Second-Order Three-Neighbor Number-Conserving Cellular Automata

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In this paper, the number-conserving rules for second-order three-neighbor binary cellular automata (SOCAs) are discussed. The SOCAs include the well-known elementary cellular automata and elementary reversible cellular automata and have a total of 2^{64} local rules. We show that the number of number-conserving SOCAs is limited to 74 through a polynomial representation of the local rules. A concrete description of the polynomial representation for 15 representative rules is provided. In addition, we establish a necessary and sufficient condition for each SOCA to be number conserving. This condition can be regarded as a discrete version of the flux form in fluid dynamics.

Keywords: second-order cellular automata; number conservation; polynomial representation; flux form

1. Introduction

A cellular automaton (CA) is a discrete dynamical system that models complex phenomena using simple local rules. Among various types of cellular automata (CAs), number-conserving CAs have attracted considerable attention for their ability to simulate systems in which quantities such as particles or energy are conserved. This property makes them particularly useful in modeling physical systems, traffic flow and granular materials. A fundamental property of the number-conserving CAs is that they follow a conservation law, that is, the sum of states of cells in the initial configuration remains constant across time. This

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property is essential in systems such as traffic flow models, where it satisfies the realistic condition that the total number of vehicles be constant. Rule 184 is a basic example of an elementary CA (ECA) that models vehicle movements simply and effectively.

Hattori and Takesue [1] derive necessary and sufficient conditions for a one-dimensional binary CA to have general additive conserved quantities. Boccara and Fuks [2] also derive necessary and sufficient conditions for a multi-state CA to be number conserving. In the case of two or more dimensions or other settings, necessary and sufficient conditions for a CA to be number conserving are also formulated [3–5].

In this paper, the number-conserving rules for the second-order binary three-neighbor CA (SOCA) are discussed. A well-known SOCA subclass is the elementary reversible CA (ERCA) [6–8], which is shown to have computational universality [9]. The number of additive conserved quantities for the ERCA is discussed and enumerated in [1]. The main aim of the present paper is to enumerate all the SOCA rules that satisfy the number conservation law and to give a necessary and sufficient condition for each SOCA to be number conserving. As a typical application, the number-conserving rules of the SOCA are shown to include the slow-to-start model [10, 11], which is a more practical traffic flow model than ECA 184.

The present study first introduces a polynomial representation of a CA, which is a key idea in the latter part of this paper. The polynomial representation of a CA derived through the disjunctive normal form in the Boolean algebra is applied to the analysis of a one-dimensional CA with number conservation and to the investigation of mathematical structures of a fuzzy CA, which are continuous-state counterparts of a binary CA [12–14].

This paper is organized as follows. A concrete description of the general form of the polynomial representation of local rules for the SOCA is derived in Section 2. In Section 3, a polynomial representation is used to enumerate 74 number-conserving rules out of the 2^{64} SOCA rules. The necessary and sufficient condition for each SOCA to be number conserving is formulated in Section 4. The 74 number-conserving SOCA rules are classified into 30 equivalence classes by the well-known equivalence relations of CAs. Section 5 lists representatives of each class and shows the spatiotemporal diagrams. Finally, conclusions along with some remarks are presented in Section 6.

2. Polynomial Representation for Binary Three-Neighbor Cellular Automata

In this section, we introduce the polynomial representation for the first- and second-order three-neighbor binary CAs with a periodic boundary condition. Let $Q = \{0, 1\}$ be the set of states and N be the number of cells. Then the state of cell $i \in \mathbb{Z}/N\mathbb{Z}$ at the discrete time t is denoted by $u_i^t \in Q$. The local rule $f : Q^3 \to Q$ determines the evolutions of cells, that is,

$$u_i^{t+1} = f(u_{i-1}^t, u_i^t, u_{i+1}^t).$$

This CA is called a first-order binary three-neighbor CA or elementary CA (ECA) [6]. The local rule f of the ECA is defined by the rule table shown in Table 1. Here, $a_j \in Q$, $j = 0, 1, \ldots, 7$, and the rule number of the ECA is given by $\sum_{j=0}^{7} a_j 2^j$. For simplicity, we denote u_{i-1}^t, u_i^t and u_{i+1}^t by x, y and z, respectively. By applying the disjunctive normal form in the Boolean algebra, the local rule f is expressed by

$$\begin{aligned} f(x, y, z) &= a_7 xyz + a_6 xy(1-z) + \\ &\qquad a_5 x(1-y)z + a_4 x(1-y)(1-z) + \\ &\qquad a_3(1-x)yz + a_2(1-x)y(1-z) + \\ &\qquad a_1(1-x)(1-y)z + a_0(1-x)(1-y)(1-z) \\ &= a_0 + (-a_0 + a_4)x + (-a_0 + a_2)y + \\ &\qquad (a_0 - a_2 - a_4 + a_6)xy + (-a_0 + a_1)z + \\ &\qquad (a_0 - a_1 - a_4 + a_5)xz + (a_0 - a_1 - a_2 + a_3)yz \\ &\qquad (-a_0 + a_1 + a_2 - a_3 + a_4 - a_5 - a_6 + a_7)xyz. \end{aligned} \tag{1}$$

Note that the operations are ordinary addition and multiplication, not those in Boolean algebra in equation (1). Monomials that appear in equation (1) are denoted by

$$\begin{aligned} &M_j = x^{b_2} y^{b_1} z^{b_0}, \\ &\quad j = b_2 \cdot 2^2 + b_1 \cdot 2^1 + b_0 \cdot 2^0, \end{aligned}$$

where $b_0, b_1, b_2 \in \{0, 1\}$. Then the polynomial $f(x, y, z)$ is rewritten as:

$$f(x, y, z) = \sum_{j=0}^{7} c_j M_j. \tag{2}$$

xyz	111	110	101	100	011	010	001	000
$f(x, y, z)$	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0

Table 1. Rule table of the ECA.

