevance to information processing in spatiotemporal physical systems remain unanswered. Here, we will focus on the realizability of parity functionality in one-dimensional (circularly joined) range-r Boolean CAs, which can be expressed by an update rule

$$x_0 = \overline{x}_0 f_{\mathrm{E}}(x, r) + x_0 \overline{f}_{\mathrm{I}}(x, r) \tag{1}$$

where $x_0 \in \{0, 1\}$ denotes the value of a cell, and f_{E} and f_{I} denote excitatory and inhibitory Boolean functions acting on the local neighborhood values $x_{-r:+r}$, as depicted in Figure 1. For example, the famous rule 30 automaton has $f_{\mathrm{E}} = x_{-1}\overline{x}_{+1} + x_{+1}\overline{x}_{-1}$ and $f_{\mathrm{I}} = x_{-1}\overline{x}_{+1} + x_{-1}x_{+1}$. For expressing higher-range automata, we will use terms such as x_{-123} to compactly denote $x_{-1}x_{-2}x_{-3}$ and so on.

The naive approach to realize parity by taking $f_{\mathrm{E}} = f_{\mathrm{I}} = x_{\pm 1}$ does not work for odd N, and for even N always ends up in all-0 just after computing the parity [4]. All working solutions to compute a stable parity output are rather nonsynergistic circuits-based approaches such as employing a nonuniform cellular automaton (CA) with a mixture of right-shift operators and XOR rules [5], a time-dependent nonuniform CA with rule 60 [6] to solve in N steps, and a time-dependent uniform CA designed to solve in N^2 steps.

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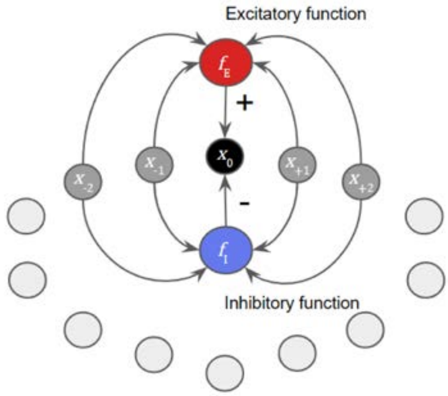

Figure 1. A depiction of the excitatory and inhibitory feedback interactions of a range $r = 2$ automaton with $N = 14$ cells in a ring, all updated in parallel, following the same mixed feedback rule.

The only known promising result to tackle the parity problem synergistically comes from [7], where the authors introduce a novel mode of equilibriation (dubbed the accordion effect) by means of an $r = 4$ CA, with an excitation function

$$\begin{aligned} f_{\mathrm{E}} = {} & x_{-123}\bar{x}_{+1} + x_{-234}\bar{x}_{-1} + \\ & \bar{x}_{-23}x_{-1}\bar{x}_{+1} + \bar{x}_{-134}x_{-2} + \bar{x}_{-2}x_{-1}x_{+1}\bar{x}_{+23} \end{aligned} \tag{2}$$

and an inhibition function

$$\begin{aligned} f_{\mathrm{I}} = {} & x_{-234}\bar{x}_{-1} + x_{-2}\bar{x}_{-13,+2} + x_{-1}\bar{x}_{-2,+1} + \\ & x_{-1,+23}\bar{x}_{+14} + x_{+1}\bar{x}_{-1,+2} + x_{+13}\bar{x}_{+2}. \end{aligned} \tag{3}$$

In order to investigate the working mechanism of the given rule, we plot state transition diagrams (i.e., functional graphs) and introduce a novel symmetry-based compression scheme to depict functional graphs with equilibrium-point behavior as a "sympressed" tree. Note that our sympression method can be generalized to be a hybrid of symmetry-based vertex-compression [8] and modularity-based edge-compression [9] methods, but that is beyond the scope of our present paper. Thus instead of providing a formal definition of sympression, we outline it in Figure 2 and provide Wolfram Language code at [10].

2. Investigation

Figures 3 to 7 confirm that the CA rule realizes a parity function as intended for $N = 3, 5, 7, 9, 11$, but this fails at $N = 13$, as shown in Figure 8. In particular, for $N = 13$, the state 0001110101001 can be manually verified to result in a loop.

