A Small Collatz Rule without the Plus One

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The Collatz rule is one of the earliest examples of a simple, deterministic system that produces chaotic behavior. The rule takes any odd positive integer n to $3n+1$ and any even positive integer n to $n/2$. Iterating this rule yields complex sequences whose dynamics are poorly understood; for example, it is unknown whether all such sequences reach 1 (the Collatz conjecture). It is reasonable to suspect that this complexity derives from the interplay of multiplication ($3n$) and addition ($+1$). However, in 2002, Monks was able to drop the +1 by constructing a 1 021 020-condition rule that simulates the Collatz rule using only multiplication. Monks's rule greatly simplifies the Collatz dynamics but at the cost of an enormous rule. The current paper achieves the goal of removing addition with a significantly smaller 30-condition rule. We show how this rule replicates the Collatz process, and we place conditions on any cyclic trajectory purporting to be a counterexample to the Collatz conjecture.

Keywords: number theory; dynamical systems

1. Introduction

The *Collatz rule* is one of the earliest examples of a simple, deterministic system that produces chaotic behavior. It is attributed to Lothar Collatz, who circulated the rule in 1950 after developing similar rules as far back as 1932 [1]. The rule has two frequently used variations:

$$C(n) = \begin{cases} n/2 & \text{for even } n; \\ 3n+1 & \text{for odd } n. \end{cases}$$

$$T(n) = \begin{cases} n/2 & \text{for even } n; \\ (3n+1)/2 & \text{for odd } n. \end{cases}$$

Iterating these rules yields chaotic sequences such as

$$C(n): 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to$$
$$20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1\ldots$$

$$T(n): 7 \to 11 \to 17 \to 26 \to 13 \to$$
$$20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1\ldots$$

https://doi.org/10.25088/ComplexSystems.35.1.1

Figure 1 shows plots of the iterates of $C(n)$, starting at 7. Readers can also explore Collatz sequences at interactive sites [2, 3].

Collatz iterations give rise to many unresolved questions, including the famous *Collatz conjecture*: Does every positive integer eventually reach 1? Because $T(n)$ is simply a shortcut version of $C(n)$, the truth of the conjecture is independent of which variation we use.

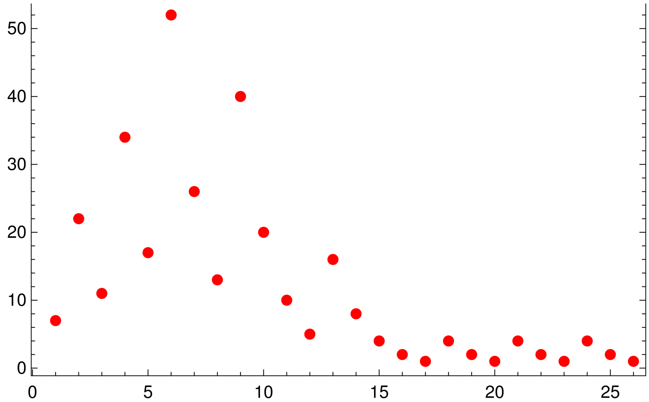

Figure 1. Iterates of the Collatz function $C(n)$, starting at 7. Each tick on the x axis represents one step of the Collatz process.

Empirically, start numbers from 1 to 10^{20} have been shown to satisfy the conjecture [4], but a proof covering all numbers still eludes researchers after 75 years. Lagarias [1, pp. 3–30] provides excellent surveys of mathematical results on Collatz dynamics.

What is it about the Collatz rule that gives rise to chaotic behavior? It is natural to try to pin the complexity on the interaction of multiplication ($3n$, $n/2$) with addition ($+1$). However, Monks [5] was able to develop a Collatz-rule variation using only multiplication, based on a formalism developed by Conway [6]. Monks's rule was quite large, though, involving more than a million conditions.