Table 2 lists the monomials M_j. The coefficient c_j in equation (2) is a linear combination of $a_0, a_1, \ldots, a_7$ and is given by

$$c_j = \sum_{k \in \mathcal{I}_j} \text{sgn}(j, k) a_k, \quad \text{sgn}(j, k) = (-1)^{|A_j \setminus A_k|},$$

where A_j is a set of variables included in a monomial M_j, that is,

$$A_j = \{b_2 x,\ b_1 y,\ b_0 z\} \backslash \{0\},$$
$$j = b_2 \cdot 2^2 + b_1 \cdot 2^1 + b_0 \cdot 2^0,$$

and $\mathcal{I}_j = \{k \mid A_k \subset A_j\}$ is the set of indices of a_k that compose c_j. For example, the specific expression in terms of a_k of the coefficient c_6 of $M_6 = xy$ can be obtained by the following procedure. Since $A_6 = \{x, y\}$, the set of indices k of a_k included in c_6 yields

$$\mathcal{I}_6 = \{k \mid A_k \subset A_6\} = \{0, 2, 4, 6\}.$$

j	7	6	5	4	3	2	1	0
M_j	xyz	xy	xz	x	yz	y	z	1

Table 2. Monomials M_j for $j = 0, 1, \ldots, 7$.

Considering $\mathrm{sgn}(6, k) = (-1)^{|A_6 \backslash A_k|}$, $k \in \mathcal{I}_6$, the coefficient c_6 can be expressed as

$$c_6 = \sum_{k \in \mathcal{I}_6} \mathrm{sgn}(6, k) a_k = a_0 - a_2 - a_4 + a_6.$$

In the following, the SOCA will be discussed. In a SOCA, the state u_i^{t+1} of cell i at discrete time $t+1$ is determined by the states u_{i-1}^{t-1}, u_i^{t-1}, u_{i+1}^{t-1}, u_{i-1}^t, u_i^t and u_{i+1}^t, that is,

$$u_i^{t+1} = f\left(u_{i-1}^{t-1}, u_i^{t-1}, u_{i+1}^{t-1}, u_{i-1}^t, u_i^t, u_{i+1}^t\right).$$

The total number of local rules $f : Q^6 \to Q$ for the SOCAs, which are shown in Table 3, is $2^{64} \approx 1.84 \times 10^{19}$, and the rule number of a SOCA is given by $\sum_{j=0}^{63} a_j 2^j$. The rule f can also be expressed by a polynomial similar to that of the ECA rule. For simplicity, let us denote $X = u_{i-1}^{t-1}$, $Y = u_i^{t-1}$, $Z = u_{i+1}^{t-1}$, $x = u_{i-1}^t$, $y = u_i^t$ and $z = u_{i+1}^t$. Then f is expressed as:

$$\begin{aligned} &f(X, Y, Z, x, y, z) = \\ &\quad a_{63} XYZxyz + a_{62} XYZxy(1-z) + \cdots + a_0 (1-X) \\ &\qquad (1-Y)(1-Z)(1-x)(1-y)(1-z) = \sum_{j=0}^{63} c_j M_j. \end{aligned} \tag{3}$$

Here, we consider the same symbols used for the ECAs for the SOCAs. The monomials M_j in equation (3) are defined as

$$M_j = X^{b_5} Y^{b_4} Z^{b_3} x^{b_2} y^{b_1} z^{b_0},$$
$$j = b_5 \cdot 2^5 + b_4 \cdot 2^4 + b_3 \cdot 2^3 + b_2 \cdot 2^2 + b_1 \cdot 2^1 + b_0 \cdot 2^0,$$

XYZ	111	111	111	111	111	111	111	111
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{63}	a_{62}	a_{61}	a_{60}	a_{59}	a_{58}	a_{57}	a_{56}
XYZ	110	110	110	110	110	110	110	110
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{55}	a_{54}	a_{53}	a_{52}	a_{51}	a_{50}	a_{49}	a_{48}
XYZ	101	101	101	101	101	101	101	101
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{47}	a_{46}	a_{45}	a_{44}	a_{43}	a_{42}	a_{41}	a_{40}
XYZ	100	100	100	100	100	100	100	100
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{39}	a_{38}	a_{37}	a_{36}	a_{35}	a_{34}	a_{33}	a_{32}
XYZ	011	011	011	011	011	011	011	011
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{31}	a_{30}	a_{29}	a_{28}	a_{27}	a_{26}	a_{25}	a_{24}
XYZ	010	010	010	010	010	010	010	010
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{23}	a_{22}	a_{21}	a_{20}	a_{19}	a_{18}	a_{17}	a_{16}
XYZ	001	001	001	001	001	001	001	001
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_{15}	a_{14}	a_{13}	a_{12}	a_{11}	a_{10}	a_9	a_8
XYZ	000	000	000	000	000	000	000	000
xyz	111	110	101	100	011	010	001	000
$f(X, Y, Z, x, y, z)$	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0

Table 3. Rule table for the SOCAs.

where $b_0, b_1, \ldots, b_5 \in \{0, 1\}$. A list of the monomials M_j for $j = 0, 1, \ldots, 63$ is shown in Table 4. The coefficients c_j in equation (3) are expressed by

$$c_j = \sum_{k \in \mathcal{I}_j} \operatorname{sgn}(j, k) a_k, \quad j = 0, 1, \ldots, 63, \qquad (4)$$
$$\operatorname{sgn}(j, k) = (-1)^{|A_j \backslash A_k|},$$

with

$$A_j = \{b_5 X, b_4 Y, b_3 Z, b_2 x, b_1 y, b_0 z\} \backslash \{0\},$$
$$\mathcal{I}_j = \{k \mid A_k \subset A_j\}, \quad j = 0, 1, \ldots, 63.$$

