Exploring the Containment Hierarchy of Subrings over Rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$ in Two-Dimensional Cellular Automata

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This paper investigates the structural properties of two-dimensional cellular automata (2DCAs) over rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$, focusing on rule composition, subring hierarchy and linear evolution. We prove that rule composition is commutative across all rings and that matrix transformations preserve hierarchical relationships. The subring containment is determined by the divisors of n: prime rings have only trivial subrings, while composite rings exhibit structured hierarchies. Let $I \in \mathbb{Z}_k^{n\times n}$ be a configuration matrix. For rings in $\mathfrak{S}_1 = \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$, repeated elementwise addition modulo k maps the entries of I to their immediate subrings in the containment hierarchy. For $\mathfrak{S}_2 = \{\mathbb{Z}_3, \mathbb{Z}_9\}$, triple summation modulo k similarly restricts the values to their corresponding subrings. Furthermore, we extend Moore neighborhood-based two-dimensional cellular automaton (2DCA) rules to rings from $\mathbb{Z}_2$ to $\mathbb{Z}_5$ and $\mathbb{Z}_7$ to $\mathbb{Z}_9$, proving that at time step t, rule matrices generate multiple non-overlapping replicas of the initial configuration across each ring and its subrings.

Experiments in C++ across various rings and image sizes revealed two key patterns requiring further mathematical explanation. Rings with only trivial subrings—$\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$—replicate the initial image at time steps $t = p^k$, where $p \in \{2, 3, 5, 7\}$ and $k \geq 0$. In contrast, rings with nontrivial subrings—$\mathbb{Z}_4$ and $\mathbb{Z}_8$—generate multiple replicas at $t = 2^k$, while $\mathbb{Z}_9$ exhibits replication at $t = 3^k$, producing

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both the original image and its subrings. The ring $\mathbb{Z}_6$, with disjoint subrings and no clear hierarchy, shows no such structured replication.

Keywords: two-dimensional cellular automata; rings; subrings; Moore neighborhood; linear rules; $\mathbb{Z}_2$ to $\mathbb{Z}_9$; image processing

1. Introduction

A cellular automaton (CA for brevity, plural cellular automata, CAs) is an abstract parallel machine that consists of a regular grid of sequences of either finite or infinite cells in an n-dimensional space. All cells interact locally and operate in parallel on an n-dimensional grid. On an n-dimensional grid, all cells are of the same shape, such as squares in a two-dimensional grid. In the evolution of a CA, each cell on the grid is in one state from a finite set, and the cell's state is updated based on the states of its neighbors and itself. In Figure 1, the current state of cell C_{ij} is black, and it changes to white in the next time step by interacting with the states of its nearest neighbors. Nature has several mechanisms that work together to maintain everything in balance and ensure that the system functions properly as a whole. Understanding natural phenomena such as the balance of carbon dioxide and oxygen, sun-induced Earth heating, rainfall, water evaporation, cloud formation and other natural processes is a difficult problem for researchers. The CA is a simulation model that makes understanding these natural phenomena easier, due to its structure. John von Neumann introduced the CA model in the late 1940s and early 1950s to model biological self-reproduction [1]. CAs are a good source of parallel processing; recently, researchers across various computer science areas (modeling various physical and biological processes, mathematics, cryptography, image processing, language recognition, computer architectures, machine learning, deep learning, fractals, chaos, etc.) have explored CAs for diverse applications.

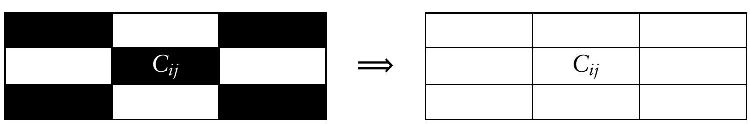

Figure 1. The black state of central cell C_{ij} changes to the white state in the 2DCA Moore neighborhood.

Wolfram [2] conducted an in-depth investigation of a series of one-dimensional cellular automaton (1DCA) rules and demonstrated that even the simplest rules can exhibit complicated behavior. The study employed polynomial algebra to analyze 1DCA properties. Das [3] investigated 1DCA characterization using linear algebra. In the 1960s, John H. Conway's [4] Game of Life exemplified a two-dimensional cellular automaton (2DCA). Within the past few years, researchers have shown a significant amount of interest in two-dimensional cellular automata (2DCAs). The literature [5] presents simple yet precise mathematical models that utilize matrix algebra over the ring $\mathbb{Z}_2$ to characterize the behavior of 2DCA linear rules under von Neumann and Moore neighborhoods with null and periodic boundary conditions.

Ceccherini-Silberstein and Coornaert [6] explore the mathematical foundations of cellular automata (CAs), focusing on their relationship to group theory, amenability and algebraic structures. Their work addresses crucial subjects such as residual finiteness, surjunctive and sofic groups, the Garden of Eden theorem and linear CAs, with a focus on the relationship between dynamical systems and group algebraic features. In [7], CAs with finite, potentially noncommutative group states are examined, extending additive CAs to noncommutative groups. Building on the results of Kitchens and Schmidt, the study examines their structural characteristics and develops algorithmic approaches to determine properties that would otherwise be undecidable.

Self-replication is the behavior of a dynamical system that results in the creation of an identical or similar replica of itself. Self-replicating patterns are an intriguing topic and research domain in nonlinear science and technology. The papers [8–16] studied self-replicating structures in one- and two-dimensional CAs. In [14], self-replicating patterns were studied under periodic, adiabatic and reflexive boundary conditions over the field $\mathbb{Z}_2$ for all linear 2DCA rules as well as image processing applications. Well-known image processing applications include translation, multiple replicas, zooming in, zooming out, thickening, thinning and edge detection, including noise suppression and convex hull construction. These applications are investigated using 2DCA linear rules with the null boundary condition on a Moore neighborhood in [14, 17–23]. Uguz et al. [18] investigated the linear rules of 2DCAs within the Moore neighborhood, applying both periodic and null boundary conditions in the ternary field $\mathbb{Z}_3$. Additionally, the authors investigated multiple non-overlapping replicas of an image over the ring $\mathbb{Z}_3$ (three-color image) for the 2DCA linear rules.

This paper investigates the generation of multiple non-overlapping replicas of an initial image across the rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$, including their subrings within the containment hierarchy. The organization of the paper is as follows. Section 2 provides an overview of modular rings

$\mathbb{Z}_n$, their subrings and the containment hierarchy in composite rings. It also introduces fundamental definitions of CAs and key notations relevant to 2DCAs. Section 3 presents a concise technical overview of 2DCAs with a Moore neighborhood between rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$. We establish the commutativity of primary rules under null boundary conditions and analyze subring containment in modular arithmetic. Additionally, we study rule matrix evolution in 2DCAs over finite rings, ensuring non-overlapping replicas. These results form the basis for understanding algebraic transformations in modular systems, as discussed in Section 4. In Section 5, we present an experimental analysis of self-replicating patterns in image processing using 2DCAs with null boundary conditions. Finally, Section 6 presents the conclusions and future work.

2. Algebraic Preliminaries and Pattern Replication in Two-Dimensional Cellular Automata

This section presents the foundational algebraic structures, specifically rings and their subrings, along with the basic definitions of 2DCAs relevant to image replication patterns. Non-overlapping image replicas are defined as replicated versions of the initial image that remain distinct and do not overlap on the two-dimensional grid.

2.1 Ring and Its Subring Hierarchies

We consider finite rings of the form $\mathbb{Z}_n$, where arithmetic operations are performed modulo n. If d is a divisor of n, then

$$d\,\mathbb{Z}_n = \{d\,k \bmod n \mid k \in \mathbb{Z}\}$$

is a subring of $\mathbb{Z}_n$. This divisibility condition naturally induces a hierarchical structure among rings:

1. Prime rings such as $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}_7$ contain only trivial subrings (i.e., the zero ring and the ring itself).
2. Composite rings like $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$ exhibit structured hierarchies, containing proper subrings aligned with their divisors. For instance, $\mathbb{Z}_8$: $\{0\}$, $\{0, 4\}$, $\{0, 2, 4, 6\}$ and $\mathbb{Z}_8$ illustrates nested containment.

The containment hierarchy illustrates the layering of these subrings within one another. The illustration is presented in Table 1.

However, some rings such as $\mathbb{Z}_6$ do not exhibit a coherent subring hierarchy, as their substructures are disjoint.

This hierarchical perspective is key to understanding the varying behavior of CAs across different rings, particularly in relation to rule composition and image replication.

