Phase Transitions in Complexity: Active Cell Growth, Shannon Entropy and Kolmogorov Complexity in Dependent Cell Ranges of Mobile Automata with Non-local Rules

Akshaj Devireddy
Wolfram Institute
akshaj.devireddy@gmail.com

This paper investigates phase transitions in complexity within mobile automata governed by non-local rules. Unlike traditional cellular automata, mobile automata involve a single active cell navigating and updating a one-dimensional array of binary-state cells based on a rule set. By varying the number and symmetry of dependent cells in non-local rules, we observe abrupt changes in system behavior that we identify as computational phase transitions. Using active cell growth and three complementary metrics (Shannon block entropy and estimates of Kolmogorov–Chaitin complexity based on block decomposition and lossless compression), we quantitatively analyze the complexity of automata across a range of dependent cell configurations. Our results reveal that certain increases in non-locality trigger dramatic shifts in entropy and compressibility, while other expansions produce negligible or even simplifying effects. We categorize the observed transitions into categories based on their entropy and growth patterns and demonstrate that complexity does not scale linearly with non-locality. This paper provides a formal foundation for understanding structural complexity in mobile automata and contributes to the broader theory of emergent computation in simple rule-based systems.

Keywords: mobile automata; non-local rules; phase transitions; Shannon block entropy; Kolmogorov–Chaitin complexity; active cell growth; emergent behavior; cellular automata

1. Introduction

Mobile automata are a subset of one-dimensional cellular automata in which a single active cell moves through a linear array of binary-state cells, updating the configuration based on a specified rule. Originally explored in [1], mobile automata exemplify how simple, local mechanisms can generate surprisingly rich and complex behavior. According to Wolfram's Principle of Computational Equivalence,

https://doi.org/10.25088/ComplexSystems.35.1.63

even these minimal systems are capable of exhibiting computational behaviors comparable to those of universal Turing machines [2].

The broader study of cellular automata has long provided a foundation for exploring emergent computation. Cook's landmark 2004 proof of the universality of rule 110 [3] confirmed that deterministic, rule-based systems can simulate any algorithmic process. In [4], it was demonstrated how cellular automata serve as idealized laboratories for investigating self-organization and complexity in both biological and physical systems. These systems have been shown to evolve through a wide spectrum of behaviors (from fixed points to periodicity to chaos), depending on the structure of their update rules.

While most prior research has focused on local interactions, this paper turns to a less-explored but highly consequential domain: nonlocal rules in mobile automata. In these systems, the active cell is allowed to base its update on cells at arbitrary distances, expanding the computational landscape. Non-locality has been previously examined in contexts such as Li's work on non-local cellular automata [5], where long-range interactions were shown to dramatically affect emergent patterns. Similarly, the role of non-locality in complex systems has been studied more generally in the context of global dependency and distributed behavior [6].

In this paper, we explore the emergence of phase transitions in complexity as the dependent cell range is varied in mobile automata with non-local rules. By employing Shannon block entropy and Kolmogorov–Chaitin complexity estimates as quantitative metrics, we analyze how increasing the number, range and symmetry of dependent cells alters the automaton's behavior. This builds upon earlier studies in self-organized criticality and complexity measures [7], while connecting to computational modeling traditions in domains like systems biology and fluid mechanics [8].

By formalizing and organizing phase transitions in this space, we aim to contribute a novel framework for understanding how computational complexity arises and shifts in mobile automata.

2. Construction of Non-local Rules

In traditional mobile automata, the state of an active cell at time t is determined by its own state and the states of its immediate neighbors (dependent cells) at time $t-1$. Non-local rules introduce an extended range of interaction, where the active cell can be influenced by cells located several steps away by increasing the range of dependent cells.

For example, consider an automaton where the active cell at position n is influenced by cells at positions $n-4$, $n-2$, $n+2$, $n+4$ (see Figure 1). This type of rule defines a broader neighborhood for decision-making, allowing more complex dynamics to emerge.

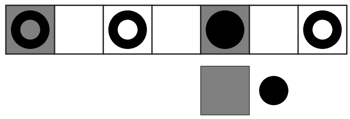


Figure 1. A non-local rule plot showing a mobile automaton with dependent cells four cells to the left, two cells to the left, the active cell itself and two cells to the right (relative to the active cell). In this case, the state of the active cell stays as black while it moves one cell to the right for the next timestep.

Mathematically, non-local rules can be formalized as a tuple:

$$\text{Rule} = \{s, d, \{p_1, p_2, \ldots, p_k\}\} \tag{1}$$

- s is the state of the active cell,
- d determines the direction in which the active cell moves, and
- p_i are the positions relative to the active cell that influence its next state.

The rule set is constructed as follows:

- The state s of the active cell is updated based on the states of cells located at p_i, which may be positive or negative offsets from the active cell.
- The movement d of the active cell is determined by the result of the rule application, which depends on the configuration of cells at these non-local positions.

The non-local mobile automaton model exhibits intriguing behaviors as we vary the range of dependent cells (Figure 2). By adjusting

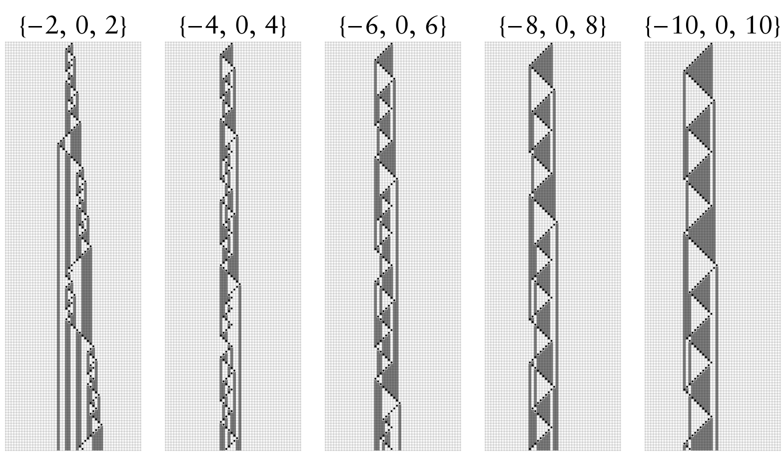


Figure 2. The behavior of a mobile automaton with the same initial conditions using different dependent cell ranges. As the range gets increased with consistent rules, the oscillation of the active cell gets into larger and larger intervals.

the range of non-local interactions, the automaton's dynamics become more complex, often generating patterns that extend across a larger spatial domain. These behaviors highlight the significance of the range parameter in influencing how the automaton evolves over time. In some contexts, the system's oscillations tend to grow, and the spatial patterns formed by the active cells change, reflecting the underlying influence of non-local dependencies on cellular interactions.

3. Complexity Measures for Phase Transitions

3.1 Active Cell Growth

Active cell growth in mobile automata represents the number of cells that transition from an inactive (white) state to an active (black) state over time. In our study of mobile automata with non-local rules, we focus on how the number of active cells increases as the automaton evolves. This is particularly relevant when analyzing automata with multiple active cells, where the growth rate of active cells can highlight the impact of non-local dependencies on the automaton's complexity (Figure 3).

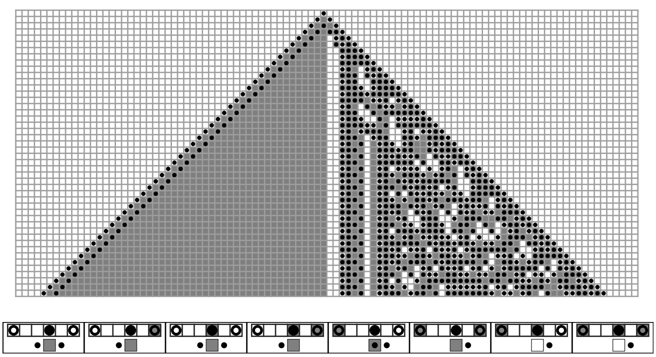


Figure 3. A non-local mobile automaton with multiple active cells that shows considerable complexity. Its dependent cells are three cells to the left of the active cell, the active cell itself and two cells to the right of the active cell.