For example, since the set of variables A_{37} and the index set $\mathcal{I}_{37}$ corresponding to the monomial $M_{37} = xXz$ are $A_{37} = \{X, x, z\}$ and

$$\mathcal{I}_{37} = \{k \mid A_k \subset A_{37}\} = \{0, 1, 4, 5, 32, 33, 36, 37\},$$

j	63	62	61	60	59	58	57	56
M_j	$XYZxyz$	$XYZxy$	$XYZxz$	$XYZx$	$XYZyz$	$XYZy$	$XYZz$	XYZ
j	55	54	53	52	51	50	49	48
M_j	$XYxyz$	$XYxy$	$XYxz$	XYx	$XYyz$	XYy	XYz	XY
j	47	46	45	44	43	42	41	40
M_j	$XZxyz$	$XZxy$	$XZxz$	XZx	$XZyz$	XZy	XZz	XZ
j	39	38	37	36	35	34	33	32
M_j	$Xxyz$	Xxy	Xxz	Xx	Xyz	Xy	Xz	X
j	31	30	29	28	27	26	25	24
M_j	$YZxyz$	$YZxy$	$YZxz$	YZx	$YZyz$	YZy	YZz	YZ
j	23	22	21	20	19	18	17	16
M_j	$Yxyz$	Yxy	Yxz	Yx	Yyz	Yy	Yz	Y
j	15	14	13	12	11	10	9	8
M_j	$Zxyz$	Zxy	Zxz	Zx	Zyz	Zy	Zz	Z
j	7	6	5	4	3	2	1	0
M_j	xyz	xy	xz	x	yz	y	z	1

Table 4. Monomials M_j for $j = 0, 1, \ldots, 63$.

respectively, the coefficient c_{37} is given by

$$\begin{aligned} c_{37} &= \sum_{k\in\mathcal{I}_{37}} \operatorname{sgn}(37, k) a_k \\ &= -a_0 + a_1 + a_4 - a_5 + a_{32} - a_{33} - a_{36} + a_{37}. \end{aligned}$$

Let

$$\boldsymbol{c} = [c_0, c_1, \ldots, c_{63}]^\top \in \mathbb{Z}^{64}$$

and

$$\boldsymbol{a} = [a_0, a_1, \ldots, a_{63}]^\top \in \{0, 1\}^{64}.$$

Then equation (4) can be viewed as the system of linear equations

$$\boldsymbol{c} = V\boldsymbol{a}, \tag{5}$$

where $V \in \{0, \pm 1\}^{64\times 64}$. Figure 1 shows a plot of the nonzero entries for the matrix V. In Figure 1, we can observe that the distribution of the nonzero entries is similar to the Sierpinski gasket.

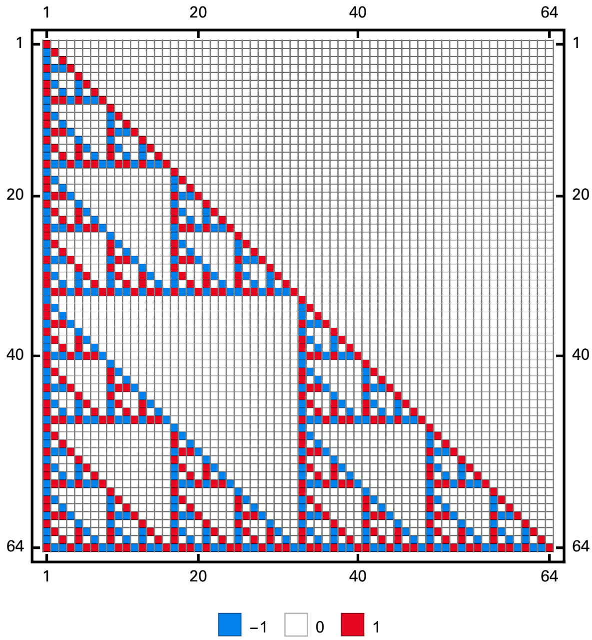

Figure 1. The nonzero entries for the matrix V.

3. Second-Order Two-State Three-Neighbor Cellular Automata with Conserved Quantities

In this section, we enumerate the number-conserving SOCA rules. A CA is number conserving if

$$\sum_{i=0}^{N-1} u_i^t = \sum_{i=0}^{N-1} u_i^{t+1} \tag{6}$$

holds for any t and for any initial configuration. Using equation (3), the right-hand side in equation (6) can be rewritten as

$$\sum_{i=0}^{N-1} u_i^{t+1} = \sum_{i=0}^{N-1} f\left(u_{i-1}^{t-1}, u_i^{t-1}, u_{i+1}^{t-1}, u_{i-1}^t, u_i^t, u_{i+1}^t\right) = \sum_{i=0}^{N-1} \sum_{j=0}^{63} c_j M_j(i).$$

Here, $M_j(i)$ is the monomial M_j with $X = u_{i-1}^{t-1}$, $Y = u_i^{t-1}$, $Z = u_{i+1}^{t-1}$, $x = u_{i-1}^t$, $y = u_i^t$ and $z = u_{i+1}^t$ substituted. For convenience of notation, we denote $S_j = \sum_{i=0}^{N-1} M_j(i)$ for $j = 0, 1, \ldots, 63$. Then equation (6) can be rewritten as

$$S_2 = \sum_{j=0}^{63} c_j S_j. \tag{7}$$

On the right-hand side of equation (7), the dependence of S_j on j arises due to the periodic boundary conditions. For example, it holds that

$$\sum_{i=0}^{N-1} u_{i-1}^t = \sum_{i=0}^{N-1} u_i^t = \sum_{i=0}^{N-1} u_{i+1}^t,$$

which denotes $S_1 = S_2 = S_4$. Similarly, the following equalities with respect to S_j hold:

$$\begin{array}{lllll} S_1 = S_2 = S_4, & S_3 = S_6, & S_8 = S_{16} = S_{32}, & & \\ S_9 = S_{18} = S_{36}, & S_{10} = S_{20}, & S_{11} = S_{22}, & S_{17} = S_{34}, & \\ S_{19} = S_{38}, & S_{24} = S_{48}, & S_{25} = S_{50}, & S_{26} = S_{52}, & S_{27} = S_{54}. \end{array}$$