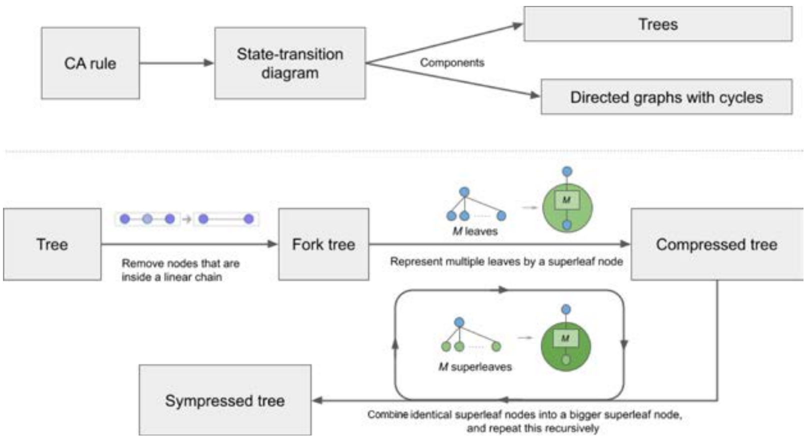

Figure 2. A directed acyclic graph component of a state transition diagram for a CA can be mapped to a tree, whose root node corresponds to a state of equilibrium. This tree can be symmetrically compressed to efficiently represent the parallel modes of equilibriation.

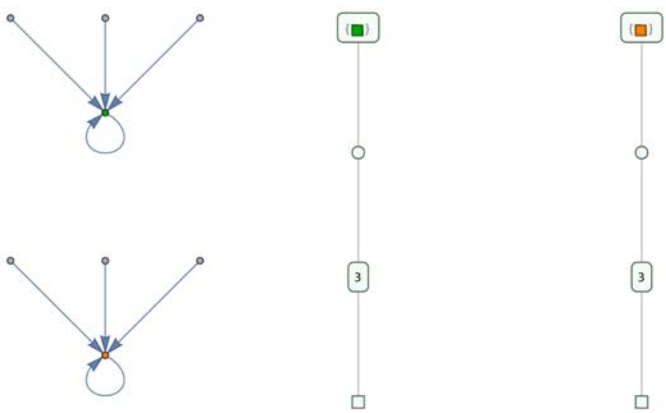

Figure 3. The state transition diagram for the CA rule from [7] at $N = 3$. All initial states of odd parity equilibriate to the all-1 state (orange) and those of even parity to the all-0 state (green). There are three parallel modes of equilibriation for both parities.

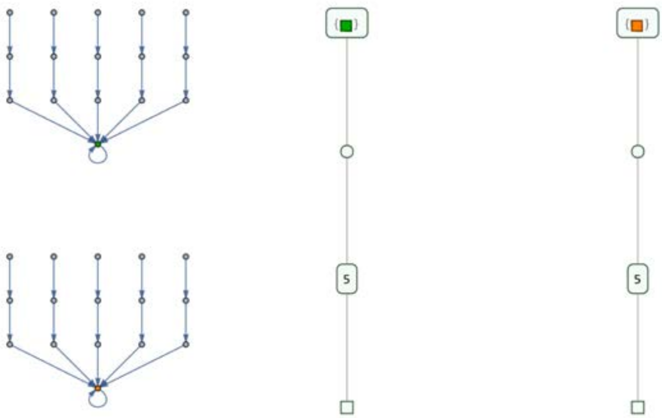

Figure 4. The state transition diagram when $N = 5$. There are five parallel modes of equilibriation for both parities.