When there is only one active cell, the growth of active cells tends to be more predictable, depending on the range and configuration of the non-local rule set (Figure 4). However, when multiple active cells are introduced, the behavior becomes more complex and less deterministic, with new patterns emerging from the interactions between the cells.

To plot and determine the number of active cells at each step, we can scan each step of the automaton for any cells that contain the black dot (which indicates an active cell).

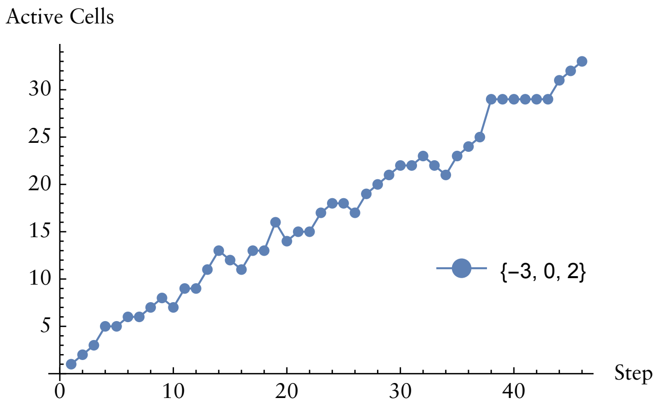


Figure 4. The active cell chart for the mobile automaton in Figure 3. The seemingly random distribution of active cells in the mobile automaton plot is visually quantifiable in the seemingly random dips and peaks of the graph.

To continue the discussion on active cell growth, an essential element of complexity in mobile automata is the pattern created by the interaction between active and dependent cells. These patterns usually become more intricate as the number of dependent cells increases or the range of interaction between them broadens. This complexity manifests visually in a widening array of active cells as rules evolve, giving rise to more dynamic behavior over time.

By incrementally adjusting the parameters for non-local rules, we can observe the impact on the automaton's evolution. The visualizations shown in Figure 5 depict the automaton's behavior under different dependent cell ranges, providing insights into the critical point (known as the threshold of complexity) where simple behavior gives way to complex, nonlinear patterns.

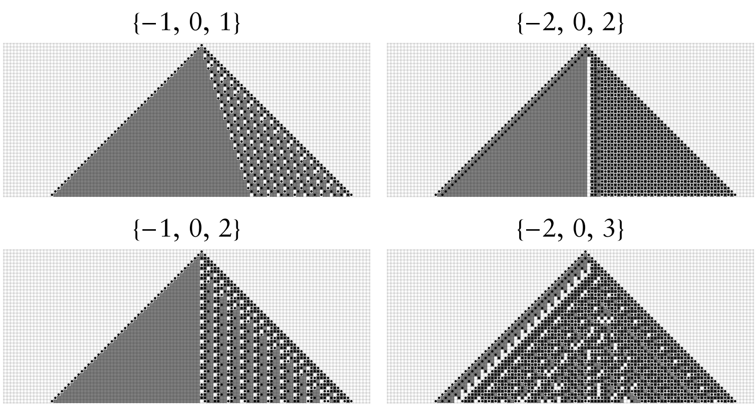


Figure 5. (*continues*).

https://doi.org/10.25088/ComplexSystems.35.1.63

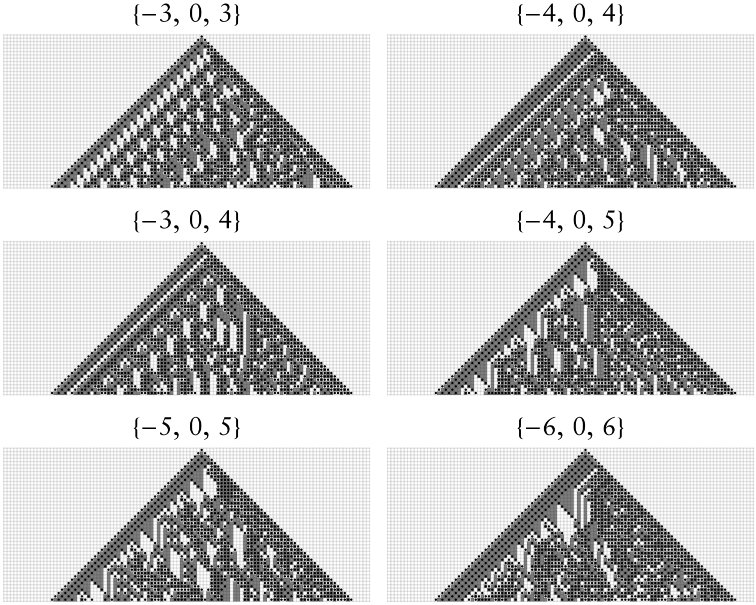


Figure 5. The evolution of a non-local mobile automaton with multiple active cells for different dependent cell ranges. The behavior suddenly switches from simple and repetitive to complex and dynamic at the point when the dependent cell range is {−2, 0, 3}, indicating a threshold of complexity at that point.

Through plotting the active cell charts for all of the mobile automata, we can visualize the threshold for complexity in a graphical format as shown in Figure 6.

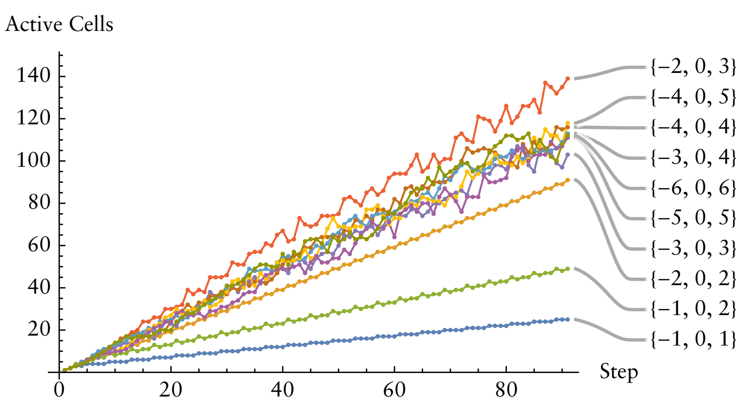


Figure 6. The active cell chart for Figure 5. We can see the threshold of complexity as the regular lines changing into seemingly random peaks and dips as soon as the dependent cell range gets past a certain point.

As we examine the behavior of mobile automata with non-local rules, distinct patterns emerge depending on the configuration of dependent cell ranges. In some cases, when the dependent cells are symmetric relative to the active cell, such as in ranges like {−2, 0, 2}, we often observe complex, non-repetitive behavior (Figure 7). The symmetry leads to a chaotic growth pattern where the automaton explores a variety of configurations without settling into a repeating cycle. This complexity is a key feature of mobile automata when the rule structure remains balanced on both sides of the active cell.