Additionally, by substituting $t-1$ for t in equation (6), $S_2 = S_{16}$ also holds true. Taking the given equalities into account, the sum on the right-hand side in equation (7) can be expressed as:

$$\begin{aligned} \sum_{j=0}^{63} c_j S_j = {} & c_0 S_0 + (c_1 + c_2 + c_4 + c_8 + c_{16} + c_{32}) S_1 + (c_3 + c_6) S_3 + c_5 S_5 + \\ & c_7 S_7 + (c_9 + c_{18} + c_{36}) S_9 + (c_{10} + c_{20}) S_{10} + (c_{11} + c_{22}) S_{11} + \\ & c_{12} S_{12} + c_{13} S_{13} + c_{14} S_{14} + c_{15} S_{15} + (c_{17} + c_{34}) S_{17} + \\ & (c_{19} + c_{38}) S_{19} + c_{21} S_{21} + c_{23} S_{23} + (c_{24} + c_{48}) S_{24} + \\ & (c_{25} + c_{50}) S_{25} + (c_{26} + c_{52}) S_{26} + (c_{27} + c_{54}) S_{27} + c_{28} S_{28} + \\ & c_{29} S_{29} + c_{30} S_{30} + c_{31} S_{31} + c_{33} S_{33} + c_{35} S_{35} + c_{37} S_{37} + \\ & c_{39} S_{39} + c_{40} S_{40} + c_{41} S_{41} + c_{42} S_{42} + c_{43} S_{43} + c_{44} S_{44} + c_{45} S_{45} + \\ & c_{46} S_{46} + c_{47} S_{47} + c_{49} S_{49} + c_{51} S_{51} + c_{53} S_{53} + c_{55} S_{55} + c_{56} S_{56} + \\ & c_{57} S_{57} + c_{58} S_{58} + c_{59} S_{59} + c_{60} S_{60} + c_{61} S_{61} + c_{62} S_{62} + c_{63} S_{63}. \end{aligned}$$

Each S_j represents the total number of occurrences of a distinct local pattern, and by locally designing the configuration, these quantities can be varied independently by appropriate choices of configurations. Therefore, since equation (7) holds for all configurations, the coefficients c_j must satisfy these linear equalities:

$$\begin{cases} c_0 = 0, \\ c_1 + c_2 + c_4 + c_8 + c_{16} + c_{32} = 1, \\ c_3 + c_6 = 0, \\ c_9 + c_{18} + c_{36} = 0, \\ c_{10} + c_{20} = 0, \\ c_{11} + c_{22} = 0, \\ c_{17} + c_{34} = 0, \\ c_{19} + c_{38} = 0, \\ c_{24} + c_{48} = 0, \\ c_{25} + c_{50} = 0, \\ c_{26} + c_{52} = 0, \\ c_{27} + c_{54} = 0, \\ c_\ell = 0, \quad \begin{array}{l} \ell{=}5,7,12,13,14,15,21,23,28,29,30,31,33, \\ 35,37,39,40,41,42,43,44,45,46,47,49,51, \\ 53,55,56,57,58,59,60,61,62,63. \end{array} \end{cases} \tag{8}$$

Let $\boldsymbol{b} = [0, 1, 0, \ldots, 0]^\top \in \{0, 1\}^{48}$. Then equation (8) can be written as

$$U\boldsymbol{c} = \boldsymbol{b}, \tag{9}$$

where $U \in \{0, 1\}^{48\times 64}$ is a coefficient matrix in equation (8). Substituting equation (5) into (9) and letting $W = UV$, the linear equation (9) yields

$$W\boldsymbol{a} = \boldsymbol{b}. \tag{10}$$

The number-conserving SOCA rules are enumerated by solving equation (10) under the condition $\boldsymbol{a} \in \{0, 1\}^{64}$. The augmented coefficient matrix $[W \mid \boldsymbol{b}]$ is transformed into the reduced row echelon form $[W' \mid \boldsymbol{b}']$ illustrated in Figure 2. Here, the computation was performed using the Wolfram Language function `RowReduce`. Observing the reduced row echelon form, the general solution of equation (10) can be obtained. Let $\boldsymbol{p} \in \mathrm{R}^{64}$ be the particular solution of equation (10) and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{16} \in \mathrm{R}^{64}$ be the basis vectors of the kernel of W, which are illustrated in Figure 3. Then the solution vector for the linear equation (10) is

$$\begin{aligned} \boldsymbol{a} = \boldsymbol{p} &+ a_{36}\boldsymbol{v}_1 + a_{37}\boldsymbol{v}_2 + a_{38}\boldsymbol{v}_3 + a_{39}\boldsymbol{v}_4 + a_{44}\boldsymbol{v}_5 + \\ & a_{45}\boldsymbol{v}_6 + a_{46}\boldsymbol{v}_7 + a_{47}\boldsymbol{v}_8 + a_{52}\boldsymbol{v}_9 + a_{53}\boldsymbol{v}_{10} + a_{54}\boldsymbol{v}_{11} + \\ & a_{55}\boldsymbol{v}_{12} + a_{59}\boldsymbol{v}_{13} + a_{60}\boldsymbol{v}_{14} + a_{61}\boldsymbol{v}_{15} + a_{62}\boldsymbol{v}_{16}. \end{aligned} \tag{11}$$

Since $\boldsymbol{p}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{16}$ is a vector with all entries integers, and the coefficients $a_{36}, a_{37}, \ldots, a_{62} \in \{0, 1\}$ hold, all the entries in the solution vector $\boldsymbol{a}$ in equation (11) are integers. However, in order to

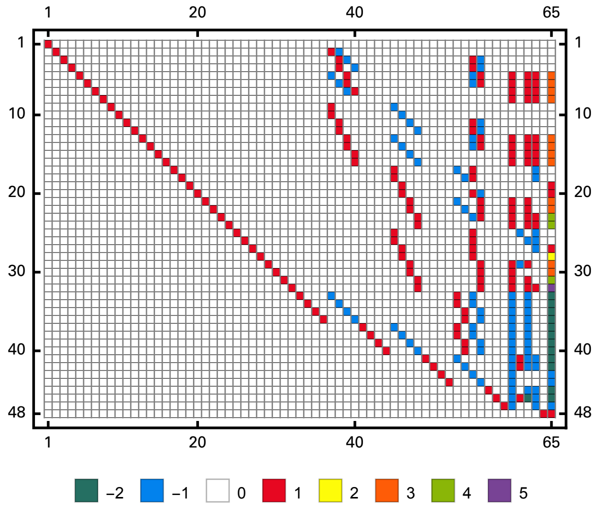

Figure 2. The reduced row echelon form $[W' \mid \boldsymbol{b}']$ for $[W \mid \boldsymbol{b}]$.