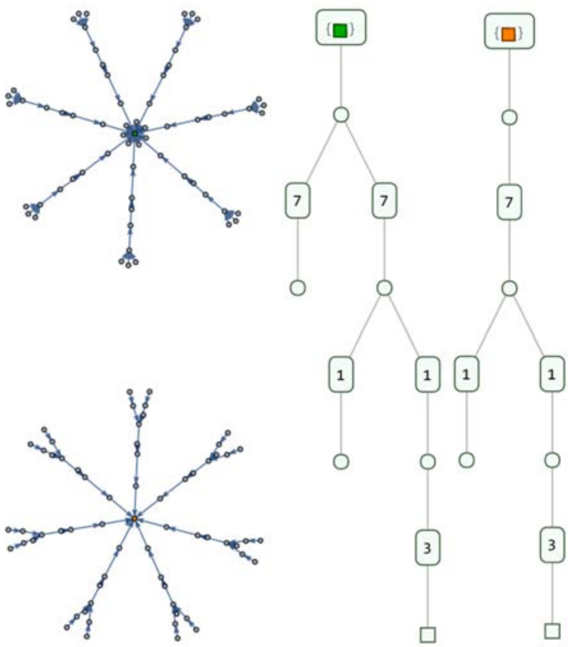

Figure 5. The state transition diagram when $N = 7$. The modes of equilibriation are different per parity and are depicted on the right by a compressed tree structure with roots denoting the all-0 or all-1 equilibrium state and weights denoting the number of parallel modes.

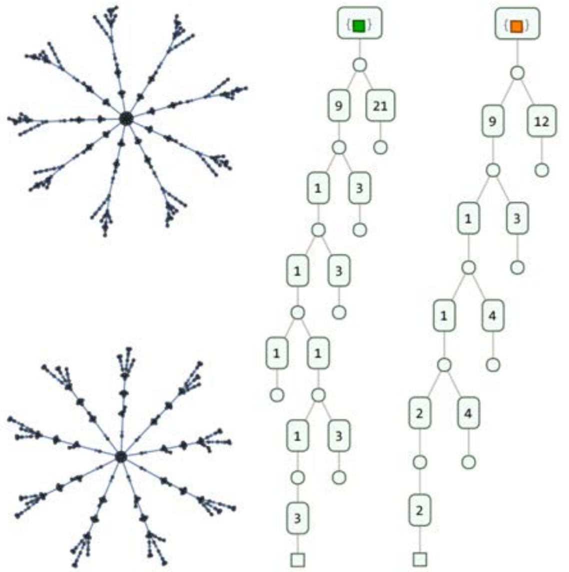

Figure 6. The state transition diagram when $N = 9$. The modes of equilibriation are different per parity, as expected for an asymmetric range-4 rule.

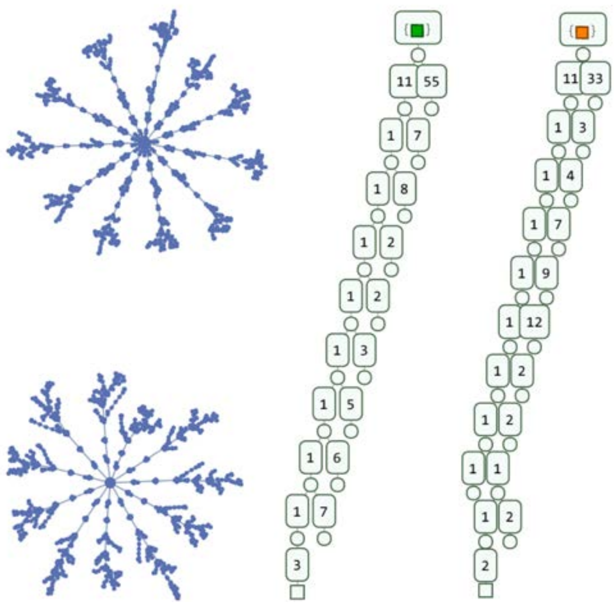

Figure 7. The state transition diagram when $N = 11$. This is a proof of concept that it is possible for uniform CAs to realize logic by synergizing in a system larger than their local range.