In this paper, we contribute a simple Collatz rule without the "+1" that uses only 30 conditions. We present it in the spirit of other simple devices that compute Collatz trajectories, such as de Mol's tag system [7], Korec's cellular automaton [8], Yolcu et al.'s rewrite system [9] and Stérin's Wang tiles [10]. We also analyze conditions under which a start number might cycle back to itself and not reach 1.

2. Main Result

The following rule $K(n)$ simulates the Collatz $C(n)$ rule:

$$K(n) = \begin{cases} (375/2)n, & \text{if } n \equiv 2 \text{ or } 8 \pmod{30}; \\ (125/2)n, & \text{if } n \equiv 0 \pmod{30}; \\ (5/3)n, & \text{if } n \equiv 3, 9, 15, 21 \text{ or } 27 \pmod{30}; \\ (2/5)n, & \text{if } n \equiv 5, 10, 20 \text{ or } 25 \pmod{30}; \\ (3/4)n, & \text{if } n \equiv 4, 6, 12, 16, 18 \text{ or } 24 \pmod{30}. \end{cases}$$

$K(n)$ simplifies Collatz trajectories by eliminating the $+1$, which is a source of difficulties for analyzing Collatz dynamics. $K(n)$ works on trajectory terms coded as powers of 2. Here, it simulates the $C(n)$ trajectory $3 \to 10$:

$$\boxed{2^3 = 8} \to 1500 \to 93\,750 \to 5\,859\,375 \to 9\,765\,625 \to 3\,906\,250 \to 1\,562\,500 \to 625\,000 \to 250\,000 \to 100\,000 \to 40\,000 \to 16\,000 \to 6400 \to 2560 \to \boxed{2^{10} = 1024}.$$

Figure 2 shows plots of the iterates of $K(n)$, starting at 2^7. Unlike $C(n)$, the trajectory of $K(n)$ is a piecewise assembly of line segments with slopes drawn from a finite set,

$$\left\{\frac{375}{2}, \frac{125}{2}, \frac{5}{3}, \frac{2}{5}, \frac{3}{4}\right\}.$$

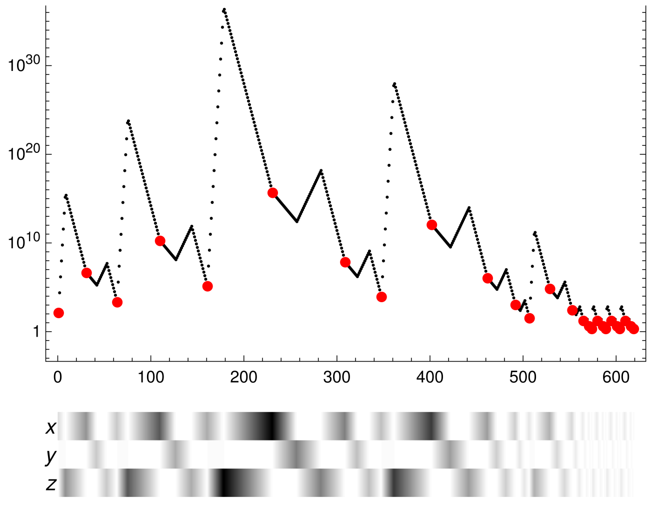

Figure 2. Iterates of $K(n)$, starting at 2^7, plotted on a log scale. Perfect powers of 2 are shown in red; these correspond to the iterates of $C(n)$. Section 4 shows that every $K(n)$ takes the form $2^x 3^y 5^z$; values of the exponents are shown in the heat map below the plot.