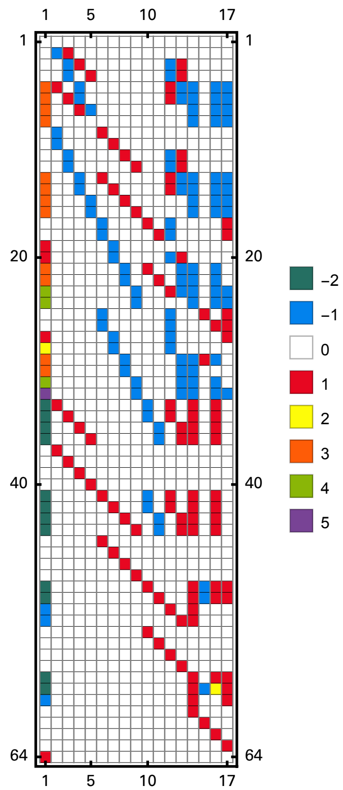

Figure 3. Column 1 corresponds to $\boldsymbol{p}$, and columns 2 through 17 correspond to $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{16}$.

determine the local rule of the SOCA, $\boldsymbol{a}$ should be constrained to 0 or 1. Therefore, we evaluate whether $\boldsymbol{a} \in \{0, 1\}^{64}$ holds for all combinations of the coefficients

$$(a_{36}, a_{37}, a_{38}, a_{39}, a_{44}, a_{45}, a_{46}, a_{47},\\ a_{52}, a_{53}, a_{54}, a_{55}, a_{59}, a_{60}, a_{61}, a_{62}) \in \{0, 1\}^{16}.$$

As a result, 74 number-conserving rules are enumerated, which are listed in Table 5.

♯	Rule Number (decimal)	♯	Rule Number (decimal)
1	12297829382473034410	30	16339284774680838370
2	13310591802206107832	31	16339322011845193420
3	13310592094263871692	32	16339322157874078434
4	13310666568188361912	33	16348841532183203020
5	13310666860246125772	34	16348841678212088034
6	13329679117910325432	35	16348878915376443084
7	13329679409968089292	36	16348879061405328098
8	13329753883892579512	37	17361641481138401520
9	13329754175950343372	38	18374966859414961920
10	13329754321979228386	39	18374967954631557120
11	14738232885223405752	40	18379749717816967168
12	14738233177281169612	41	18384524858162806784
13	14738307651205659832	42	18389306621348216832
14	14738307943263423692	43	18394055270335774720
15	14747864846794673356	44	18398837033521184768
16	14757320200927623352	45	18403612173867024384
17	14757320492985387212	46	18408393937052434432
18	14757357876178627276	47	18408412628699582720
19	14757394966909877432	48	18413036620606865408
20	14757395258967641292	49	18417818383792275456
21	14757395404996526306	50	18422593524138115072
22	14757432642160881356	51	18427375287323525120
23	14757432788189766370	52	18427450053305779200
24	14766952162498890956	53	18432123936311083008
25	14766952308527775970	54	18436905699496493056
26	14766989545692131020	55	18436943082689733120
27	14766989691721016034	56	18441680839842332672
28	16339284336594189496	57	18441830370189936640
29	16339284628651953356	58	18446462603027742720

Table 5. (*continues*).

♯	Rule Number (decimal)	♯	Rule Number (decimal)
59	18446481294674891008	67	18446630825022494976
60	18446499986220982784	68	18446649516568586752
61	18446518677868131072	69	18446668208215735040
62	18446537369009996800	70	18446686899357600768
63	18446556060657145088	71	18446705591004749056
64	18446574752203236864	72	18446724282550840832
65	18446593443850385152	73	18446742974197989120
66	18446612133375346688	74	18446744069414584320

Table 5. The 74 number-conserving rules in the SOCA and their reference numbers.

4. Number Conservation and Flux Form in the SOCA

In this section, a necessary and sufficient condition for the SOCA rules to be number conserving is given, which is a generalization of the result in [1] to the SOCA.

Proposition 1. The number of rules in the SOCA that can be expressed as

$$f(X, Y, Z, x, y, z) = y + J(X, Y, x, y) - J(Y, Z, y, z) \tag{12}$$

or

$$f(X, Y, Z, x, y, z) = Y + J(X, Y, x, y) - J(Y, Z, y, z) \tag{13}$$

by using some $J : \{0, 1\}^4 \to \mathbb{R}$ is 74 in total.

Proof. First, consider the enumeration for the SOCA that satisfies equation (12). Then $f(X, Y, Z, x, y, z)$ satisfies these conditions:

$$\begin{cases} \text{(C0)} \quad f(X, Y, Z, 0, 0, 0) = J(X, Y, 0, 0) - J(Y, Z, 0, 0), \\ \text{(C1)} \quad f(X, Y, Z, 0, 0, 1) = J(X, Y, 0, 0) - J(Y, Z, 0, 1), \\ \text{(C2)} \quad f(X, Y, Z, 0, 1, 0) = \\ \qquad 1 + J(X, Y, 0, 1) - J(Y, Z, 1, 0), \\ \text{(C3)} \quad f(X, Y, Z, 0, 1, 1) = \\ \qquad 1 + J(X, Y, 0, 1) - J(Y, Z, 1, 1), \\ \text{(C4)} \quad f(X, Y, Z, 1, 0, 0) = J(X, Y, 1, 0) - J(Y, Z, 0, 0), \\ \text{(C5)} \quad f(X, Y, Z, 1, 0, 1) = J(X, Y, 1, 0) - J(Y, Z, 0, 1), \\ \text{(C6)} \quad f(X, Y, Z, 1, 1, 0) = \\ \qquad 1 + J(X, Y, 1, 1) - J(Y, Z, 1, 0), \\ \text{(C7)} \quad f(X, Y, Z, 1, 1, 1) = \\ \qquad 1 + J(X, Y, 1, 1) - J(Y, Z, 1, 1). \end{cases} \tag{14}$$