Prime Rings	Composite Rings
$\mathbb{Z}_2$: $\{0\} \subset \mathbb{Z}_2$	$\mathbb{Z}_4$: $\{0\} \subset \{0, 2\} \subset \mathbb{Z}_4$
$\mathbb{Z}_3$: $\{0\} \subset \mathbb{Z}_3$	$\mathbb{Z}_6$: $\{0, 3\}$, $\{0, 2, 4\}$ (disjoint subrings)
$\mathbb{Z}_5$: $\{0\} \subset \mathbb{Z}_5$	$\mathbb{Z}_8$: $\{0\} \subset \{0, 4\} \subset \{0, 2, 4, 6\} \subset \mathbb{Z}_8$
$\mathbb{Z}_7$: $\{0\} \subset \mathbb{Z}_7$	$\mathbb{Z}_9$: $\{0\} \subset \{0, 3, 6\} \subset \mathbb{Z}_9$

Table 1. Containment hierarchy of rings from $\mathbb{Z}_2$ to $\mathbb{Z}_9$.

2.2 Basic Definitions and Image Replication Behavior in Cellular Automata

The formal definition of a CA is a 4-tuple (G, Q, f, N), where G is an n-dimensional grid (finite or infinite), Q is a finite state set, f is a local transition function (or rule), and N is a neighborhood for each cell on the grid. The neighborhood refers to the set of cells surrounding a particular cell that are considered when determining that cell's next state, such as the Moore neighborhood (see Figure 1). The neighborhood can vary depending on the rules of the CA. The way the whole configuration of a CA changes from one time step to the next is described by the global function, also known as the global update function. This is the result of applying the local transition function to every cell in the grid simultaneously. The following definitions are frequently used to characterize CAs.

Definition 1. A CA is said to be a finite CA if the total number of cells in a CA grid is finite.

Definition 2. A uniform CA applies the same rule to every cell, whereas a hybrid CA allows different cells to follow different rules.

Definition 3. If every boundary cell of a CA is connected to the zero state, then the CA is a null boundary CA.

Definition 4. A rule is considered linear (or additive) if it involves only the modulo operator.

Throughout this paper, we adopt the following notation to formally describe a finite 2DCA.

- $(2\text{DCA})_{m\times n}$: A two-dimensional grid composed of m rows and n columns.
- $C_{i,j}$: The cell located at row i and column j within $(2\text{DCA})_{m\times n}$, where $1 \le i \le m$ and $1 \le j \le n$.
- $C_{i,j}^{(t)}$: The state of cell $C_{i,j}$ at discrete time step $t \in \mathbb{N}$.

Consider two cells, $C_{i,j}$ and $C_{x,y}$, located at positions (i, j) and (x, y) on the two-dimensional square grid. Their spatial relationships are classified as follows:

(i) Horizontal cells: $C_{i,j}$ and $C_{x,y}$ are in the same row $i = x$, and the absolute difference in columns $|y - j|$ is 1 or 2. Cells are *adjacent* if $|y - j| = 1$ and *nonadjacent* if $|y - j| = 2$.

(ii) Vertical cells: $C_{i,j}$ and $C_{x,y}$ are in the same column $j = y$, and the absolute difference in rows $|i - x||$ is 1 or 2. Cells are *adjacent* if $|i - x| = 1$ and *nonadjacent* if $|i - x| = 2$.

(iii) Diagonal cells: $C_{i,j}$ and $C_{x,y}$ satisfy either $|i - x| = |j - y| = 1$ or $|i - x| = |j - y| = 2$. Cells are *adjacent* if $|i - x| = |j - y| = 1$ and *nonadjacent* if $|i - x| = |j - y| = 2$.

(iv) Combination cells: $C_{i,j}$ and $C_{x,y}$ satisfy either $|i - x| = 1$ and $|j - y| = 2$ or $|i - x| = 2$ and $|j - y| = 1$.

In CAs, certain linear rules generate repeated patterns of the initial configuration across the grid. To formalize this behavior, we introduce the concept of multiple non-overlapping replicas within a finite 2DCA.

Definition 5. (Multiple non-overlapping replicas on a finite 2DCA grid). Let $I_0 \in \mathbb{Z}_k^{n\times n}$ be an initial configuration on a 2DCA grid, and let $I_t \in \mathbb{Z}_k^{M\times M}$ be the configuration at time step t, where $M \geq n$. We say that I_t contains non-overlapping replicas of I_0 if there exists a finite set of index pairs

$$\mathcal{P} = \{(r_1, c_1), (r_2, c_2), \ldots, (r_m, c_m)\}$$

such that for each $(r_i, c_i) \in \mathcal{P}$, the submatrix of I_t defined by

$$I_t[r_i : r_i + n - 1, c_i : c_i + n - 1]$$

is equal to I_0, and all such submatrices are pairwise disjoint, that is,

$$[r_i, r_i + n - 1]\times[c_i, c_i + n - 1] \bigcap \\ [r_j, r_j + n - 1]\times[c_j, c_j + n - 1] = \emptyset \quad \text{for all } i \neq j.$$

Example 1. Let the initial configuration be:

$$I_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathbb{Z}_5^{2\times 2}.$$

Suppose that at time step $t = 3$, the 2DCA configuration becomes:

$$I_3 = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 2 \\ 3 & 4 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 2 \\ 3 & 4 & 0 & 0 & 3 & 4 \end{bmatrix} \in \mathbb{Z}_5^{6\times 6}.$$

We observe that I_3 contains four non-overlapping replicas of I_0, located at:

$$\mathcal{P} = \{(1, 1), (1, 5), (5, 1), (5, 5)\}.$$

Each corresponding 2×2 submatrix matches I_0, and all are disjoint in position. Therefore, I_3 contains four non-overlapping replicas of I_0.

3. Two-Dimensional Cellular Automata over the Rings from $\mathbb{Z}_2$ to $\mathbb{Z}_9$

A finite 2DCA consists of an $m\times n$ grid of cells arranged in m rows and n columns, where each cell takes a value from the modular rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$. It is denoted as $(\text{2DCA})_{m\times n}$, with the possible states per cell determined by the chosen modular ring. The configuration at time step $t+1$ is governed by a local linear rule, where the state of each cell depends on the states of some neighboring cells at time step t. While various neighborhood structures exist in 2DCA theory, this paper focuses on Moore neighborhoods (see Figure 1). A $(\text{2DCA})_{3\times 3}$ with a null boundary is illustrated in Table 2. A local transition function (or linear rule) $f : (\mathbb{Z}_k)^9 \to \mathbb{Z}_k$ updates the cell state $C_{i,j}^{(t)}$ at the next time $(t+1)$ as:

$$C_{i,j}^{(t+1)} = f\big(C_{i-1,j-1}^{(t)}, C_{i-1,j}^{(t)}, C_{i-1,j+1}^{(t)}, C_{i,j-1}^{(t)}, C_{i,j}^{(t)}, C_{i,j+1}^{(t)}, C_{i+1,j-1}^{(t)}, C_{i+1,j}^{(t)}, C_{i+1,j+1}^{(t)}\big) \quad (1)$$

$$C_{i,j}^{(t+1)} = \alpha_1 C_{i-1,j-1}^{(t)} \oplus \alpha_2 C_{i-1,j}^{(t)} \oplus \alpha_3 C_{i-1,j+1}^{(t)} \oplus \alpha_4 C_{i,j-1}^{(t)} \oplus \alpha_5 C_{i,j}^{(t)} \oplus \alpha_6 C_{i,j+1}^{(t)} \oplus \alpha_7 C_{i+1,j-1}^{(t)} \oplus \alpha_8 C_{i+1,j}^{(t)} \oplus \alpha_9 C_{i+1,j+1}^{(t)} (\text{mod } k) \quad (2)$$

where α_1 to $\alpha_9 \in \mathbb{Z}_k$ and the value $k \in \{2, 3, 4, 5, 6, 7, 8, 9\}$.

0	0	0	0	0
0	$C_{i-1,j-1}^{(t)}$	$C_{i-1,j}^{(t)}$	$C_{i-1,j+1}^{(t)}$	0
0	$C_{i,j-1}^{(t)}$	$C_{i,j}^{(t)}$	$C_{i,j+1}^{(t)}$	0
0	$C_{i+1,j-1}^{(t)}$	$C_{i+1,j}^{(t)}$	$C_{i+1,j+1}^{(t)}$	0
0	0	0	0	0

Table 2. Null boundary condition in a $(\text{2DCA})_{3\times 3}$.