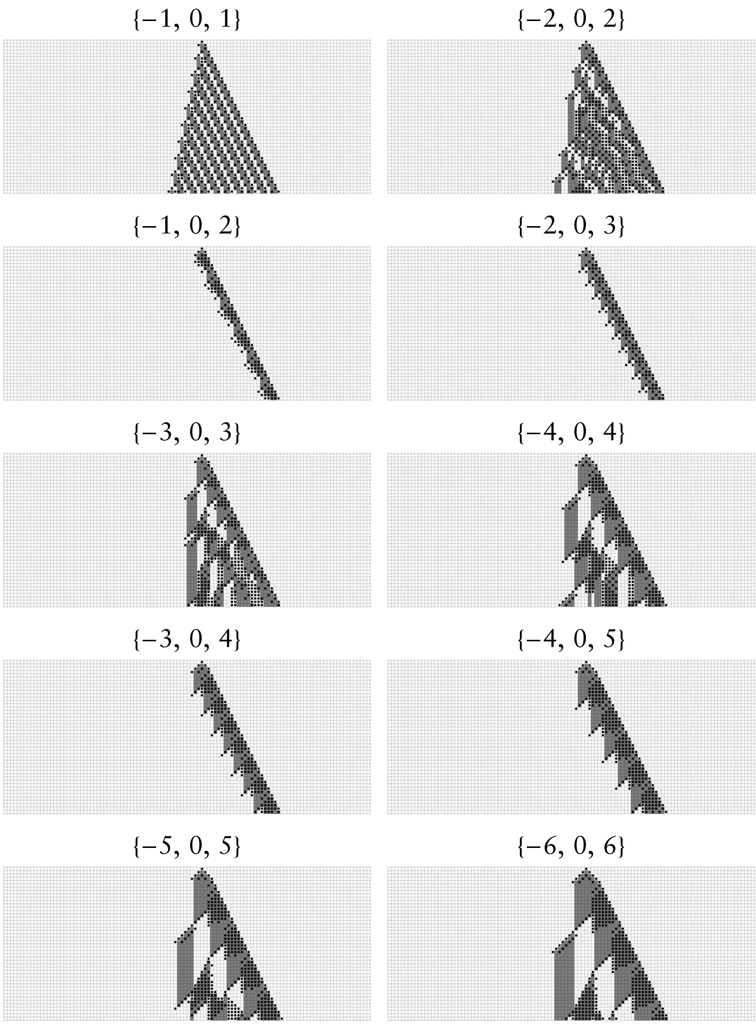


Figure 7. A specific initial condition for a mobile automaton where symmetric dependent cell ranges {−2, 0, 2} lead to complex behavior, while asymmetric ranges {−2, 0, 3} result in simpler, repetitive patterns.

In contrast, when the dependent cell ranges are asymmetric, for instance, {−2, 0, 3}, the automaton tends to exhibit a simpler, repetitive behavior. Here, the lack of symmetry reduces the complexity of interactions between cells, often leading to regular and predictable patterns (Figure 8). The shift from chaotic to ordered behavior is significant in understanding how the placement and selection of dependent cells drive the overall system's behavior.

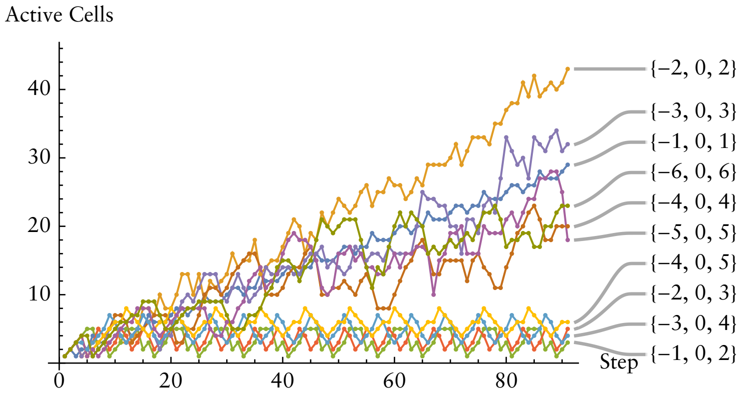


Figure 8. The active cell graph for Figure 7. The automata with asymmetric dependent cell ranges show very periodic and predictable behavior in terms of active cell growth, while the symmetric dependent cell ranges exhibit more complex, seemingly random behavior with active cell growth.

However, as shown in Figure 9, increasing the dependent cell range does not necessarily lead to more complex behavior. In fact, small adjustments to the range (such as increasing it by a single step on one side) can entirely simplify the behavior, resulting in simple, linear growth rather than chaotic expansion. This counterintuitive result highlights the sensitivity of mobile automata to minor changes in their rule structures, where increasing the range by even one unit leads to complete behavioral simplification.

While we might intuitively expect that increasing the dependent cell range introduces greater complexity, Figure 9 demonstrates a case where the opposite occurs. The initial configuration with a small, local rule set (−1, 0, 1) generates dynamic and seemingly chaotic behavior, whereas even a slight increase in the dependent cell range results in a rapid suppression of complexity. This trend is further quantified in Figure 10, where the active cell chart shows that only the (−1, 0, 1) case maintains any degree of variability, while all larger dependent cell ranges exhibit a trivial and nearly constant number of active cells. This suggests that non-locality can, in some cases, enforce strict determinism and limit the unpredictability of the automaton's evolution.

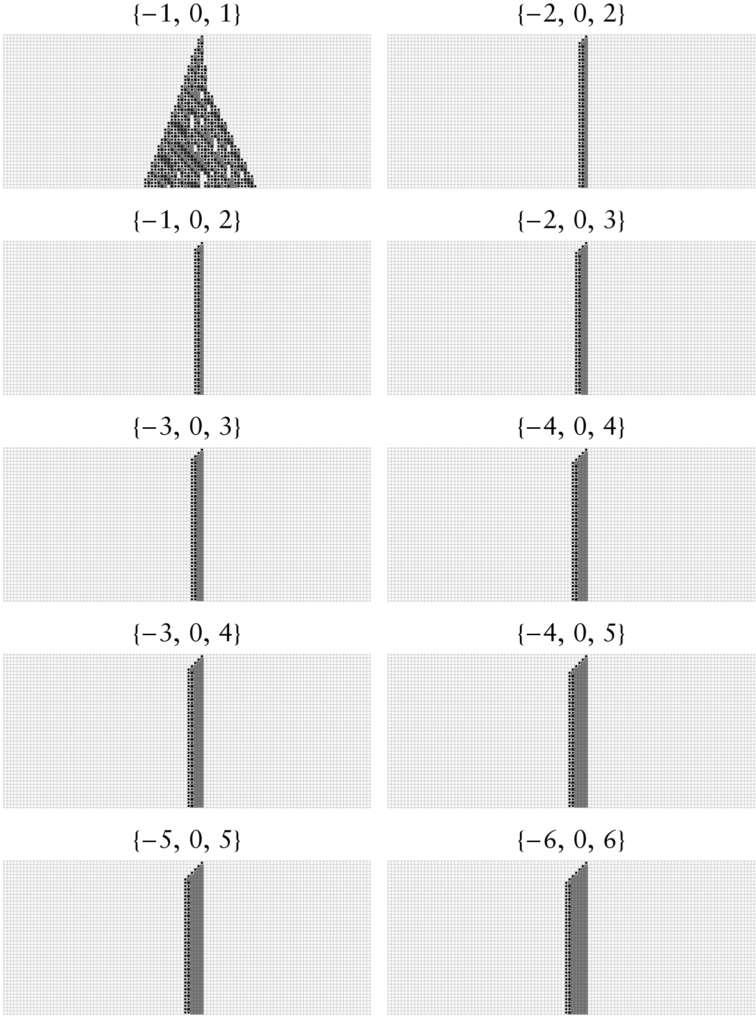


Figure 9. An example in which increasing the dependent cell range does not always increase complexity. In these examples, extending the range by just one unit simplifies the initially complex behavior of the automaton.

So far, we have looked at non-local mobile automata with three dependent cells, but non-local rules can expand to any n number of dependent cells, governed by 2^n configuration rules. Figure 11 presents a mobile automaton with a larger dependent cell range of $(-3, -1, 0, 2)$, and a visualization of all 2^4 possible rules governing its behavior is shown in Figure 12. This rule space provides insight into how different update functions interact with expanded non-local dependencies.

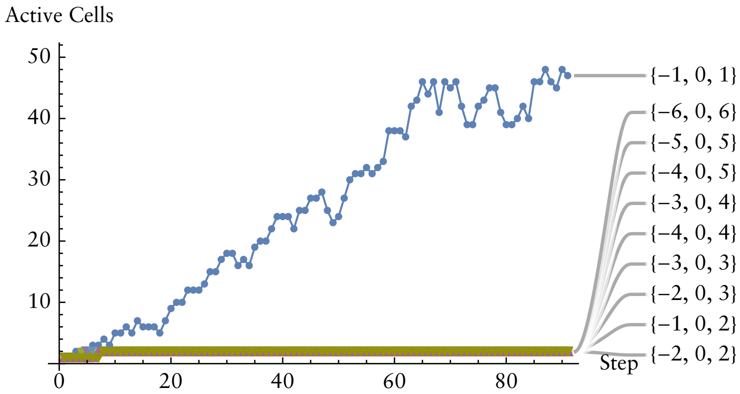


Figure 10. The active cell chart for Figure 9. Only the first dependent cell range, which uses only local rules, shows any kind of complexity with active cell growth. However, increasing the dependent cell range immediately simplifies and reduces the chaos of the active cell growth.