Assume that $J(A, B, 0, 0) < J(A', B', 0, 0)$ holds for some two tuples $(A, B), (A', B') \in \{0, 1\}^2$. Then inequality

$$J(A, B, 0, 0) < J(B, A', 0, 0)$$

or

$$J(B, A', 0, 0) < J(A', B', 0, 0)$$

holds. This is because if neither of these is true, then $J(A, B, 0, 0) \geq J(B, A', 0, 0) \geq J(A', B', 0, 0)$ holds, which contradicts the assumption. If $J(A, B, 0, 0) < J(B, A', 0, 0)$ holds, then, by considering (C0), $f(A, B, A', 0, 0, 0) = J(A, B, 0, 0) - J(B, A', 0, 0) < 0$ leads to a contradiction, due to the fact that f induces a binary CA. A similar contradiction arises in the case of $J(B, A', 0, 0) < J(A', B', 0, 0)$. Therefore, there exists a constant L such that $J(A, B, 0, 0) = L$ identically holds, and in particular $f(X, Y, Z, 0, 0, 0) = 0$ holds. By a similar discussion for (C7), there exists a constant K such that $J(A, B, 1, 1) = K$ identically holds, and $f(X, Y, Z, 1, 1, 1) = 1$. Then the conditions of equation (14) can be rewritten as:

$$\begin{cases} \text{(C0)} & f(X, Y, Z, 0, 0, 0) = 0, \\ \text{(C1)} & f(X, Y, Z, 0, 0, 1) = L - J(Y, Z, 0, 1), \\ \text{(C2)} & f(X, Y, Z, 0, 1, 0) = \\ & 1 + J(X, Y, 0, 1) - J(Y, Z, 1, 0), \\ \text{(C3)} & f(X, Y, Z, 0, 1, 1) = 1 + J(X, Y, 0, 1) - K, \\ \text{(C4)} & f(X, Y, Z, 1, 0, 0) = J(X, Y, 1, 0) - L, \\ \text{(C5)} & f(X, Y, Z, 1, 0, 1) = J(X, Y, 1, 0) - J(Y, Z, 0, 1), \\ \text{(C6)} & f(X, Y, Z, 1, 1, 0) = 1 + K - J(Y, Z, 1, 0), \\ \text{(C7)} & f(X, Y, Z, 1, 1, 1) = 1. \end{cases} \tag{15}$$

Since f is a local rule of a binary CA, and considering (C1), (C3), (C4) and (C6) in equation (15), it holds that

$$J(A, B, 1, 0) \in \{K, K+1\} \cap \{L, L+1\}$$

and

$$J(A, B, 0, 1) \in \{K-1, K\} \cap \{L-1, L\}$$

for $(A, B) \in \{0, 1\}^2$. This implies that both $\{K, K+1\} \cap \{L, L+1\}$ and $\{K-1, K\} \cap \{L-1, L\}$ are nonempty. Let us introduce $\Delta \in \{-1, 0, 1\}$ and $\delta_{10}, \delta_{01} : \{0, 1\}^2 \to \{0, 1\}$ so that L, $J(A, B, 1, 0)$ and $J(A, B, 0, 1)$ are expressed as

$$L = K + \Delta, \tag{16}$$

$$J(A, B, 1, 0) = K + \delta_{10}(A, B), \tag{17}$$

$$J(A, B, 0, 1) = K - \delta_{01}(A, B), \tag{18}$$

respectively. By substituting equations (16), (17) and (18) into equation (15), the conditions can be rewritten as

$$\begin{cases} \text{(C0)} \quad f(X, Y, Z, 0, 0, 0) = 0, \\ \text{(C1)} \quad f(X, Y, Z, 0, 0, 1) = \Delta + \delta_{01}(Y, Z), \\ \text{(C2)} \quad f(X, Y, Z, 0, 1, 0) = 1 - \delta_{01}(X, Y) - \delta_{10}(Y, Z), \\ \text{(C3)} \quad f(X, Y, Z, 0, 1, 1) = 1 - \delta_{01}(X, Y), \\ \text{(C4)} \quad f(X, Y, Z, 1, 0, 0) = \delta_{10}(X, Y) - \Delta, \\ \text{(C5)} \quad f(X, Y, Z, 1, 0, 1) = \delta_{10}(X, Y) + \delta_{01}(Y, Z), \\ \text{(C6)} \quad f(X, Y, Z, 1, 1, 0) = 1 - \delta_{10}(Y, Z), \\ \text{(C7)} \quad f(X, Y, Z, 1, 1, 1) = 1. \end{cases} \tag{19}$$

Note that

$$J(A, B, a, b) = \\ K + a(1-b)\delta_{10}(A, B) - (1-a)b\delta_{01}(A, B) + (1-a)(1-b)\Delta.$$

In the following, we discuss individual cases based on the value of $\Delta \in \{-1, 0, 1\}$.