In general, the next state of a cell may not depend on all nine neighboring cells. The rule number of a 2DCA over the rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$ represents the number of neighboring cells that influence the value of each cell. Tables 2 and 3 illustrate the standard method for defining a rule number in a linear 2DCA. In [3, 5, 17], the authors used a particular rule convention over ring $\mathbb{Z}_2$, illustrated in Table 3, where the central cell is designated as 1, and its adjacent cells to the right, bottom-right, bottom, bottom-left, left, top-left, top and top-right are assigned the values 2, 4, 8, 16, 32, 64, 128 and 256, respectively. The number within each cell represents the rule number that characterizes the central cell and is influenced by its particular neighbor cell. These are the primary rules over the ring $\mathbb{Z}_2$. The current cell (i.e., the cell being considered) updates its state with its own present state and is represented as $\mathbb{Z}_2$ rule 1; the current cell updates its state based only on its left neighbor and is represented as $\mathbb{Z}_2$ rule 32. If the present cell updates its state using two or more neighboring cells, the rule number will be the arithmetic total of the neighboring cells' marked numbers. For example, the current cell updates its state based on its four neighboring cells (left, bottom, right and itself) and is called $\mathbb{Z}_2$ rule 43 (32 + 8 + 2 + 1). The number of such linear rules is

$$9_{C_1} + \cdots + 9_{C_9} = 511$$

(excluding $\mathbb{Z}_2$ rule 0, which is trivially linear). Similarly, Uguz et al. [18] adopted the same rule convention for the ring $\mathbb{Z}_3$. In this paper, we follow the identical rule-indexing approach for $\mathbb{Z}_3$ and extend it to the rings $\mathbb{Z}_4$ through $\mathbb{Z}_9$. Specifically, the neighbor-weight numbering scheme shown in Table 3, in which each Moore-neighborhood position is assigned a power-of-two weight, is consistently applied to the rings $\mathbb{Z}_3$ through $\mathbb{Z}_9$.

$C_{i-1,j-1}^{(t)}$ **(64)**	$C_{i-1,j}^{(t)}$ **(128)**	$C_{i-1,j+1}^{(t)}$ **(256)**
$C_{i,j-1}^{(t)}$ **(32)**	$C_{i,j}^{(t)}$ **(1)**	$C_{i,j+1}^{(t)}$ **(2)**
$C_{i+1,j-1}^{(t)}$ **(16)**	$C_{i+1,j}^{(t)}$ **(8)**	$C_{i+1,j+1}^{(t)}$ **(4)**

Table 3. Numbering of rule conventions with respect to neighbors over ring $\mathbb{Z}_2$.

Table 4 illustrates the Moore neighborhood surrounding the cell $C_{i,j}^{(t)}$ in the ring $\mathbb{Z}_3$ at time step t. The table also defines the numbering convention used for the neighborhood within $\mathbb{Z}_3$.

$C_{i-1,j-1}^{(t)}$ **(729)**	$C_{i-1,j}^{(t)}$ **(2187)**	$C_{i-1,j+1}^{(t)}$ **(6561)**
$C_{i,j-1}^{(t)}$ **(243)**	$C_{i,j}^{(t)}$ **(1)**	$C_{i,j+1}^{(t)}$ **(3)**
$C_{i+1,j-1}^{(t)}$ **(81)**	$C_{i+1,j}^{(t)}$ **(27)**	$C_{i+1,j+1}^{(t)}$ **(9)**

Table 4. Numbering of rule conventions with respect to neighbors over ring $\mathbb{Z}_3$.

A local transition function (or linear rule) $f:(\mathbb{Z}_3)^9 \to \mathbb{Z}_3$ updates the state of a cell $C_{i,j}^{(t)}$ at the next time step $t+1$, yielding $C_{i,j}^{(t+1)}$. This function f depends on the states of the cell and its Moore neighborhood in $\mathbb{Z}_3$ and is formally defined in equation (2). The primary rules over the ring $\mathbb{Z}_3$ include 1, 3, 9, 27, 81, 243, 729, 2187 and 6561.

Tables 5 through 7 illustrate the Moore neighborhood around the cell $C_{i,j}^{(t)}$ in the rings $\mathbb{Z}_4$ to $\mathbb{Z}_9$ at time step t. These tables also establish the numbering conventions for the neighborhood in each respective ring. The function f, which determines the next state of the cell based on its own state and that of its Moore neighborhood, is formally defined in equation (2).

Ring $\mathbb{Z}_4$			Ring $\mathbb{Z}_5$		
4096	16384	65536	15625	78125	390625
1024	1	4	3125	1	5
256	64	16	625	125	25

Table 5. Numbering of rule conventions over rings $\mathbb{Z}_4$ and $\mathbb{Z}_5$.

Ring $\mathbb{Z}_6$			Ring $\mathbb{Z}_7$		
46656	279936	1679616	117649	823543	5764801
7776	1	6	16807	1	7
1296	216	36	2401	343	49

Table 6. Numbering of rule conventions over rings $\mathbb{Z}_6$ and $\mathbb{Z}_7$.

Ring $\mathbb{Z}_8$			Ring $\mathbb{Z}_9$		
262144	2097152	16777216	531441	4782969	43046721
32768	1	8	59049	1	9
4096	512	64	6561	729	81

Table 7. Numbering of rule conventions over rings $\mathbb{Z}_8$ and $\mathbb{Z}_9$.

The study of 2DCAs over the rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$ provides a structured framework for defining rule conventions and transition functions. By extending the traditional numbering system from $\mathbb{Z}_2$ to higher modular rings, we ensure consistency in analyzing CA behavior. The use of Moore neighborhoods and local linear rules enables systematic state evolution based on selected neighboring influences. This approach lays the foundation for further exploration of complex dynamics in multivalued CAs.

4. Properties of Primary Rules in Modular Matrices with Null Boundary Conditions

Modular rings exhibit intricate structural properties, influencing rule compositions and containment hierarchies. In this section, we establish the commutativity of primary rules under null boundary conditions and analyze the containment hierarchy of subrings in modular arithmetic. Additionally, we explore the evolution of rule matrices in 2DCAs over the rings, ensuring non-overlapping replicas across containment hierarchies. These results provide a foundation for understanding algebraic transformations in modular systems.

The definitions of the auxiliary matrices T_1 and T_2 for the null boundary follow:

$$T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ & \cdots & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In [17], the next-state representation of all primary rules over $\mathbb{Z}_2$ (1, 2, 4, 8, 16, 32, 64, 128 and 256) under null boundary conditions is expressed using the auxiliary matrices T_1 and T_2. In this paper, we extend this approach to determine the next state of all primary rules for other rings, specifically defining it for $\mathbb{Z}_8$.

Using the previously defined auxiliary matrices T_1 and T_2, the next-state representation of all primary rules (1, 8, 64, 512, 4096, 32768, 262144, 2097152 and 16777216) over $\mathbb{Z}_8$ under null boundary conditions is given by:

$$\begin{aligned} \mathbb{Z}_8 \text{ rule } 1\ N(\mathcal{R}_1) &: [I_{t+1}] = [I_t] \\ \mathbb{Z}_8 \text{ rule } 8\ N(\mathcal{R}_2) &: [I_{t+1}] = [X_t][T_2] \\ \mathbb{Z}_8 \text{ rule } 64\ N(\mathcal{R}_3) &: [I_{t+1}] = [T_1][I_t][T_2] \\ \mathbb{Z}_8 \text{ rule } 512\ N(\mathcal{R}_4) &: [I_{t+1}] = [T_1][I_t] \\ \mathbb{Z}_8 \text{ rule } 4096\ N(\mathcal{R}_5) &: [I_{t+1}] = [T_1][I_t][T_1] \\ \mathbb{Z}_8 \text{ rule } 32\,768\ N(\mathcal{R}_6) &: [I_{t+1}] = [I_t][T_1] \\ \mathbb{Z}_8 \text{ rule } 262\,144\ N(\mathcal{R}_7) &: [I_{t+1}] = [T_2][I_t][T_1] \\ \mathbb{Z}_8 \text{ rule } 2\,097\,152\ N(\mathcal{R}_8) &: [I_{t+1}] = [T_2][I_t] \\ \mathbb{Z}_8 \text{ rule } 16\,777\,216\ N(\mathcal{R}_9) &: [I_{t+1}] = [T_2][I_t][T_2]. \end{aligned} \tag{3}$$

Similarly, the next state of all primary rules under the null boundary condition can be determined for the remaining rings $\mathbb{Z}_2$ to $\mathbb{Z}_7$ and $\mathbb{Z}_9$. Subsequently, we prove the commutativity of primary rule compositions within modular rings under null boundary conditions. The results ensure that the order of rule applications does not affect the ultimate result, hence maintaining consistency in algebraic and computational systems.