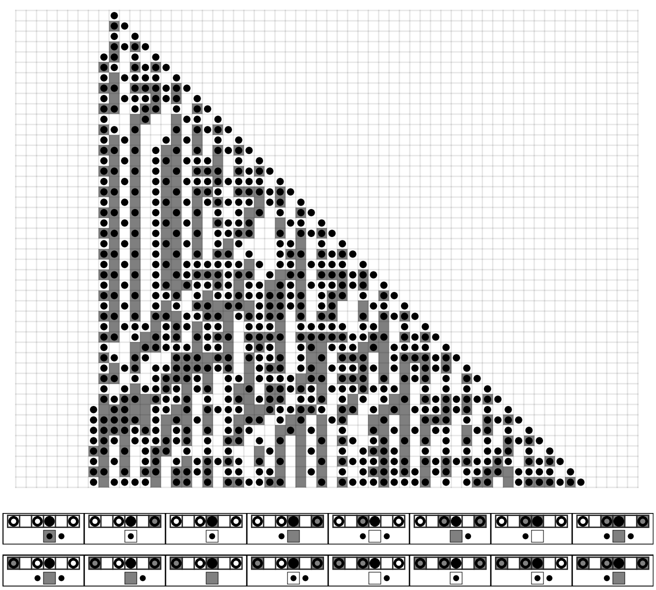


Figure 11. The mobile automaton evolution for a rule with four dependent cells (−3, −1, 0, 2), alongside a visualization of all possible 2^4 update rules. The rule plot provides a comprehensive view of the automaton's decision space, revealing patterns that emerge as dependency structures become more distributed.

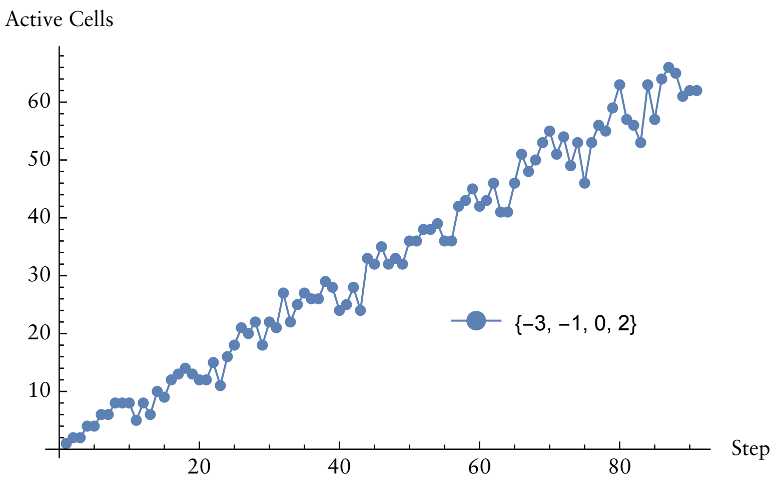


Figure 12. The active cell chart for the mobile automaton in Figure 11, with four dependent cells (−3, −1, 0, 2). Unlike automata with simpler dependent cell ranges, this configuration exhibits more fluctuating growth in the number of active cells, suggesting a phase transition toward more complex behavior.

The active cell charts, as a metric of complexity, illustrate a clear distinction between complex and simple behaviors based on the structure of dependent cell ranges. Local rules, such as (−1, 0, 1), generate irregular, fluctuating growth in the number of active cells, while increasing the range of dependencies often leads to stabilization and periodicity, or even complete suppression of activity. However, while active cell growth provides insight into macroscopic trends, it does not fully capture the underlying information content of the system.

To further quantify the complexity of mobile automata with nonlocal rules, we turn to Shannon block entropy, which measures the unpredictability and diversity of local configurations within the automaton's evolution. Unlike active cell growth, which tracks only the number of active sites, Shannon entropy allows us to assess the richness of patterns within the automaton's state space. In the next section, we examine how entropy changes as a function of the dependent cell range and investigate whether it aligns with the complexity thresholds observed in the active cell analysis.

3.2 Shannon Block Entropy

While the previous section analyzed complexity through the growth of active cells, this alone does not fully capture the information-theoretic complexity of the system. A system with a high number of active cells could still exhibit highly regular behavior, whereas a system with few active cells could generate intricate, unpredictable patterns. To provide a more detailed quantification of complexity, we analyze Shannon block entropy, a metric that measures the diversity and unpredictability of local configurations within the automaton's evolution.

Shannon entropy for block structures of length m is computed as:

$$S_m = -\sum_i P(i) \log_2 P(i) \tag{2}$$

where $P(i)$ represents the probability of encountering a specific block of size m in the automaton's evolution [9]. By varying m, we can examine how the automaton's complexity scales across different pattern lengths.

We apply this measure to mobile automata with different dependent cell ranges to investigate whether the previously observed thresholds of complexity in active cell growth correspond to structural complexity within the automaton's configuration space. The following results demonstrate how Shannon block entropy evolves as the dependent cell range increases.

Figure 13 presents the Shannon entropy values for increasing dependent cell ranges for the rule in Figure 5. This analysis helps determine whether increasing non-locality consistently leads to greater complexity or whether certain transitions in dependent cell ranges mark a threshold between simple and complex behavior. Here, the threshold of complexity is clearer, as the Shannon block entropy suddenly jumps when the dependent cell range hits (−2, 0, 3).

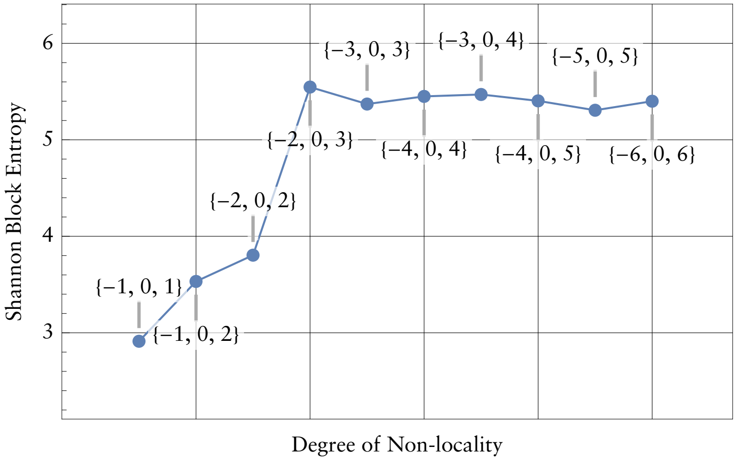


Figure 13. Shannon block entropy for the mobile automaton shown in Figure 5 with varying dependent cell ranges. The x axis represents the degree of non-locality (the range of dependent cells), while the y axis represents Shannon block entropy. After the sudden jump in complexity, the entropy of the mobile automaton levels off and stabilizes.

Figure 13 established how Shannon block entropy evolves as the degree of non-locality increases. We observed distinct shifts in entropy, suggesting possible phase transitions where complexity either rises sharply or stabilizes. However, one rule alone does not fully capture the structural differences in automaton behavior. In Figure 14, which depicts the Shannon entropy for the rule in Figure 7, the figure shows a different picture that matches up with our observations on the active cell charts. Mainly, in some rules, the complexity of a mobile automaton depends on whether the dependent cell range is symmetric or asymmetric.