In the case where $\Delta = -1$, from (C1) and (C4) in equation (19), $\delta_{01}(A, B)$ and $\delta_{10}(A, B)$ are identically 1 and 0, respectively. Therefore, equation (19) uniquely determines the CA with

$$J(A, B, a, b) = K - 1 + a. \tag{20}$$

In the case where $\Delta = 1$, from (C1) and (C4) in equation (19), $\delta_{01}(A, B)$ and $\delta_{10}(A, B)$ are identically 0 and 1, respectively. Therefore, equation (19) uniquely determines the CA with

$$J(A, B, a, b) = K + 1 - b. \tag{21}$$

In the case where $\Delta = 0$, equation (19) can be expressed as

$$\begin{cases} \text{(C0)} \quad f(X, Y, Z, 0, 0, 0) = 0, \\ \text{(C1)} \quad f(X, Y, Z, 0, 0, 1) = \delta_{01}(Y, Z), \\ \text{(C2)} \quad f(X, Y, Z, 0, 1, 0) = 1 - \delta_{01}(X, Y) - \delta_{10}(Y, Z), \\ \text{(C3)} \quad f(X, Y, Z, 0, 1, 1) = 1 - \delta_{01}(X, Y), \\ \text{(C4)} \quad f(X, Y, Z, 1, 0, 0) = \delta_{10}(X, Y), \\ \text{(C5)} \quad f(X, Y, Z, 1, 0, 1) = \delta_{10}(X, Y) + \delta_{01}(Y, Z), \\ \text{(C6)} \quad f(X, Y, Z, 1, 1, 0) = 1 - \delta_{10}(Y, Z), \\ \text{(C7)} \quad f(X, Y, Z, 1, 1, 1) = 1. \end{cases} \tag{22}$$

In the following, we discuss three individual cases: (a) $\delta_{10}(A, B)$ is identically 0, (b) $\delta_{01}(A, B)$ is identically 0, and (c) is neither (a) nor (b).

In the case of (a), by considering the conditions of equation (22), f is the local rule of a binary CA for any of $2^4 = 16$ functions $\delta_{01} : \{0, 1\}^2 \to \{0, 1\}$. That is, 16 different functions

$$J(A, B, a, b) = K - (1-a)b\delta_{01}(A, B) \tag{23}$$

determine 16 different CAs. Similarly, in the case of (b), the CAs that satisfy the conditions of equation (22) are given by the 16 types of function

$$J(A, B, a, b) = K + a(1 - b)\delta_{10}(A, B). \tag{24}$$

Note here that the case where $\delta_{10}(A, B) = \delta_{01}(A, B) = 0$, that is, $J(A, B, a, b) = K$, is counted redundantly.

In the case of (c), $\delta_{10}(A, B)$ takes the value of 1 for at least one (A, B).

1. In the case where $\delta_{10}(1, 1) = 1$, from (C2) and (C5) in equation (22), it holds that
   $f(1, 1, 1, 0, 1, 0) = -\delta_{01}(1, 1)$,
   $f(0, 1, 1, 0, 1, 0) = -\delta_{01}(0, 1)$,
   $f(1, 1, 1, 1, 0, 1) = 1 + \delta_{01}(1, 1)$,
   $f(1, 1, 0, 1, 0, 1) = 1 + \delta_{01}(1, 0)$.

   Therefore, $\delta_{01}(1, 1) = \delta_{01}(1, 0) = \delta_{01}(0, 1) = 0$. Noting that $\delta_{01}(A, B)$ is not identically 0, $\delta_{01}(0, 0) = 1$ holds. Then, a function J is determined as:

   $$J(A, B, a, b) = K + a(1 - b)AB - (1 - a)b(1 - A)(1 - B). \tag{25}$$

2. In the case where $\delta_{10}(1, 0) = 1$,
   $f(1, 0, 1, 1, 0, 1) = 1 + \delta_{01}(0, 1)$,
   $f(1, 0, 0, 1, 0, 1) = 1 + \delta_{01}(0, 0)$,
   $f(1, 1, 0, 0, 1, 0) = -\delta_{01}(1, 1)$,
   $f(0, 1, 0, 0, 1, 0) = -\delta_{01}(0, 1)$

   holds. Therefore, $\delta_{01}(1, 1) = \delta_{01}(0, 1) = \delta_{01}(0, 0) = 0$ and $\delta_{01}(1, 0) = 1$. Then, a function J is determined as:

   $$J(A, B, a, b) = K + (a - b)A(1 - B). \tag{26}$$

3. In the case where $\delta_{10}(0, 1) = 1$,
   $f(0, 1, 1, 1, 0, 1) = 1 + \delta_{01}(1, 1)$,
   $f(0, 1, 0, 1, 0, 1) = 1 + \delta_{01}(1, 0)$,
   $f(1, 0, 1, 0, 1, 0) = -\delta_{01}(1, 0)$,
   $f(0, 0, 1, 0, 1, 0) = -\delta_{01}(0, 0)$

   holds. Therefore $\delta_{01}(1, 1) = \delta_{01}(1, 0) = \delta_{01}(0, 0) = 0$ and $\delta_{01}(0, 1) = 1$. Then, a function J is determined as:

   $$J(A, B, a, b) = K + (a - b)(1 - A)B. \tag{27}$$

4. In the case where $\delta_{10}(0, 0) = 1$,
   $f(0, 0, 1, 1, 0, 1) = 1 + \delta_{01}(0, 1)$,
   $f(0, 0, 0, 1, 0, 1) = 1 + \delta_{01}(0, 0)$,
   $f(1, 0, 0, 0, 1, 0) = -\delta_{01}(1, 0)$,
   $f(0, 0, 0, 0, 1, 0) = -\delta_{01}(0, 0)$

holds. Therefore $\delta_{01}(1, 0) = \delta_{01}(0, 1) = \delta_{01}(0, 0) = 0$ and $\delta_{01}(1, 1) = 1$. Then, a function J is determined as:

$$J(A, B, a, b) = K + a(1 - b)(1 - A)(1 - B) - (1 - a)bAB. \tag{28}$$

To summarize the discussion so far, there are 37 CA rules that satisfy equation (12). For the CAs that satisfy equation (13), by exchanging the role of discrete time t for $t - 1$, a similar discussion yields 37 CA rules. That is, by swapping (X, Y, Z) and (x, y, z) in equation (12), the number-conserving rules satisfying equation (14) are obtained from those satisfying (12). Therefore, the total 74 CA rules are obtained. □

Since the proof of Proposition 1 is constructive, all the functions J are obtained, and hence all the corresponding local rules f of the SOCA can be written down concretely. Table 6 lists the functions J obtained in the proof of Proposition 1, along with the corresponding rule numbers. The first column of Table 6 gives the reference number of the equation from which J in the second column was obtained, and the third column shows the reference number of the rule number in Table 5 corresponding to the function J. The fourth column shows the reference number corresponding to the function f that satisfies equation (13), where the function $J(A, B, a, b)$ is replaced by $J(a, b, A, B)$. Therefore, Table 6 shows that the local rules f that satisfy equations (12) or (13) have one-to-one correspondence with the number-conserving SOCA rules obtained in Section 3. Theorem 1, which is the main result of this paper, is derived.