Theorem 1. Let $I_t \in \mathbb{Z}_k^{n\times n}$ denote the state of a 2DCA at time step t. The null boundary extension of the state I_t is denoted by $\tilde{I}_t \in \mathbb{Z}_k^{(n+2)\times(n+2)}$ and is defined as:

$$(\tilde{I}_t)_{i,j} = \begin{cases} 0 & \text{if } i = 1 \text{ or } i = n+2 \text{ or } j = 1 \text{ or } j = n+2 \\ (I_t)_{i-1,j-1} & \text{otherwise} \end{cases} \tag{4}$$

Let the shift matrices T_1, T_2 be redefined as:

$$(T_1)_{i,j} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}, \quad (T_2)_{i,j} = \begin{cases} 1 & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then the following shift properties hold for the extended configuration:

1. $\tilde{I}T_1$ shifts the entire configuration I_t one step to the right (→), moving each element one position right.
2. $T_1\tilde{I}$ shifts the entire configuration I_t one step upward (↑), moving each element one position up.
3. $\tilde{I}T_2$ shifts the entire configuration I_t one step to the left (←), moving each element one position left.
4. $T_2\tilde{I}$ shifts the entire configuration I_t one step downward (↓), moving each element one position down.

Proof. We prove the four shift properties using matrix multiplication rules along with equations (4) and (5).

1. Rightward configuration shift via $\tilde{I}_t T_1$ multiplication: Consider the product $\tilde{I}_t T_1$. For any position (i, j), we have:

   $$\left(\tilde{I}_t T_1\right)_{i,j} = \sum_{k=1}^{n+2} \tilde{I}_t(i, k) T_1(k, j) = \tilde{I}_t(i, j-1)$$

   since $T_1(k, j) = 1$ if and only if $k = j - 1$. Thus, each column is replaced by its left column, producing a rightward shift. The first column becomes 0 due to the null boundary.

2. Upward configuration shift via $T_1\tilde{I}_t$ multiplication: Consider the product $T_1\tilde{I}_t$. For any (i, j), we get:

   $$\left(T_1\tilde{I}_t\right)_{i,j} = \sum_{k=1}^{n+2} T_1(i, k)\tilde{I}_t(k, j) = \tilde{I}_t(i+1, j)$$

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since $T_1(i, k) = 1$ if and only if $k = i + 1$. Each row is replaced by the row below it, producing an upward shift. The last row is zeroed due to the null boundary.

3. Left configuration shift via $\tilde{I}_t T_2$ multiplication: Consider $\tilde{I}_t T_2$. Then:

$$\left(\tilde{I}_t T_2\right)_{i,j} = \sum_{k=1}^{n+2} \tilde{I}_t(i, k) T_2(k, j) = \tilde{I}_t(i, j + 1)$$

because $T_2(k, j) = 1$ if and only if $k = j + 1$. Each column moves to the left, with the last column becoming 0.

4. Downward configuration shift via $T_2 \tilde{I}_t$ multiplication: Consider $T_2 \tilde{I}_t$. We compute:

$$\left(T_2 \tilde{I}_t\right)_{i,j} = \sum_{k=1}^{n+2} T_2(i, k) \tilde{I}_t(k, j) = \tilde{I}_t(i - 1, j)$$

because $T_2(i, k) = 1$ if and only if $k = i - 1$. Each row moves downward, and the first row is padded with 0. □

This concludes the proof that matrix compositions involving $\tilde{I}$, T_1 and T_2 produce directional shifts of I_t, where the shift (right, left, up or down) is determined by the order of multiplication.

Theorem 2. (Shift matrix commutativity under null boundary condition over $\mathbb{Z}_n$.) Let $\tilde{I}_t \in \mathbb{Z}_k^{(n+2)\times(n+2)}$ denote the extended configuration matrix of a 2DCA with null boundary conditions, as defined in equation (4). Let T_1 and T_2 be the shift matrices defined by equation (5). Then the shift matrices commute under null boundary conditions:

$$T_1 T_2 \tilde{I}_t = T_2 T_1 \tilde{I}_t.$$

Furthermore, for any non-negative integers $m, n \in \mathbb{N}$,

$$T_1^m T_2^n \tilde{I}_t = T_2^n T_1^m \tilde{I}_t.$$

Proof. From Theorem 1, we know the effects of the shift matrices on the extended configuration $\tilde{I}_t$: the product $\tilde{I}_t T_1$ shifts the configuration one step to the right, $\tilde{I}_t T_2$ shifts it one step to the left, $T_1 \tilde{I}_t$ shifts it upward and $T_2 \tilde{I}_t$ shifts it downward.

Now consider the composition $T_1 T_2 \tilde{I}_t$. The matrix T_2 first shifts $\tilde{I}_t$ downward by one step, and then T_1 shifts the result upward. Conversely, in $T_2 T_1 \tilde{I}_t$, the configuration is first shifted upward and then downward. In both cases, the vertical shifts cancel out, returning the configuration to its original vertical position. Since no horizontal shift occurs during these operations, the overall effect remains the same:

$$T_1 T_2 \tilde{I}_t = T_2 T_1 \tilde{I}_t.$$

The commutativity of T_1 and T_2 extends naturally to higher powers by the associativity of matrix multiplication and the principle of mathematical induction.

We begin with the base case $m = n = 1$, which has already been verified

$$T_1 T_2 \tilde{I}_t = T_2 T_1 \tilde{I}_t.$$

Assume that for some $m, n \in \mathbb{N}$, the following relation holds

$$T_1^m T_2^n \tilde{I}_t = T_2^n T_1^m \tilde{I}_t.$$

Then, for $m+1$, we compute

$$T_1^{m+1} T_2^n \tilde{I}_t = T_1 T_1^m T_2^n \tilde{I}_t = T_1 T_2^n T_1^m \tilde{I}_t = T_2^n T_1 T_1^m \tilde{I}_t = T_2^n T_1^{m+1} \tilde{I}_t.$$

Here, we used the inductive hypothesis and the associativity of matrix multiplication.

Thus, by induction, the commutation relation holds for all $m, n \in \mathbb{N}$

$$T_1^m T_2^n \tilde{I}_t = T_2^n T_1^m \tilde{I}_t. \square$$

Theorem 3. For arbitrary primary rules under the null boundary condition over the rings, the composition of rules is commutative.

Proof. Since all primary rules under the null boundary condition exhibit a structurally identical form across the rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$, we may, without loss of generality, select any representative ring. Consider a 2DCA defined over $\mathbb{Z}_8$, where the evolution of the configuration matrix $[I_t] \in \mathbb{Z}_8^{n\times n}$ is based on local update rules. The null boundary extension of the state I_t is denoted by $\tilde{I}_t \in \mathbb{Z}_8^{(n+2)\times(n+2)}$.

Each rule corresponds to a specific neighborhood direction and is defined using horizontal and vertical shift matrices T_1 and T_2, respectively. All such rules can be uniformly expressed as

$$F_k\big([\tilde{I}_t]\big) = A_k \cdot [\tilde{I}_t] \cdot B_k$$

where A_k and B_k are composed of products of T_1, T_2 or the identity matrix $\mathcal{I}$. For instance, rule $\mathcal{R}_1$ (see equation (3)) has $A = \mathcal{I}$, $B = \mathcal{I}$, while rule $\mathcal{R}_5$ has $A = T_1$, $B = T_1$.

Let

$$F_i\big([\tilde{I}_t]\big) = A_i \cdot [\tilde{I}_t] \cdot B_i \quad \text{and} \quad F_j\big([\tilde{I}_t]\big) = A_j \cdot [\tilde{I}_t] \cdot B_j.$$

Then, the composition of rules becomes

$$\mathcal{R}_j \circ \mathcal{R}_i\big([\tilde{I}_t]\big) = F_j\big(F_i\big([\tilde{I}_t]\big)\big) = A_j \cdot \big(A_i \cdot [\tilde{I}_t] \cdot B_i\big) \cdot B_j = (A_j A_i) \cdot [\tilde{I}_t] \cdot (B_i B_j).$$

Since A_i, A_j, B_i and B_j are composed of products of the shift matrices T_1 and T_2, and these matrices commute by Theorem 2, it follows that

$$(A_i A_j)\cdot \tilde{I}_t \cdot (B_j B_i) = A_i \cdot \left(A_j \cdot \tilde{I}_t \cdot B_j\right)\cdot B_i.$$

This implies

$$F_j\left(F_i(\tilde{I}_t)\right) = F_i\left(F_j(\tilde{I}_t)\right) \Rightarrow \mathcal{R}_j \circ \mathcal{R}_i = \mathcal{R}_i \circ \mathcal{R}_j. \square$$

Example 2. Consider a configuration over $\mathbb{Z}_8$

$$[I] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \tilde{I} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rule $\mathcal{R}_1$: $F_1(\tilde{I}) = \mathcal{I}\cdot \tilde{I}\cdot \mathcal{I} = \tilde{I}$, where $\mathcal{I}$ is the identity matrix.