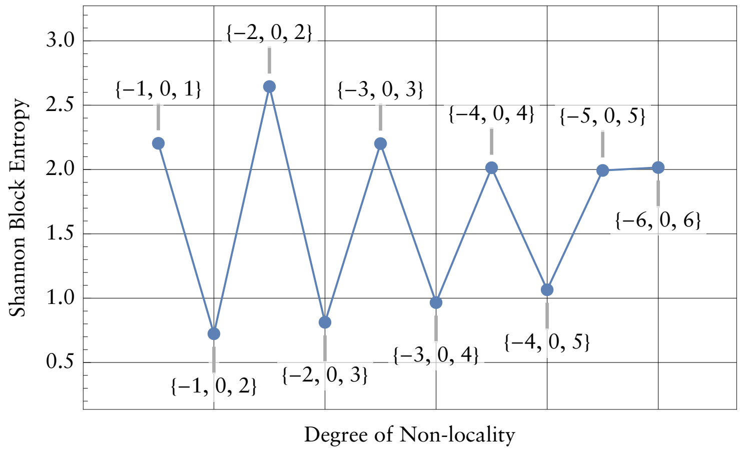


Figure 14. Shannon block entropy for the mobile automaton shown in Figure 7 with varying dependent cell ranges. The entropy alternates between complexity stages as the dependent cell ranges switch between symmetric and asymmetric.

The oscillatory pattern in Figure 12 illustrates how the alternation between symmetric and asymmetric dependent cell ranges leads to distinct shifts in Shannon entropy. This suggests that symmetry plays a fundamental role in determining whether a mobile automaton exhibits structured complexity or repetitive behavior. To further investigate this phenomenon, Figure 13 presents another rule configuration where increasing the degree of non-locality results in a rapid collapse of entropy. Unlike the previous case, where complexity alternated in a structured way, this rule demonstrates an immediate transition to minimal entropy, highlighting that certain rule structures inherently resist complexity growth beyond a specific threshold. This contrast reinforces the idea that different mobile automata exhibit qualitatively different phase transitions, depending on how their rules interact with increasing non-locality.

To contrast the behavior in Figure 13, Figure 14 presents a case where the entropy remains virtually unchanged regardless of the degree of non-locality. This highlights a different kind of phase transition, one where increasing the dependent cell range has no significant effect on the system's complexity. The automaton's behavior is invariant under changes in non-locality, implying that certain rules are inherently resistant to shifts in complexity.

The analysis now shifts from examining the Shannon entropy of mobile automata with three dependent cells (Figures 13 through 16) to those with four dependent cells (Figure 17). In the case of three-dependent-cell automata, entropy remains largely unchanged across different dependent cell ranges, indicating a stable complexity level. However, for four-dependent-cell automata, we observe a more dynamic entropy distribution. Notably, the entropy values for certain configurations with four dependent cells are not necessarily higher than those for three-dependent-cell automata. This suggests that increasing the number of dependent cells does not always correlate with increased complexity. Rather, the structure introduced by additional dependencies can sometimes reduce entropy by enforcing more rigid, less chaotic behavior.

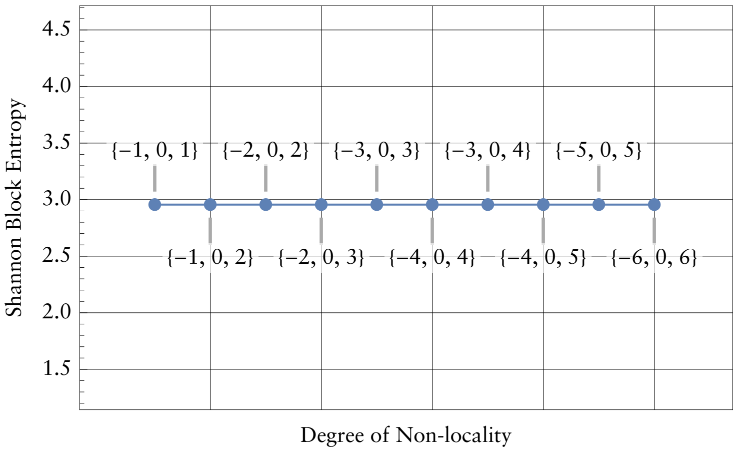


Figure 15. Shannon block entropy for a mobile automaton with varying dependent cell ranges. The dependent cell range does not significantly affect the automaton's entropy.

While Shannon block entropy provides insights into the statistical complexity of mobile automata by measuring the diversity of state transitions over time, it does not fully capture the underlying algorithmic complexity of the system. Specifically, Shannon entropy is limited to quantifying unpredictability based on probability distributions,

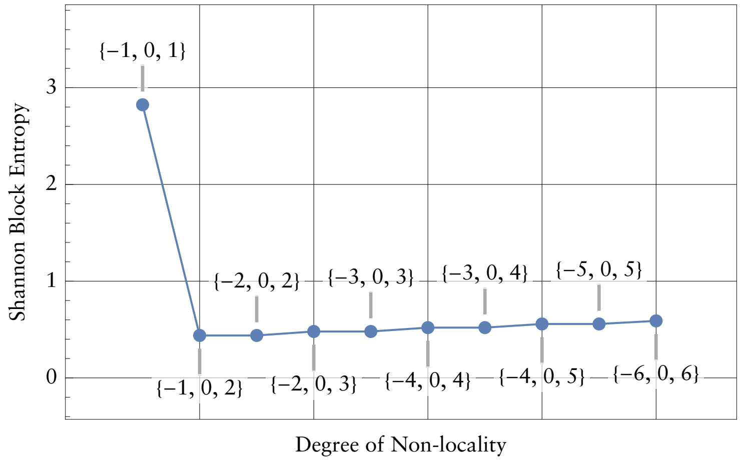


Figure 16. Shannon block entropy for the mobile automaton shown in Figure 9 with varying dependent cell ranges. The entropy immediately plunges as soon as the dependent cell range is non-local (i.e., > (−1, 0, 1)).

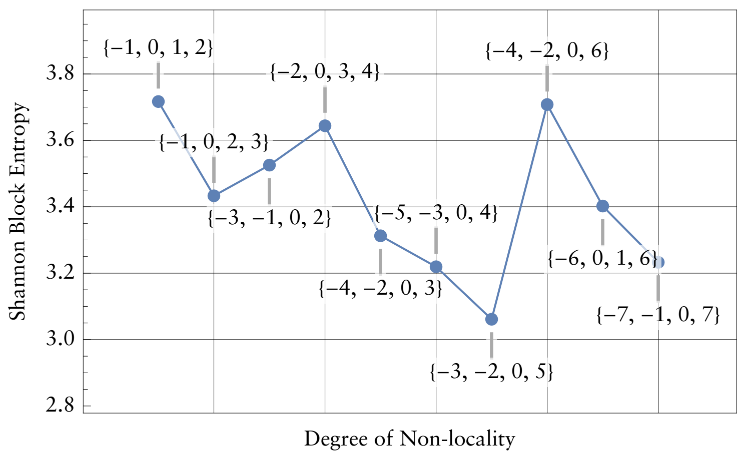


Figure 17. Shannon block entropy for the mobile automata shown in Figure 11 with varying dependent cell ranges. The entropy does not exhibit a strictly increasing trend as the number of dependent cells increases, highlighting that the addition of more dependent cells does not necessarily lead to greater complexity.

whereas a metric like Kolmogorov–Chaitin complexity offers a different perspective by assessing the minimal description length required to encode the automaton's evolution. To further formalize our observations, we now transition to analyzing the Kolmogorov–Chaitin complexity of these mobile automata. This will allow us to compare how well Shannon entropy aligns with a measure that considers the computational irreducibility of the automaton's behavior, helping to identify deeper structural phase transitions.

3.3 Kolmogorov–Chaitin Complexity Estimation Using the Block Decomposition Method

The Kolmogorov–Chaitin complexity $K(x)$ of a finite string x is defined as the length of the shortest program that outputs x on a fixed universal Turing machine. In essence, it measures the minimal description length required to fully reproduce a string and thus reflects the string's degree of compressibility or structure [10].

Unlike Shannon entropy, Kolmogorov–Chaitin complexity is incomputable; there exists no general algorithm capable of determining $K(x)$ for arbitrary strings. This incomputability arises from its relationship with the halting problem and the unbounded nature of general program search, which has motivated several modern approximation frameworks, including those described in [11]. In this paper, we adopt two approximations of $K(x)$ for longer sequences: the block decomposition method (BDM) and lossless compression, following the framework summarized in [12].