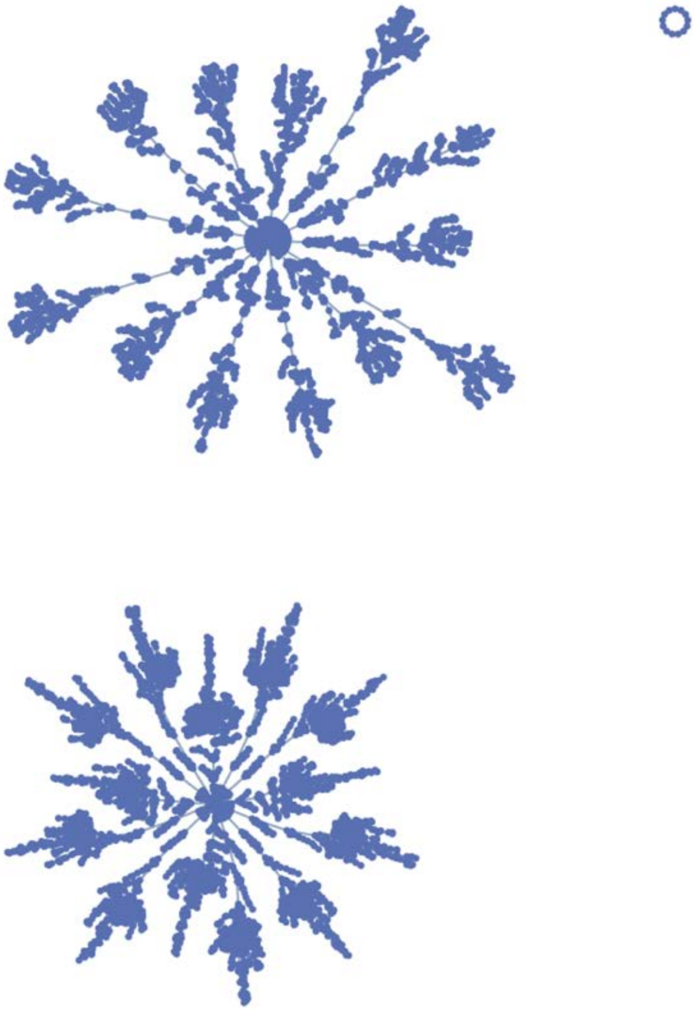

Figure 8. The state transition diagram when $N = 13$. Beyond modes of equilibriation for both parities, there also exists a cycle (check top right). This invalidates the claim made toward arbitrary-N parity in [7].

The first version of this paper was submitted to the ASCAT 2023 conference and received some noteworthy reviews. Reviewer 1 acknowledged that they too could verify that a parity function is realized by the rule for $N = 3, 5, 7, 9, 11$ but fails at $N = 13$. To add to the surprise, they claimed to find that the rule results in a parity function again for $N = 15, 17, 19, 21, 23, 25, 27, 29, 31$. They question whether the rule will fail for $N = 33$ or higher, and they suggest identifying which of the lemmas in the original proof is invalid to identify large N counterexamples. This question is easily answered, because for $N = 13P$, the state $(0\,001\,110\,101\,001)^P$ would also always result in a loop. Studying the synchronization problem over CAs [11] and the insight that a state with a periodic configuration always transitions to another state with a configuration of the same period (or its divisor) helped in seeing that answer.

Reviewer 3 asked for a stronger justification of the Boolean bottleneck conjecture by analyzing the results over more varieties of CAs, finding the theory behind every result and then reporting it. It would be efficient to do so as a team endeavor, and I am willing to coordinate among interested parties. However, proving the Boolean bottleneck conjecture may be difficult or require the development of new mathematical techniques.

3. Conclusion

In hindsight, it is not surprising that our analysis here has identified a flaw in the realizability of arbitrary-N parity logic by uniform cellular automata (CAs). This is because the parity function is nontrivial, and there is no good reason why it should be realizable over other $\approx 2^{2^N}$ (when $N \to \infty$) nontrivial functions that depend on all their variables. Thus, due to this Boolean bottleneck, I conjecture that there can be no uniform CAs that can realize the parity function for all N. However, as a corollary of the unspecial status thrown to the parity function, there could exist small-range rules that realize high-N parity logic. It seems most fruitful to chase such a local-interactions synergizing to globally nontrivial equilibrium-point functionality by studying analog dynamical systems. Nevertheless, the uniform Boolean automata of equation (1) can continue to inform us.

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