Rule $\mathcal{R}_7$: $F_2(\tilde{I}) = T_2 \cdot \tilde{I}\cdot T_1$.

First, compute the composition $\mathcal{R}_7 \circ \mathcal{R}_1(\tilde{I})$

$$\mathcal{R}_7 \circ \mathcal{R}_1(\tilde{I}) = T_2 \cdot \tilde{I}\cdot T_1.$$

Apply the right shift T_1 followed by the down shift T_2

$$\tilde{I}\cdot T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow T_2 \cdot \left(\tilde{I}\cdot T_1\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

Now compute the reverse composition $\mathcal{R}_1 \circ \mathcal{R}_7(\tilde{I})$

$$\mathcal{R}_1 \circ \mathcal{R}_7(\tilde{I}) = \mathcal{I}\cdot \left(T_2 \cdot \tilde{I}\cdot T_1\right)\cdot \mathcal{I} = T_2 \cdot \tilde{I}\cdot T_1.$$

Thus, we conclude

$$\mathcal{R}_7 \circ \mathcal{R}_1(\tilde{I}) = \mathcal{R}_1 \circ \mathcal{R}_7(\tilde{I}).$$

Hence, the two primary rules commute under the null boundary condition.

Lemma 1 establishes a containment hierarchy of subrings within the modular rings $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, $\mathbb{Z}_8$ and $\mathbb{Z}_9$, based on the behavior of matrix addition. For rings in $\mathfrak{S}_1 = \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$, if a matrix I has entries from any subring in $\mathfrak{S}_1$, then adding it to itself modulo k (where $k \in \{2, 4, 8\}$) maps those entries to the next smaller subring in the hierarchy. Similarly, for $\mathfrak{S}_2 = \{\mathbb{Z}_3, \mathbb{Z}_9\}$, performing triple summation modulo k (with $k \in \{3, 9\}$) produces a similar reduction. This pattern also holds for $\mathbb{Z}_5$ and $\mathbb{Z}_7$, where repeated summation five or seven times modulo the respective ring shifts elements within their algebraic structure. These consistent reduction patterns reveal deeper structural properties in modular arithmetic and have notable implications in algebra and combinatorial matrix theory.

Lemma 1. Let $\mathfrak{S}_1 = \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$ and $\mathfrak{S}_2 = \{\mathbb{Z}_3, \mathbb{Z}_9\}$. The containment hierarchy of subrings of rings of S_1 and S_2 is $\mathbb{Z}_2 : \{0\} \subset \mathbb{Z}_2$, $\mathbb{Z}_4 : \{0\} \subset \{0, 2\} \subset \mathbb{Z}_4$, $\mathbb{Z}_8 : \{0\} \subset \{0, 4\} \subset \{0, 2, 4, 6\} \subset \mathbb{Z}_8$, $\mathbb{Z}_9 : \{0\} \subset \{0, 3, 6\} \subset \mathbb{Z}_9$. Let

$$I = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & x_{n,3} & \cdots & x_{n,n} \end{bmatrix}.$$

The containment hierarchy holds the following properties:

(A) Select any ring $\mathbb{Z}_k$ from the set $\mathfrak{S}_1$, where $k \in \{2, 4, 8\}$. Let A be any subring appearing in the containment hierarchy of $\mathbb{Z}_k$. Then A is either the ring $\mathbb{Z}_k$ itself or one of its proper subrings. If all elements of the matrix I lie in A, then the elements of $(I + I) \bmod k$ belong to the immediate predecessor subring of A in the hierarchy.

(B) Select any ring $\mathbb{Z}_x$ from the set $\mathfrak{S}_2$, where $x \in \{3, 9\}$. Let B be any subring in the containment hierarchy of $\mathbb{Z}_x$. If all elements of the matrix I lie in B, then the elements of $(I + I + I) \bmod x$ will belong to the immediate predecessor subring of B in the hierarchy.

(C) If the matrix I is defined over $\mathbb{Z}_5$ or $\mathbb{Z}_7$, then the repeated sum $I + I + \cdots + I$ (taken 5 times modulo 5 or 7 times modulo 7, respectively) results in the trivial subring $\{0\}$. This subring is the smallest element in the containment hierarchy of $\mathbb{Z}_5$ or $\mathbb{Z}_7$.

Proof. Let $I \in \mathbb{Z}_k^{n\times n}$ be a matrix with entries in a modular ring $\mathbb{Z}_k$, where $k \in \{2, 3, 4, 5, 7, 8, 9\}$. We verify the three containment properties for rings in $\mathfrak{S}_1$, $\mathfrak{S}_2$ and for $\mathbb{Z}_5$, $\mathbb{Z}_7$ individually.

(A) For $\mathfrak{S}_1 = \{\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_8\}$: Suppose all entries of matrix I belong to a subring $A \subseteq \mathbb{Z}_k$, where $k \in \{2, 4, 8\}$. By the structure of these rings, repeated addition modulo k reduces elements to smaller subrings:

- In $\mathbb{Z}_4$, if $I \subseteq \{0, 2\}$, then $(I + I) \bmod 4 \subseteq \{0\}$.
- In $\mathbb{Z}_8$, if $I \subseteq \{0, 4\}$, then $(I + I) \bmod 8 \subseteq \{0\}$, and if $I \subseteq \{0, 2, 4, 6\}$, then $(I + I) \bmod 8 \subseteq \{0, 4\}$.

Thus, summing the matrix with itself modulo k maps elements to the immediate predecessor subring of A in the hierarchy.

(B) For $\mathfrak{S}_2 = \{\mathbb{Z}_3, \mathbb{Z}_9\}$: Let matrix I contain entries from a subring $B \subseteq \mathbb{Z}_x$, with $x \in \{3, 9\}$. The containment hierarchy implies that:

- In $\mathbb{Z}_9$, if $I \subseteq \{0, 3, 6\}$, then $(I + I + I) \bmod 9 \subseteq \{0\}$.

Hence, performing a triple summation modulo x maps elements of B to its immediate predecessor subring.

(C) For $\mathbb{Z}_5$ and $\mathbb{Z}_7$: These rings are fields and contain only the trivial subring $\{0\}$. If all entries of matrix I lie in $\mathbb{Z}_5$ (respectively, $\mathbb{Z}_7$), then adding the matrix to itself 5 times modulo 5 (respectively, 7 times modulo 7) yields:

$$(I + I + \cdots + I) \bmod 5 = 5\, I \bmod 5 = 0, \quad \text{or} \quad 7\, I \bmod 7 = 0,$$

implying I maps to the trivial subring $\{0\}$.

This confirms the containment behavior of matrix summation across the defined modular rings. □

In Theorem 4, we analyze the behavior of 2DCAs over rings, defined by linear rules in the Moore neighborhood. We demonstrate that for any ring $\mathbb{Z}_k$ in the set $L = \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_9\}$, the rule matrix produces non-overlapping configuration replicas within a containment hierarchy. These replicas, corresponding to $\mathbb{Z}_k$ and its subrings, preserve the dimensions of the initial configuration over time.

Theorem 4. Let $L = \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_9\}$ be a finite set of rings. Consider a linear 2DCA with a Moore neighborhood and null boundary conditions, defined over each ring $\mathbb{Z}_k \in L$. Let $\mathcal{R}_x$ be any 2DCA rule formed as a binary linear combination of primary rules $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_9$ over $\mathbb{Z}_k$, and let $I_0 \in \mathbb{Z}_k^{n\times n}$ denote the initial configuration. Then, at some time step t, the rule $\mathcal{R}_x$ produces non-overlapping replicas of both $\mathbb{Z}_k$ and its subrings within the containment hierarchy, each replica matching the dimensions of I_0.

Proof. Let $\mathcal{R}_x = \sum_{i=1}^{9} \beta_i \mathcal{R}_i \bmod k$, where $\beta_i \in \{0, 1\}$ indicates whether the primary rule $\mathcal{R}_i$ is active in the composition. Since each β_i is binary, there are $2^9 = 512$ distinct linear rule combinations over $\mathbb{Z}_k$.

Let $I_0 \in \mathbb{Z}_k^{n\times n}$ be the initial configuration. The automaton evolves as:

$$I_{t+1} = \mathcal{R}_x(\tilde{I}_t),$$

where $\tilde{I}_t$ denotes I_t extended with a null boundary. Assume $\mathcal{R}_x$ includes m active primary rules. Then, at iteration t, the configuration results from all ordered compositions of these m rules:

$$\mathcal{R}_{j_1} \circ \mathcal{R}_{j_2} \circ \cdots \circ \mathcal{R}_{j_t}(I_0),$$

with $\mathcal{R}_{j_i} \in \mathcal{R}_x$ for each $i \in \{1, 2, \ldots, t\}$.