The BDM provides a computable approximation of Kolmogorov–Chaitin complexity by decomposing a structure into smaller subblocks whose algorithmic complexities can be estimated from empirical data. Following the work in [12], the method combines algorithmic probability and Shannon-like aggregation. Each block r_i of size $d \times d$ is assigned a complexity value derived from the coding theorem method (CTM),

$$K_m(r_i) = -\log_2 m(r_i) \quad \text{where } m(r_i) = \sum_{p:U(p)==r_i} 2^{-|p|} \tag{3}$$

and $m(r_i)$ represents the algorithmic probability that a universal Turing machine U outputs r_i. The overall BDM complexity of an array or string X is then computed as the sum of the CTM-derived values of its non-overlapping blocks, with a logarithmic term penalizing repeated patterns:

$$\text{BDM}(X) = \sum_i [K_m(r_i) + \log_2 n_i] \tag{4}$$

where n_i is the number of occurrences of block r_i [12]. This approach merges local algorithmic information (captured by CTM) with global repetition structure (captured by the frequency term), allowing BDM to approximate both fine-grained randomness and large-scale regularities.

In the context of mobile automata, we find that estimated Kolmogorov–Chaitin complexity follows a trajectory similar to that of Shannon block entropy. Both metrics capture changes in structural regularity and emergent behavior as we vary the dependent cell range of the automaton. In Figure 18, we present a consolidated view of the Kolmogorov–Chaitin complexity results using BDM, which closely parallel the established entropy trends.

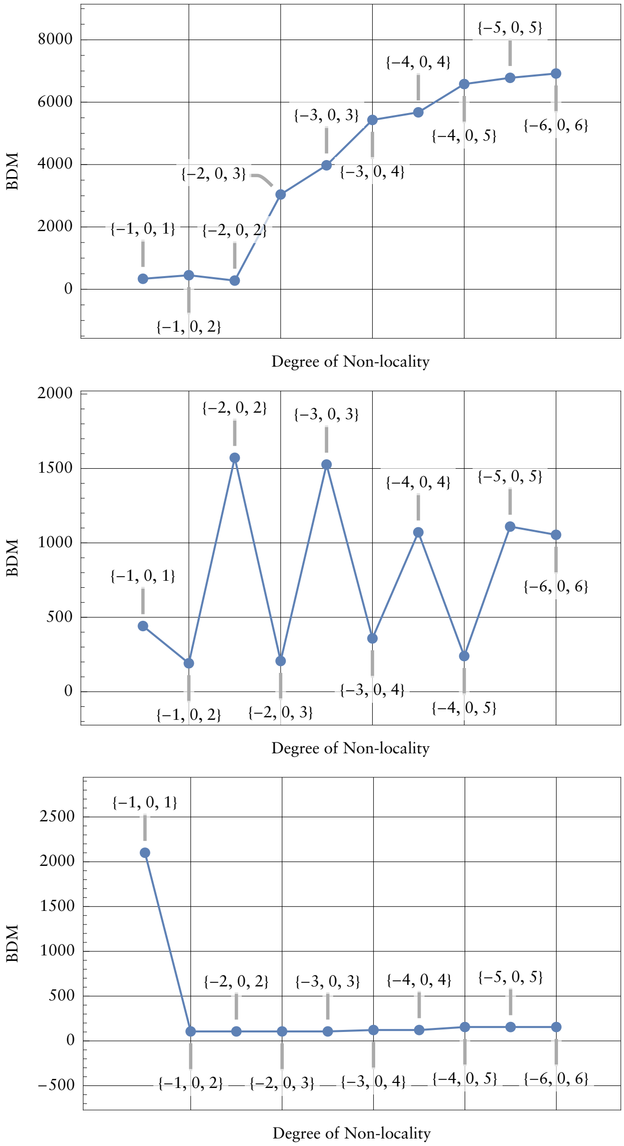


Figure 18. (*continues*).

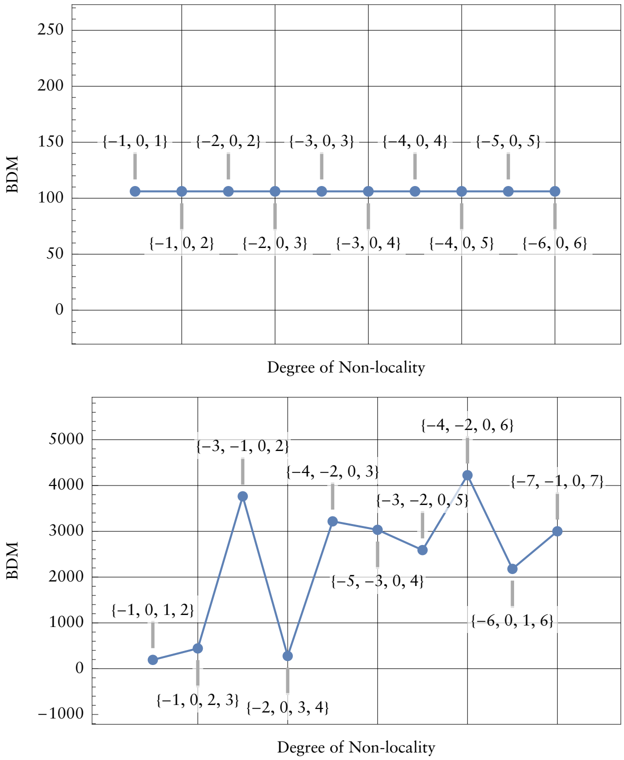


Figure 18. Kolmogorov–Chaitin complexity using BDM over increasing dependent cell ranges for each of the mobile automaton rules discussed in Sections 3.1 and 3.2.

In Figure 18, with each plot corresponding to a different rule, the overall patterns closely mirror those observed in the Shannon block entropy plots for the same rule, highlighting similar phase transitions, complexity jumps and saturation behavior. These estimates, derived from BDM, reinforce the consistency between algorithmic and statistical complexity measures in capturing structural change. This specific finding echoes a result in Lindgren and Nordahl [13], where Kolmogorov–Chaitin complexity was found to closely relate to Shannon entropy for long sequences generated by stationary stochastic processes, although in general, the present paper relies on the modern review and BDM-based framework in [12].

However, when comparing Shannon entropy and the Kolmogorov–Chaitin complexity estimate derived via BDM, an interesting difference first appears in the case of the first mobile automaton rule explored (top-left of Figure 18). While the Shannon entropy plot (Figure 13) shows a sharp increase as the dependent cell range expands (indicating a pronounced rise in local unpredictability and a consequent threshold of complexity), the BDM-based Kolmogorov–Chaitin estimate grows more gradually. This suggests that while local symbol distributions become less regular, the overall generative structure of the spacetime evolution remains compressible under BDM, implying that much of the apparent randomness may still arise from repetitive algorithmic processes.

Additionally, when comparing the Shannon block entropy and Kolmogorov–Chaitin complexity plot for the mobile automaton with four dependent cells, a noticeable divergence emerges between the two complexity metrics. In prior examples, both entropy and complexity tended to rise and fall together, often exhibiting the same abrupt transitions at particular dependent cell ranges. However, for this four-cell rule, that pattern breaks down. For instance, while the Shannon entropy graph shows a sharp dip at the dependent range {−3, −2, 0, 5}, indicating reduced unpredictability, the Kolmogorov–Chaitin complexity at that same point remains relatively high. This discrepancy suggests that although entropy is capturing the unpredictability in local configurations, the broader compressibility of the spacetime pattern (the aspect captured by Kolmogorov–Chaitin complexity) might be shaped by different structural features not evident in local entropy alone. This divergence between Shannon entropy and Kolmogorov complexity estimates also aligns with findings in [12], where it is emphasized that while entropy captures statistical regularity, compression-based Kolmogorov–Chaitin estimates capture deeper structural algorithmic randomness [12].