3. Background

Conway [6] situated the Collatz rule within a class of *generalized Collatz rules* that take the form

$$G(n) = \begin{cases} (a_0 n + b_0)/d, & \text{if } n \equiv 0 \pmod{d}; \\ (a_1 n + b_1)/d, & \text{if } n \equiv 1 \pmod{d}; \\ \quad \cdots & \\ (a_{d-1} n + b_{d-1})/d, & \text{if } n \equiv d-1 \pmod{d}. \end{cases}$$

When $d = 2$, $a_0 = 1$, $b_0 = 0$, $a_1 = 3$ and $b_1 = 1$, $G(n)$ implements the Collatz rule $T(n)$. Conway also constructed a generalized Collatz rule that mimics the computation of a universal Turing machine. Conway's universal rule did not require the "$+b$," being of the simpler form

$$G'(n) = \begin{cases} (a_0 n)/d, & \text{if } n \equiv 0 \pmod{d}; \\ (a_1 n)/d, & \text{if } n \equiv 1 \pmod{d}; \\ \quad \cdots & \\ (a_{d-1} n)/d, & \text{if } n \equiv d-1 \pmod{d}. \end{cases}$$

Monks [5] constructed an ingenious generalized Collatz rule—also without $+b$—that simulates the Collatz rule itself. Both Conway and Monks presented their generalized Collatz rules in a more compact rule format called Fractran [11]. Monks's rule is

$$\left(\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, \frac{7}{1}\right),$$

which is interpreted as follows: if n is divisible by 11, replace n by $(1/11)n$; otherwise, if n is divisible by 15, replace n by $(136/15)n$, and so on.

To simulate the Collatz rule, each number in a Collatz $T(n)$ trajectory is coded as a power of 2. Monks's Fractran rule is then applied to numbers of this form, while other prime factors serve as registers for intermediate computations. A sample trajectory is

$$\boxed{2^7} \to 2^5 3^1 11^1 \to 2^5 3^1 \to 2^3 3^2$$
$$11^1 \to 2^3 3^2 \to 2^1 3^3 11^1 \to 2^1 3^3 \to 3^3 5^1 \to 2^3 3^2 17^1 \to 2^3 3^2 5^1 \to 2^6$$
$$3^1 17^1 \to 2^6 3^1 5^1 \to 2^9 17^1 \to 2^9 5^1 \to \boxed{2^{11}} \to 2^9 3^1 11^1 \to$$
$$2^9 3^1 \to \cdots \to \boxed{2^{17}} \to \cdots \to \boxed{2^{26}} \to \cdots \to \boxed{2^1}.$$

The Collatz conjecture is equivalent to the proposition that every positive integer 2^m eventually reaches 2^1 under Monks's rule.

The "if-then-otherwise" action of any Fractran rule can be expanded into a generalized Collatz rule. Monks sets

$$d = \text{lcm}(11, 15, 17, 5, 21, 13, 7, 4, 2) = 1\,021\,020,$$

yielding the 1 021 020-way rule

$$M(n) = \begin{cases} 92\,820n\,/\,1\,021\,020, & \text{if } n \equiv 0 \pmod{1\,021\,020}; \\ 7\,147\,140n\,/\,1\,021\,020, & \text{if } n \equiv 1 \pmod{1\,021\,020}; \\ \ldots & \\ 7\,147\,140n\,/\,1\,021\,020, & \text{if } n \equiv 1\,021\,019 \pmod{1\,021\,020}. \end{cases}$$

4. Simulating Collatz Trajectories

The new rule $K(n)$ from Section 2 is much smaller, with only 30 conditions. This section explains its action.

Theorem 1. The rule $K(n)$ simulates the Collatz rule $C(n)$.

Proof. The logic of $K(n)$ is straightforward. $K(n)$ uses three registers (2, 3, 5) to keep track of the computational state. For example, to simulate $C(n)$ on start term 1, $K(n)$ uses six steps:

$$2^1 \xrightarrow{\frac{375}{2}} 3^1 5^3 \xrightarrow{\frac{5}{3}} 5^4 \xrightarrow{\frac{2}{5}} 2^1 5^3 \xrightarrow{\frac{2}{5}} 2^2 5^2 \xrightarrow{\frac{2}{5}} 2^3 5^1 \xrightarrow{\frac{2}{5}} 2^4.$$