There are m^t such compositions at time t, representing all possible sequences of rule applications of length t drawn from m choices.

For illustration, consider $\mathcal{R}_x = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$ over $\mathbb{Z}_k$, with $I_0 \in \mathbb{Z}_k^{n\times n}$. Then:

$$I_1 = \mathcal{R}_x(\tilde{I}_0) = (\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3)(\tilde{I}_0),$$

$$I_2 = \mathcal{R}_x(\tilde{I}_1) = \sum_{\text{2-compositions}} \mathcal{R}_i \circ \mathcal{R}_j(\tilde{I}_0),$$

$$I_3 = \mathcal{R}_x(\tilde{I}_2) = \sum_{\text{3-compositions}} \mathcal{R}_{i_1} \circ \mathcal{R}_{i_2} \circ \mathcal{R}_{i_3}(I_0).$$

With $m = 3$, there are $3^3 = 27$ distinct compositions at $t = 3$, as illustrated in Figure 2. These compositions fall into two main categories:

- Self-composition rules, of the form $\mathcal{R}_i \circ \mathcal{R}_i \circ \cdots \circ \mathcal{R}_i$, generate distinct, non-overlapping replicas of I_0, preserving the full ring structure of $\mathbb{Z}_k$. For example, $\mathcal{R}_1 \circ \mathcal{R}_1 \circ \mathcal{R}_1$, $\mathcal{R}_2 \circ \mathcal{R}_2 \circ \mathcal{R}_2$ and $\mathcal{R}_3 \circ \mathcal{R}_3 \circ \mathcal{R}_3$.
- Non-self-composition rules, such as $\mathcal{R}_i \circ \mathcal{R}_j$ with $i \neq j$, result in overlapping submatrices. As stated in Lemma 1, the modular addition of such overlapping elements yields submatrices whose entries lie in proper subrings of $\mathbb{Z}_k$.

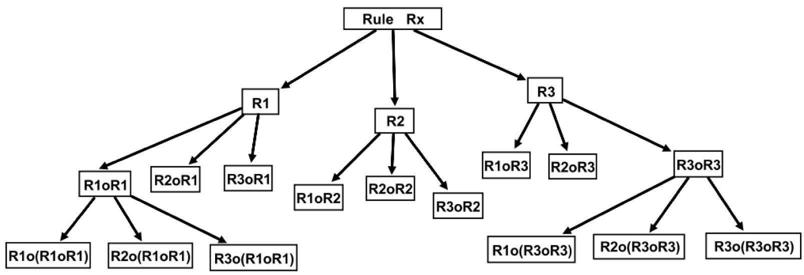

Figure 2. Hierarchical expansion of rule $\mathcal{R}_x$ via recursive composition.

Figure 3 demonstrates two iterations of $\mathcal{R}_x$ applied to a sample configuration, highlighting the emergence of both ring and subring structures within the finite 2DCA grid.

Since rule composition is commutative (Theorem 3), many composition sequences result in the same transformation. Consequently, the matrix I_t at iteration t contains:

- Non-overlapping replicas over the full ring $\mathbb{Z}_k$, produced by self-composition rules.
- Non-overlapping submatrices over proper subrings of $\mathbb{Z}_k$, arising from non-self-composition rules.

Thus, at some time step t, the rule $\mathcal{R}_x$ produces disjoint, self-replicating patterns over both $\mathbb{Z}_k$ and its subrings, with each replicated block having the same dimensions as the initial configuration I_0. □

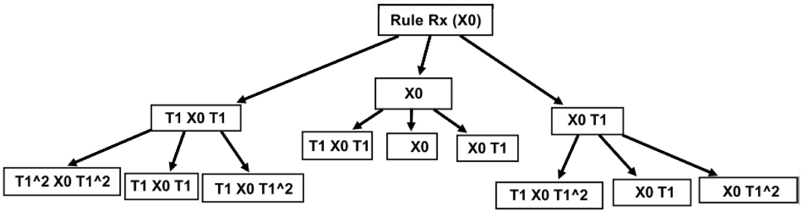

Figure 3. Recursive application of rule $\mathcal{R}_x(I_0)$ in a tree structure.

Example 3. Consider the ring $\mathbb{Z}_4$ and its containment hierarchy: $\{0\} \subset \{0, 2\} \subset \mathbb{Z}_4$. Apply the $\mathbb{Z}_4$ rule $1284N$ to the initial configuration. The next state of each cell is determined by its left neighbor, right neighbor and bottom-left neighbor. In the resulting matrix I_4, the submatrices corresponding to $\mathbb{Z}_4$ and its nontrivial subring $\{0, 2\}$ are highlighted in bold:

$$I_0 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \qquad I_1 = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 2 & 2 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 2 & 2 & 2 & 2 & 0 & 0 \\ 3 & 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad I_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 & 3 \\ 0 & 0 & 2 & 2 & 1 & 3 & 3 & 1 \\ 2 & 2 & 3 & 1 & 0 & 0 & 3 & 1 \\ 3 & 1 & 1 & 3 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{\mathbf{2}} & \underline{\mathbf{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \underline{\mathbf{3}} & \underline{\mathbf{1}} \\ 0 & 0 & 0 & 0 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & 0 & 0 & \underline{\mathbf{0}} & \underline{\mathbf{0}} \\ 0 & 0 & 0 & 0 & \underline{\mathbf{2}} & \underline{\mathbf{2}} & 0 & 0 & \underline{\mathbf{2}} & \underline{\mathbf{2}} \\ \underline{\mathbf{2}} & \underline{\mathbf{2}} & 0 & 0 & \underline{\mathbf{0}} & \underline{\mathbf{0}} & 0 & 0 & \underline{\mathbf{2}} & \underline{\mathbf{2}} \\ \underline{\mathbf{3}} & \underline{\mathbf{1}} & 0 & 0 & \underline{\mathbf{2}} & \underline{\mathbf{2}} & 0 & 0 & \underline{\mathbf{3}} & \underline{\mathbf{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Self-Replicating Patterns in Image Processing Using Two-Dimensional Cellular Automata with Null Boundary

CAs have demonstrated significant potential in image processing due to their simple yet powerful computational framework. Their lattice-based structure makes them well suited for various image processing tasks, enabling efficient implementation on hardware such as VLSI, FPGA and GPU devices. CAs are commonly used for edge detection, noise reduction, thinning and convex hull generation, leveraging matrix-based transformations to enhance images through defined neighborhood interactions. Additionally, the ability of CAs to generate complex patterns from simple initial configurations, including self-replicating structures, highlights their effectiveness in image-related computations.

One of the most fascinating features of CAs in image processing is their ability to generate self-replicating patterns—a concept deeply rooted in nonlinear science and pattern formation research. By repeatedly replicating a seed image across a defined region, a CA enables seamless tiling without gaps or overlaps, making it particularly valuable for automated pattern design, fractal analysis and texture synthesis. Additionally, CA principles such as self-organization and chaotic dynamics open avenues for developing novel image transformation techniques. Originally explored by John von Neumann [4], these self-replicating mechanisms have profound implications for digital image synthesis and computational modeling.

Self-replicating patterns in 2DCAs arise when transformation rules produce multiple non-overlapping replicas of an initial configuration. As noted in [14], this replication process is defined by Moore and von Neumann neighborhoods, where each cell updates based on its neighbors under various boundary conditions, including null, periodic, adiabatic and reflexive. Experimental studies [19] show that specific CA rules, such as rule 9840 in the ternary field $\mathbb{Z}_3$, exhibit structured self-replication after a fixed number of iterations, with 512 classified CA rules displaying diverse replication behaviors. This phenomenon plays a crucial role in image processing, enabling applications in image transformation, encryption, edge detection and noise reduction, highlighting its significance in both theoretical and practical contexts.

In this section, we examine the initial configuration of an image over the rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$ and explore the application of 2DCA linear rules in image processing. Present experimental analysis specifically focuses on each ring and its nontrivial subrings. An image of size $m \times n$ serves as the initial configuration, where each pixel color corresponds to an element in an $m \times n$ matrix over the chosen ring. The colors are numerically represented from 0 to 8, corresponding to white, black, red, blue, green, yellow, orange, pink and magenta, respectively. This implementation utilizes a custom simulation program written using raylib and C++ for image transformations (github.com/ked1108/2d-cellular-automata-simulation).