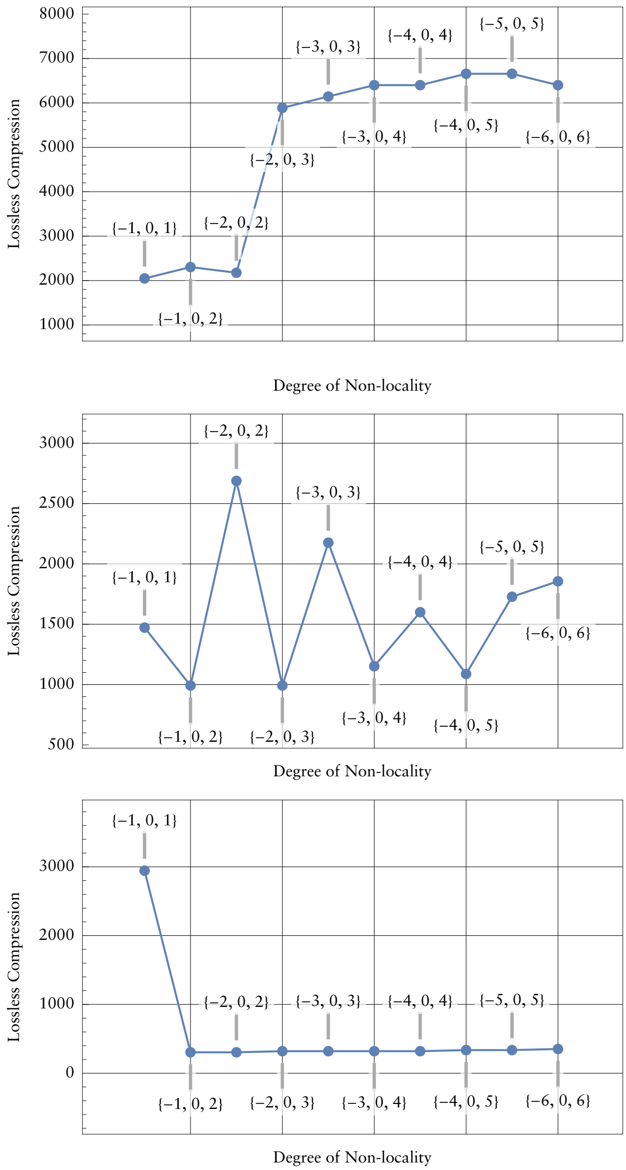
3.4 Kolmogorov–Chaitin Complexity Estimation Using Lossless Compression

While BDM provides a theoretically grounded, generative approach to approximating Kolmogorov–Chaitin complexity, it is computationally intensive and limited by the availability of precomputed algorithmic probability tables. As a supporting metric to complement this analysis, we also estimate Kolmogorov–Chaitin complexity using a more practical heuristic: lossless compression. This secondary approach does not rely on precomputed priors but instead measures compressibility directly, offering a comparative perspective on structural regularity across the same range of mobile automaton configurations.

In lossless compression algorithms, the size of the compressed string serves as an upper bound:

$$K(x) \lessapprox |C(x)|. \tag{5}$$

Here, $C(x)$ denotes a lossless compression of x (e.g., via LZ77 or DEFLATE), and $|C(x)|$ is its length. The shorter the compressed representation, the lower the estimated complexity. In this work, we adopt this compression-based approach to estimate Kolmogorov–Chaitin complexity using the function LosslessCompressionEstimate, which applies practical compression heuristics to the evolution output of mobile automata. In Figure 19, we present a consolidated view of the Kolmogorov–Chaitin complexity results using lossless compression.



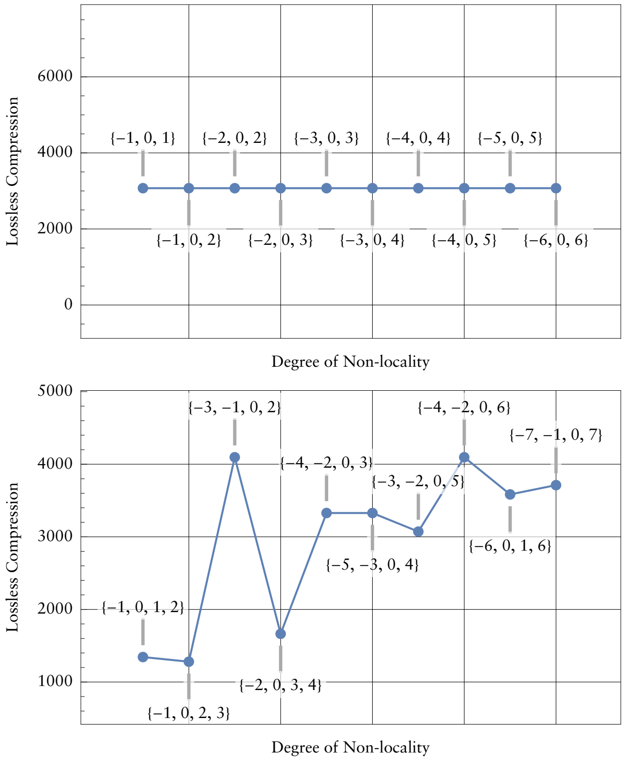


Figure 19. Kolmogorov–Chaitin complexity using lossless compression over increasing dependent cell ranges for each of the mobile automaton rules discussed in Sections 3.1 and 3.2.

When estimated using lossless compression (Figure 19), the Kolmogorov–Chaitin complexity trends closely parallel those observed through both Shannon block entropy and the BDM approximation. The overall shape of the curves remains consistent across all rules, suggesting that the underlying structural dynamics of the mobile automata are robust across estimation methods. However, unlike BDM, which (as noted earlier) exhibits a smoother growth in the first rule, the lossless compression metric shows a noticeably larger jump in complexity with increasing dependent cell range. This behavior aligns more closely with the entropy-based measure, reflecting the sensitivity of compression-based estimators to local symbol variability. Beyond this deviation, the complexity trajectories in Figure 19 mirror those of the BDM estimate, reinforcing the consistency of both approaches and providing an additional check on the BDM-based analysis.

4. Identifying and Analyzing Phase Transitions

A central goal of this paper is to uncover how the complexity of mobile automata changes as we increase the degree of non-locality, that is, as we expand the range of dependent cells in the update rules. In this section, we formalize the concept of phase transitions in mobile automata and systematically analyze them using both Shannon block entropy and Kolmogorov complexity metrics across multiple automaton rules.

4.1 Defining Phase Transitions in Mobile Automata

In statistical physics, a phase transition refers to an abrupt change in the macroscopic behavior of a system due to a small change in a control parameter. Analogously, we define a computational phase transition in mobile automata as a sudden or qualitative shift in the complexity of the automaton's evolution caused by a small change in the dependent cell range.

Let $R = \{r_1, r_2, \ldots, r_n\}$ be a sequence of dependent cell ranges ordered by increasing non-locality, and let $C(r_i)$ represent a chosen complexity metric (e.g., Shannon block entropy or Kolmogorov complexity) evaluated for an automaton with range r_i. A phase transition is identified by a non-smooth jump in $C(r)$, such as a sharp increase or drop that is not explained by noise or local fluctuations.

4.2 Evidence of Phase Transitions across Automata Rules

Figures 13 through 19 illustrate complexity measures for five distinct mobile automaton rules across sequences of dependent cell ranges. The metrics offer strong empirical evidence of phase transitions in the behavior of these automata.

In Figures 13 and 19 (top-left), we observe a clear phase transition occurring between the dependent cell ranges $\{-2, 0, 2\}$ and $\{-2, 0, 3\}$. Both the Shannon block entropy and the Kolmogorov complexity jump significantly, marking a transition from relatively low to high complexity. After this jump, the entropy and complexity stabilize, forming a plateau, indicative of a new computational phase. This stabilization implies that increasing the non-locality further does not meaningfully change the qualitative behavior of the automaton. This transition mirrors a common pattern in computational systems: a simple behavior regime (low entropy and low complexity), a rapid transition region (the "edge of chaos" or "threshold of complexity") and a complex regime that saturates at a high entropy ceiling. We can also see this concept in the active cell growth chart in Figure 6.