To simulate $C(n)$ on any odd term $2m+1$ $(m \geq 1)$, $K(n)$ uses $8m+6$ steps. It starts by repeatedly decrementing the exponent of 2 by one while simultaneously incrementing the exponent of 5 by three:

$$\begin{aligned} 2^{2m+1} &\to 2^{2m+1-1} 3^1 5^3 && \text{1 step of } (375\,/\,2) \\ &\to 2^{2m+1-2} 3^1 5^6 \to \cdots \to 3^1 5^{6m+3} && 2m \text{ steps of } (125\,/\,2) \\ &\to 5^{6m+4} && \text{1 step of } (5\,/\,3) \\ &\to 2^1 5^{6m+4-1} \to 2^2 5^{6m+4-2} \to \cdots \to 2^{6m+4} && 6m+4 \text{ steps of } (2\,/\,5). \end{aligned}$$

Finally, $K(n)$ uses $3m$ steps to simulate $C(n)$ on any even term $2m$ $(m \geq 1)$. It begins by decrementing the exponent of 2 by two while incrementing the exponent of 3 by one:

$$\begin{aligned} 2^{2m} &\to 2^{2m-2} 3^1 \to 2^{2m-4} 3^2 \to \cdots \to 3^m && m \text{ steps of } (3\,/\,4) \\ &\to 3^{m-1} 5^1 \to 3^{m-2} 5^2 \to \cdots \to 5^m && m \text{ steps of } (5\,/\,3) \\ &\to 5^{m-1} 2^1 \to 5^{m-2} 2^2 \to \cdots \to 2^m && m \text{ steps of } (2\,/\,5). \end{aligned}$$

Here, $K(n)$ uses 5^m as a stopover, instead of transiting directly from 3^m to 2^m—under the latter approach, upon encountering $2^a 3^b$, $K(n)$ would not know whether to decrement the exponent of 2 or increment it. It is fortuitous that the $5\,/\,3$ condition can be reused for different purposes. When processing odd terms, it puts the $+1$ in $3n+1$; for even terms, it transits from 3^m to 5^m.

Now we must show that the congruence classes of $K(n)$ encode the correct computational state. From the given trajectories, we see eight expressions that must be distinguished from one another: 2^{2a+1},

$2^a 3^b 5^c$, 3^b, $2^a 3^b$, $2^a 5^c$, 2^{2a}, $3^b 5^c$ and 5^c. Using tables for 2^n, 3^n and $5^n \bmod 30$, we calculate possible classes (modulo 30) for these expressions:

2^{2a+1}	2, 8	$2^a 3^b 5^c$	0	3^b	3, 9, 21, 27	$2^a 3^b$	6, 12, 18, 24
$2^a 5^c$	10, 20	2^{2a}	4, 16	$3^b 5^c$	15	5^c	5, 25

Because the classes are distinct, $K(n)$ specifies the correct action to take in each case. □

Note that the 30-condition rule $K(n)$ is not a true generalized Collatz rule, because $(3/4)n$ in the fifth condition cannot be rewritten as $(a_4 n + b_4)/30$. A 60-condition variation $K'(n)$ fixes this problem:

$$K'(n) = \begin{cases} 11\,250n/60, & \text{if } n \equiv 2, 8, 32, 38 \pmod{60}; \\ 3750n/60, & \text{if } n \equiv 0, 30; \\ 100n/60, & \text{if } n \equiv 3, 9, 15, 21, 27, 33, 39, 45, 51, 57; \\ 24n/60, & \text{if } n \equiv 5, 10, 20, 25, 35, 40, 50, 55; \\ 45n/60, & \text{if } n \equiv 4, 6, 12, 16, 18, 24, 34, 36, 42, 46, 48, 54. \end{cases}$$

5. Cycles

The trivial Collatz cycle 1-2-4-1 has a trajectory of length 15 under the $K(n)$ rule:

$$\boxed{2^1 = 2} \to 375 \to 625 \to 250 \to 100 \to 40 \to \boxed{2^4 = 16} \to$$
$$12 \to 9 \to 15 \to 25 \to 10 \to \boxed{2^2 = 4} \to 3 \to 5 \to \boxed{2^1 = 2}.$$