Initially, a color image is selected based on the chosen ring, and a null boundary condition is applied. The 2DCA linear rules are then iteratively applied, updating the configuration at each time step and generating a progressively expanding image. Extensive experiments across $\mathbb{Z}_2$ to $\mathbb{Z}_9$ reveal intriguing patterns, highlighting the potential for further mathematical analysis. Through these experiments, we observed that 2DCA linear rules generate multiple non-overlapping replicas at time steps 2^k (see Figures 4 and 5), 3^k (see Figures 6 and 7), 5^k (see Figures 8 and 9) and 7^k (see Figures 10 and 11) over the rings $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$, respectively. These rings do not have any nontrivial subrings.

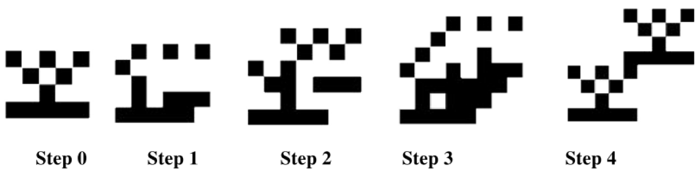

Figure 4. Non-overlapping replicas at time step 4 of rule 257 over the ring $\mathbb{Z}_2$.

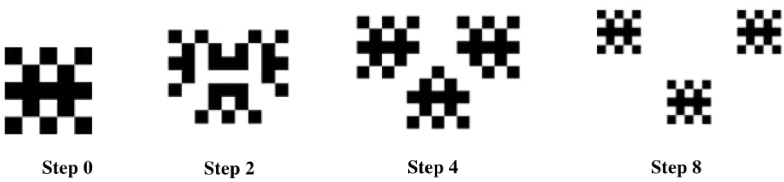

Figure 5. Non-overlapping replicas at time step 8 of rule 21 over the ring $\mathbb{Z}_2$.

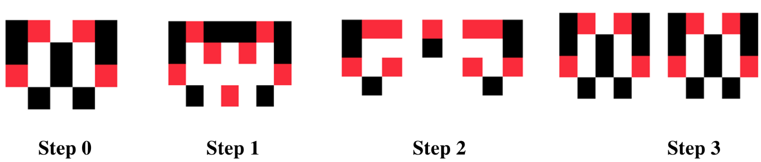

Figure 6. Non-overlapping replicas at time step 3 of rule 90 over the ring $\mathbb{Z}_3$.

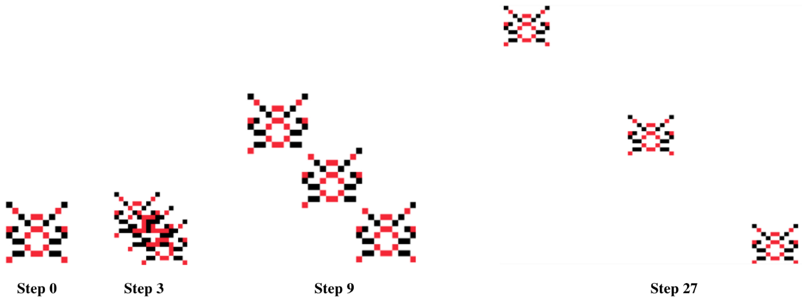

Figure 7. Non-overlapping replicas at time step 27 of rule 739 over the ring $\mathbb{Z}_3$.

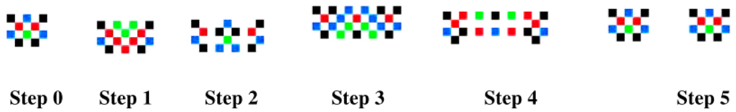

Figure 8. Non-overlapping replicas at time step 5 of rule 650 over the ring $\mathbb{Z}_5$.

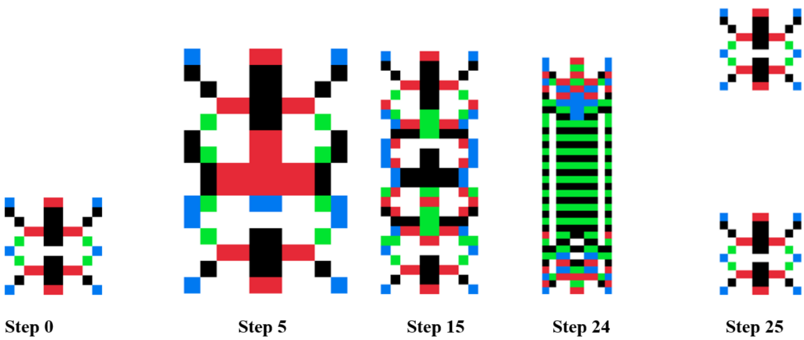

Figure 9. Non-overlapping replicas at time step 25 of rule 78 126 over the ring $\mathbb{Z}_5$.

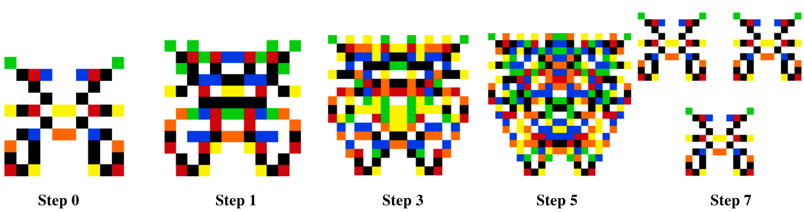

Figure 10. Non-overlapping replicas at time step 7 of rule 825 993 over the ring $\mathbb{Z}_7$.

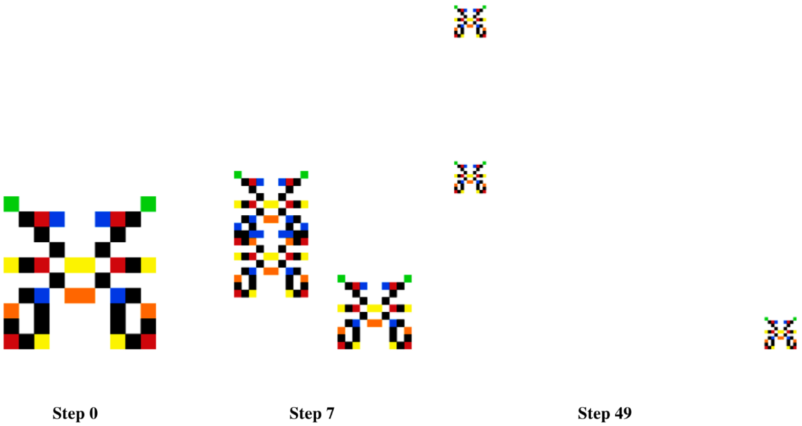

Figure 11. Non-overlapping replicas at time step 49 of rule 117 705 over the ring $\mathbb{Z}_7$.

The rings $\mathbb{Z}_4 = \mathbb{Z}_{2^2}$, $\mathbb{Z}_8 = \mathbb{Z}_{2^3}$ and $\mathbb{Z}_9 = \mathbb{Z}_{3^2}$ contain nontrivial subrings. The elements of $\mathbb{Z}_4$ are represented using the colors {0 (white), 1 (black), 2 (red), 3 (blue)}, while its nontrivial subring consists of {0 (white), 2 (red)}. Similarly, the elements of $\mathbb{Z}_8$ are represented by the colors {0 (white), 1 (black), 2 (red), 3 (blue), 4 (green), 5 (yellow), 6 (orange), 7 (pink)}, with its nontrivial subrings being {white, red, green} and {white, green}. For $\mathbb{Z}_9$, the colors used are {0 (white), 1 (black), 2 (red), 3 (blue), 4 (green), 5 (yellow), 6 (orange), 7 (pink), 8 (magenta)}, with its nontrivial subring consisting of {white, blue, green}.

When the initial image belongs to a specific subring within this hierarchical structure, the application of 2DCA linear rules leads to the emergence of multiple non-overlapping replicas of both the original image and its corresponding subrings of their subrings. This replication occurs at specific time steps, depending on the underlying ring structure. For $\mathbb{Z}_4$, these replicas appear at time steps $t = 2^k$ (see Figures 12 and 13), while for $\mathbb{Z}_8$, the same pattern is observed at $t = 2^k$ (see Figures 16 and 17). In contrast, for $\mathbb{Z}_9$, the replication follows a different progression, occurring at time steps $t = 3^k$ (see Figure 18). This key finding suggests that a uniform 2DCA linear rule with a Moore neighborhood and a null boundary condition generates multiple non-overlapping replicas over the ring $\mathbb{Z}_{p^m}$ at time steps p^k, where p is a prime and m, k are positive integers.