Figures 14 and 18 (top-right) reveal a periodic alternation in entropy as the dependent cell range toggles between symmetric and asymmetric structures. This behavior is a cyclic phase shift, where complexity does not increase or decrease monotonically with non-locality but rather depends on the spatial symmetry of the rule. For example, ranges like {−2, 0, 2} yield higher entropy than asymmetric ones like {−2, 0, 3}, suggesting that symmetry acts as a structural parameter influencing complexity phases. Interestingly, the Kolmogorov complexity (Figure 18) for this rule mimics the entropy's jagged profile, further validating the presence of alternating computational phases, though with some variation in the amplitude of changes.

In Figures 16 and 18 (middle-left), we find a striking inverse phase transition. The automaton starts with high entropy and complexity at the local rule {−1, 0, 1}, but as soon as even minor non-locality is introduced (e.g., {−1, 0, 2}), the entropy plummets and remains low across all more extended ranges. This suggests that for certain rules, adding non-local information does not enrich the behavior but rather simplifies it, possibly by overpowering local dynamics with globally stabilizing influences.

Figures 15 and 18 (middle-right) show an automaton whose entropy and Kolmogorov complexity are invariant across increasing dependent cell ranges. There are no phase transitions here; the automaton operates in a single phase regardless of locality. This result is crucial, as it demonstrates that not all automata respond to non-locality in the same way: some are structurally robust, with complexity unaffected by range modifications.

In Figures 17 and 18 (bottom-left), we study a four-dependent-cell automaton and find non-monotonic behavior in both entropy and complexity. Notably, the entropy does not exceed that of some automata with only three dependent cells (see Figure 13), suggesting that more dependency does not always yield more complexity. Here, phase transitions appear intermittently, and the relationship between rule structure and phase change is more nuanced. This provides a broader perspective: transitions depend not only on range size but also on the specific configuration of dependency.

4.3 Categories of Phase Transition Behavior

Based on the empirical behaviors observed across the five mobile automaton rules, I introduce a set of phase-transition categories that describe how complexity changes as non-locality increases:

- **Type I: Complexity-Saturating Transitions**
  A sudden increase in complexity followed by a plateau (e.g., Figure 13).
- **Type II: Oscillating Transitions**
  Complexity alternates due to symmetric versus asymmetric range structure (e.g., Figure 14).
- **Type III: Complexity-Suppressing Transitions**
  An abrupt drop in complexity as non-locality increases (e.g., Figure 16).

- **Type IV: Invariant Behavior**
  Complexity remains constant across all dependent cell ranges (e.g., Figure 15).
- **Type V: Non-Monotonic Multi-phase Dynamics**
  Complexity fluctuates in an irregular pattern across ranges (e.g., Figure 17).

These categories provide a vocabulary for describing how non-local information affects the computational phases of mobile automata, complementing but not replacing broader, canonical frameworks such as Wolfram's four classes [1]. Rather than offer a universal taxonomy, these categories capture fine-grained, range-dependent transitions specific to mobile automata with non-local update rules. In other words, they describe how complexity changes with non-locality within each rule and thus characterize transition patterns rather than global rule behavior. More generally, these categories reveal that non-locality is not a unidimensional control knob. It can yield a variety of effects, depending on how it is introduced.

5. Conclusion

This paper has revealed that mobile automata with non-local rules exhibit a rich spectrum of computational behavior as the dependent cell range increases. By analyzing both Shannon block entropy and Kolmogorov complexity across multiple rule configurations, we identified several distinct categories of computational phase transitions, where small changes in the rule's structure (especially in the arrangement and symmetry of dependent cells) led to large, qualitative shifts in complexity.

Crucially, our findings show that complexity does not necessarily increase with the size or non-locality of the rule. In some cases, extending the dependent cell range simplified the automaton's behavior, while in others, complexity alternated or plateaued. These results highlight the sensitive dependence of complexity on the structure and symmetry of non-local rules, challenging the assumption that more information necessarily results in more complex behavior.

For computational details, the interested reader is referred to [14]. By formalizing phase transitions in this context and organizing them into distinct behavioral regimes, this paper contributes a new framework for understanding complexity in mobile automata. While the focus here is on one-dimensional mobile automata for simplicity, the approach can be generalized to higher-dimensional systems. We anticipate that similar behaviors (such as the organization of phase transitions) will persist in higher dimensions, providing a solid foundation for future investigations. Future work may build on this foundation by incorporating additional complexity measures, exploring different initial conditions or expanding to higher-dimensional rule spaces. Nonetheless, the experimental evidence presented here strongly suggests that the space of non-local mobile automata is shaped by critical transition points: structural thresholds where complexity emerges, collapses or stabilizes in unexpected ways.

Acknowledgements

I would like to express my deepest gratitude to Dr. Danny Barash for his continuous guidance, insightful feedback and encouragement throughout the development and revision of this work. I would also like to thank Felipe Amorim, whose mentorship and thoughtful discussions were instrumental in initiating and shaping the early stages of this project. I am indebted to Dr. Stephen Wolfram, whose ideas on cellular automata and computational complexity inspired the central questions explored in this paper. I would also like to thank Rory Foulger and James Wiles for their support.

References

[1] S. Wolfram, *A New Kind of Science*, Champaign, IL: Wolfram Media, Inc. 2002.

[2] S. Wolfram, "Chapter 12: Principle of Computational Equivalence." Wolfram Science, 2002.
www.wolframscience.com/nks/
chap-12--the-principle-of-computational-equivalence.

[3] M. Cook, "Universality in Elementary Cellular Automata," *Complex Systems*, **15**(1), 2004 pp. 1–40. doi:10.25088/ComplexSystems.15.1.1.

[4] M. Mitchell, P. T. Hraber and J. P. Crutchfield, "Revisiting the Edge of Chaos: Evolving Cellular Automata to Perform Computations," *Complex Systems*, **7**(2), 1993 pp. 89–130.
complex-systems.com/pdf/07-2-1.pdf.

[5] W. Li, "Phenomenology of Nonlocal Cellular Automata," *Journal of Statistical Physics*, **68**(5–6), 1992 pp. 829–882.
doi:10.1007/BF01048877

[6] S. Wolfram, *Cellular Automata and Complexity: Collected Papers*, Addison-Wesley, 1994.

[7] P. Bak, C. Tang and K. Wiesenfeld, "Self-Organized Criticality: An Explanation of the 1/f Noise," *Physical Review Letters*, **59**(4), 1987 pp. 381–384. doi:10.1103/PhysRevLett.59.381.

[8] R. E. Goldstein, I. Tuval and J.-W. van de Meent, "Microfluidics of Cytoplasmic Streaming and Its Implications for Intracellular Transport," *Proceedings of the National Academy of Sciences*, **105**(10), 2008 pp. 3663–3667. doi:10.1073/pnas.0707223105.

[9] C. E. Shannon, "A Mathematical Theory of Communication," *The Bell System Technical Journal*, **27**(3), 1948 pp. 379–423. doi:10.1002/j.1538-7305.1948.tb01338.x.

[10] A. N. Kolmogorov, "Three Approaches to the Quantitative Definition of Information," *International Journal of Computer Mathematics*, **2**(1–4), 1968 pp. 157–168. doi:10.1080/00207166808803030.

[11] H. Zenil and J. Delahaye, "On the Kolmogorov–Chaitin Complexity for Short Sequences," *Randomness and Complexity, from Leibniz to Chaitin* (C. Calude, ed.), World Scientific, 2008 pp. 123–129.

[12] H. Zenil and E. Villarreal-Zapata, "Asymptotic Behaviour and Ratios of Complexity in Cellular Automata," *International Journal of Bifurcation and Chaos*, **23**(9), 2013 1350159. doi:10.1142/S0218127413501599.

[13] K. Lindgren and M. G. Nordahl, "Complexity Measures and Cellular Automata," *Complex Systems*, **2**(4), 1988 pp. 409–440. complex-systems.com/pdf/02-4-2.pdf.

[14] A. Devireddy. "Mobile Automata with Non-local Rules" from Wolfram Community—A Wolfram Web Resource. community.wolfram.com/groups/-/m/t/3214519.