The rule's coefficients are used in this manner:

$$2^1 \cdot \frac{375}{2} \cdot \frac{5}{3} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{5}{3} \cdot \frac{5}{3} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{5}{3} \cdot \frac{2}{5} = 2^1.$$

If a Collatz cycle starts at n, then $2^n \cdot p = 2^n$, where $p = 1$ is a product of fractions drawn from the set

$$\left\{\frac{375}{2}, \frac{125}{2}, \frac{5}{3}, \frac{2}{5}, \frac{3}{4}\right\},$$

with repeats. The shortest list of such fractions is

$$\frac{125}{2} \cdot \frac{5}{3} \cdot \frac{5}{3} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{3}{4} = 1.$$

However, no trajectory uses these particular fractions in any order; consider that $125/2$ is only employed directly after $375/2$. So $p = 1$ is a necessary condition for a cycle, but not a sufficient one.

We follow Monks by investigating the possible combinations of fractions used in a cycle. Suppose

$$\frac{375}{2}^a \frac{125}{2}^b \frac{5}{3}^c \frac{2}{5}^d \frac{3}{4}^e = 1.$$

In that case,

$$2^{d-a-b-2e}\, 3^{a+e-c}\, 5^{3a+3b+c-d} = 1,$$

which means

$$\begin{aligned} d-a-b-2e &= 0 \\ a+e-c &= 0 \\ 3a+3b+c-d &= 0. \end{aligned}$$

This system of equations is solved by

$$\begin{aligned} b &= e/2 - 3a/2 \\ c &= a+e \\ d &= 5e/2 - a/2. \end{aligned} \tag{1}$$

By inspecting the action of $K(n)$, we can assign interpretations to the variables. For example, $K(n)$ processes an even number n by applying $3/4$ some $n/2$ times. Therefore, e is half the sum of all even numbers in a Collatz cycle.

Theorem 2. The rule $K(n)$ takes $(5/2)m$ steps to simulate a $C(n)$ cycle, where m is the sum of even terms in the cycle.

Proof. Because e is half the sum of the even numbers in the cycle, $m = 2e$:

$$a+b+c+d+e =$$

$$a + (e/2 - 3a/2) + (a+e) + (5e/2 - a/2) + e = 5e = \frac{5}{2}m. \square$$

Thus, the length of every $K(n)$ cycle is a multiple of 5. Also, because $e = (a+b+c+d+e)/5$, the $3/4$ branch of $K(n)$ is taken 20% of the time during any cycle, neither more nor less than its fair share.

By further inspecting $K(n)$, we note that a counts the number of odd terms in the cycle, while b is the sum of odd terms minus the count of odd terms. Letting $m_1 \ldots m_k$ be the terms of any Collatz cycle, we rewrite equation (1):

$$\begin{aligned} b &= e/2 - 3a/2 \\ 2(a+b)+a &= e \\ 2\sum_{\text{odd } m_i} m_i + \sum_{\text{odd } m_i} 1 &= \frac{1}{2}\sum_{\text{even } m_i} m_i \\ \sum_{\text{odd } m_i} (2m_i+1) &= \frac{1}{2}\sum_{\text{even } m_i} m_i. \end{aligned}$$

This is a well-known property of $C(n)$ cycles. For the trivial 1-2-4-1 cycle, we have

$$2(1)+1 = \frac{1}{2}(2+4).$$

6. Conclusion

The Collatz conjecture presents an unsolved problem involving multiplication and addition. This paper presents a variation of the Collatz rule that uses multiplication only, simplifying its dynamics. Our rule is the smallest of its kind, using only 30 conditions compared to previous variations with more than one million conditions. We also show that the length of any Collatz cycle must be a multiple of 5 under this rule.

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