However, in the case of $\mathbb{Z}_6$ (see Figures 14 and 15), the presence of disjoint subrings, specifically {0, 3} and {0, 2, 4}, disrupts this structured behavior. Due to the lack of a unified subring structure, the linear rules fail to generate multiple non-overlapping replicas or establish a well-defined pattern in the time evolution of the image.

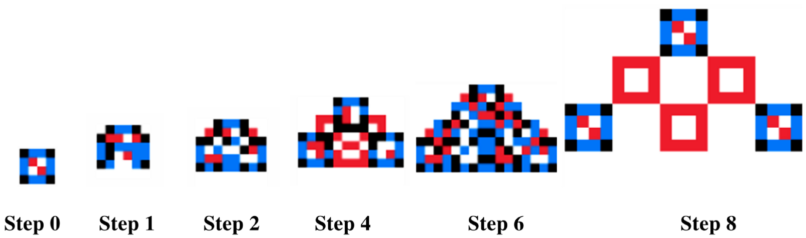

Figure 12. Replication of the original image and subring in $\mathbb{Z}_4$ at time step 8 of rule 1092.

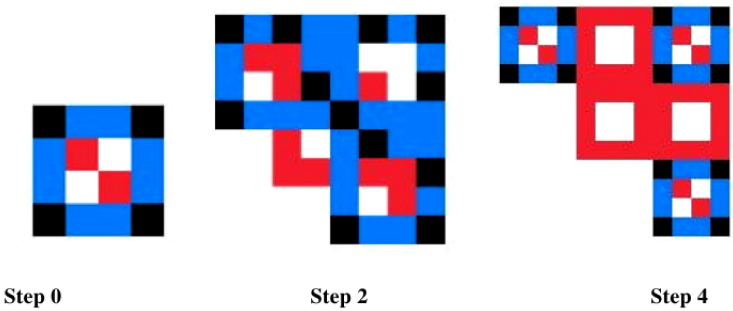

Figure 13. Replication of the original image and subring in $\mathbb{Z}_4$ at time step 4 of rule 4368.

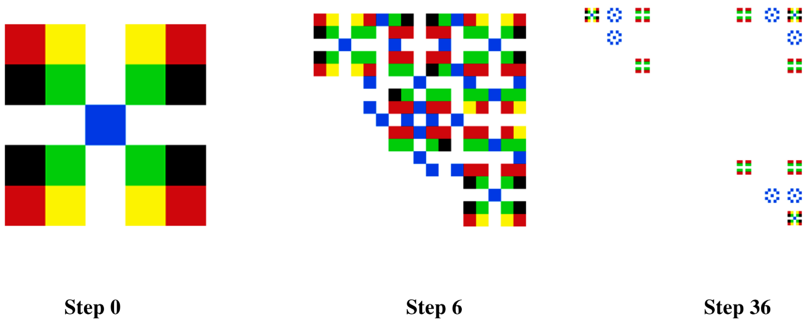

Figure 14. Non-overlapping replicas along with some beautiful patterns at time step 36 of rule 47 988 over the ring $\mathbb{Z}_6$.

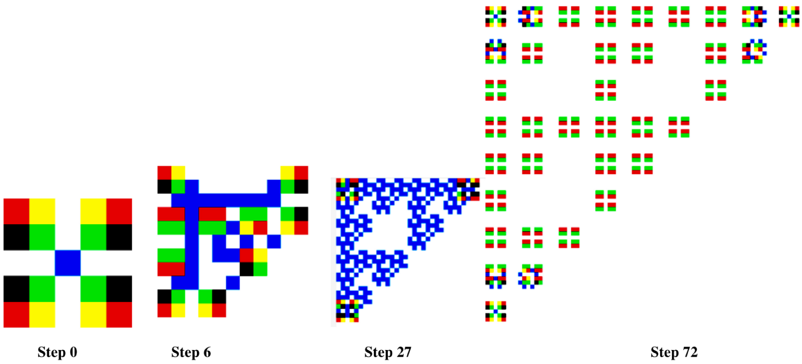

Figure 15. Non-overlapping replicas along with some beautiful patterns at time step 72 of rule 258 over the ring $\mathbb{Z}_6$.

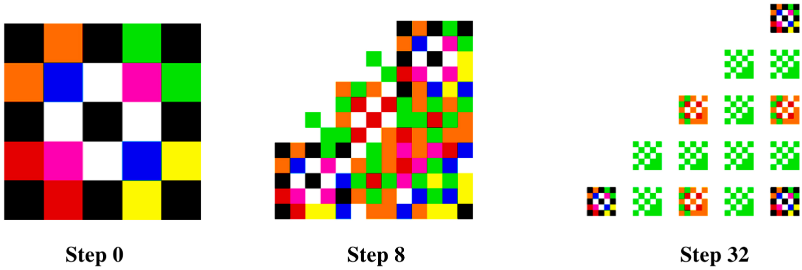

Figure 16. Replication of the original image and subrings in $\mathbb{Z}_8$ at time step 32 of rule 36 865.

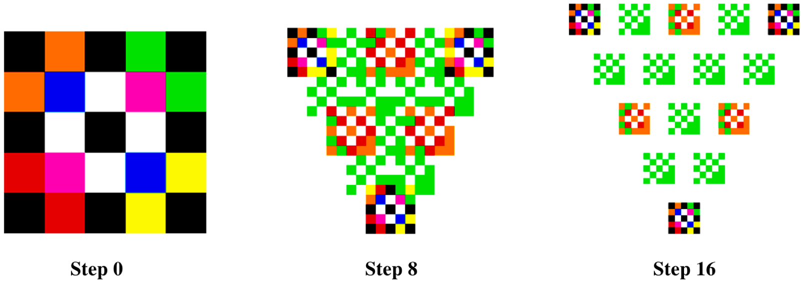

Figure 17. Replication of the original image and subring in $\mathbb{Z}_8$ at time step 16 of rule 2 101 312.

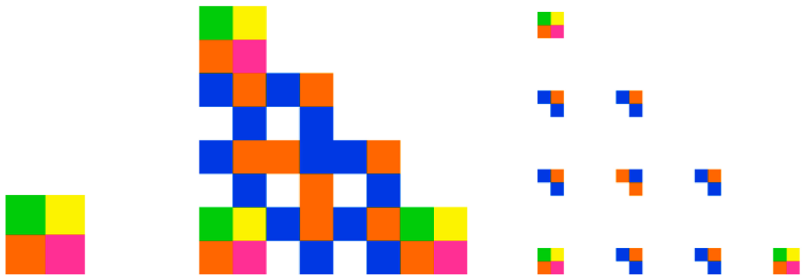

Figure 18. Replication of the original image and subring in $\mathbb{Z}_9$ at time step 9 of rule 43 578 243.

6. Conclusion and Future Work

In this paper, we explore the structural properties and evolution of two-dimensional cellular automata (2DCAs) over the rings $\mathbb{Z}_2$ to $\mathbb{Z}_9$, providing deeper insights into their containment hierarchy, rule composition and image replication. For any such linear rule $\mathcal{R}_x$, the system produces, at a certain time step t, multiple non-overlapping replicas of the initial configuration matrix over both the full ring $\mathbb{Z}_k$ and its subrings. These replicas preserve the dimensions of the original configuration and appear in a structured, self-similar pattern across the two-dimensional grid. The total number of such configurations at each iteration is determined by the number of active rules included in the composition. It is established that self-composition rules, where a single primary rule is repeatedly applied, produce exact non-overlapping copies of the initial image over the full ring. In contrast, non-self-composition rules, involving combinations of distinct primary rules, lead to overlapping intermediate results that, due to modular arithmetic, resolve into non-overlapping replicas over proper subrings of $\mathbb{Z}_k$.

Furthermore, the commutativity of rule composition ensures that the order in which transformations are applied does not affect the final configuration, thereby simplifying the analysis and reinforcing the predictable structure of replication across the grid. However, for $\mathbb{Z}_6$, the presence of disjoint subrings {0, 3} and {0, 2, 4} disrupts structured replication. The absence of a unified subring hierarchy prevents linear rules from producing non-overlapping replicas or consistent patterns during image evolution.

This paper highlights a strong connection between the algebraic structure of rings and the spatiotemporal behavior of 2DCA, providing a framework for understanding modular replication and hierarchical pattern formation in discrete dynamical systems. Future research can focus on formally characterizing the time steps at which replication occurs, especially those of the form $t = p^k$, where p is a prime and $k \geq 0$. Additionally, extending this analysis to larger rings such as $\mathbb{Z}_{p^m}$ for $m \geq 1$, or to composite rings with more complex additive structures, could uncover new algebraic properties governing the behavior of 2DCAs. Practical applications may include structured image tiling, texture synthesis and cryptographic techniques that leverage the predictable replication of ring and subring structures.

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