Equation	$J(A, B, a, b)$	(12)	(13)
(20)	$K - 1 + a$	37	74
(21)	$K + 1 - b$	1	38
(23)	K	20	58
(23)	$K - (1 - a)b(1 - A - B + AB)$	21	59
(23)	$K - (1 - a)b(B - AB)$	22	60
(23)	$K - (1 - a)b(1 - A)$	23	61
(23)	$K - (1 - a)b(A - AB)$	24	62
(23)	$K - (1 - a)b(1 - B)$	25	63
(23)	$K - (1 - a)b(A + B - 2AB)$	26	64
(23)	$K - (1 - a)b(1 - AB)$	27	65
(23)	$K - (1 - a)bAB$	29	66
(23)	$K - (1 - a)b(1 - A - B + 2AB)$	30	67
(23)	$K - (1 - a)bB$	31	68
(23)	$K - (1 - a)b(1 - A + AB)$	32	69
(23)	$K - (1 - a)bA$	33	70
(23)	$K - (1 - a)b(1 + AB - B)$	34	71

Table 6. (*continues*).

Equation	$J(A, B, a, b)$	(12)	(13)
(23)	$K-(1-a)b(A+B-AB)$	35	72
(23)	$K-(1-a)b$	36	73
(24)	$K+a(1-b)(1-A-B+AB)$	19	56
(24)	$K+a(1-b)(B-AB)$	17	54
(24)	$K+a(1-b)(1-A)$	16	53
(24)	$K+a(1-b)(A-AB)$	14	51
(24)	$K+a(1-b)(1-B)$	13	50
(24)	$K+a(1-b)(A+B-2AB)$	12	49
(24)	$K+a(1-b)b(1-AB)$	11	48
(24)	$K+a(1-b)AB$	9	46
(24)	$K+a(1-b)(1-A-B+2AB)$	8	45
(24)	$K+a(1-b)B$	7	44
(24)	$K+a(1-b)(1-A+AB)$	6	43
(24)	$K+a(1-b)A$	5	42
(24)	$K+a(1-b)(1+AB-B)$	4	41
(24)	$K+a(1-b)(A+(1-A)B)$	3	40
(24)	$K+a(1-b)$	2	39
(25)	$K+a(1-b)AB-(1-a)b(1-A)(1-B)$	10	47
(26)	$K+(a-b)A(1-B)$	15	52
(27)	$K+(a-b)(1-A)B$	18	55
(28)	$K+a(1-b)(1-A)(1-B)-(1-a)bAB$	28	57

Table 6. Correspondence between function J and the 74 number-conserving rules.

Theorem 1. The SOCA is number conserving if and only if the local rule f can be written by equation (12) or (13).

Equations (12) and (13) can be regarded as a discrete version of the flux form in fluid dynamics. Theorem 1 shows that the number conservation law corresponds to the conservation law of the flux form in the SOCA.

5. The Spatiotemporal Diagrams

In this section, we consider equivalence classes for the 74 number-conserving rules. In the ECA, 256 rules are reduced to 88 equivalence classes by considering the spatial reflection R, which swaps left and right; the Boolean conjugation C, which swaps 0s and 1s; and their composition RC. For the case of the SOCA, the relations R, C and RC are defined as

$$f_1 \sim_R f_2 \overset{\text{def}}{\Longleftrightarrow} f_1(X, Y, Z, x, y, z) = \\ f_2(Z, Y, X, z, y, x),$$

$$f_1 \sim_C f_2 \overset{\text{def}}{\Longleftrightarrow} f_1(X, Y, Z, x, y, z) = 1 - f_2(1 - X, 1 - Y, 1 - Z, 1 - x, 1 - y, 1 - z),$$

$$f_1 \sim_{RC} f_2 \overset{\text{def}}{\Longleftrightarrow} f_1(X, Y, Z, x, y, z) = 1 - f_2(1 - Z, 1 - Y, 1 - X, 1 - z, 1 - y, 1 - x).$$

Table 7 lists equivalence classes for the 74 number-conserving rules. The 74 number-conserving rules are divided into 30 equivalence classes, and the CA rules with reference number given in the leftmost column of Table 7 are the representatives of each class. Furthermore, for simplicity, the temporal swap is also considered, that is,

$$f_1 \sim_T f_2 \overset{\text{def}}{\Longleftrightarrow} f_1(X, Y, Z, x, y, z) = f_2(x, y, z, X, Y, Z).$$

Table 8 lists the polynomials for the 15 representative number-conserving SOCA rules. Since the rule numbers in the first column of Table 7 correspond to those in the sixth column by the relation $\sim_T$, the polynomial representations for the remaining 15 representative rules can be obtained by swapping (x, y, z) and (X, Y, Z) for the polynomials in Table 8. Polynomials for #1, #2 and #20 coincide with those for the ECA rules 170, 184 and 204, respectively, from Table 8. Figure 4 shows the spatiotemporal diagrams for the 15 representative rules. The SOCA with ♯3 $\sim_{RC}$ ♯11 is the slow-to-start model.

♯	R	C	RC	T	♯	R	C	RC	T
1	37	1	37	38	38	74	38	74	1
2	36	36	2	39	39	73	73	39	2
3	35	27	11	40	40	72	65	48	3
4	32	32	4	41	41	69	69	41	4
5	31	23	13	42	42	68	61	50	5
6	34	34	6	43	43	71	71	43	6
7	33	25	16	44	44	70	63	53	7
8	30	30	8	45	45	67	67	45	8
9	29	21	19	46	46	66	59	56	9
10	28	10	28	47	47	57	47	57	10
12	26	26	12	49	49	64	64	49	12
14	22	22	14	51	51	60	60	51	14
15	18	18	15	52	52	55	55	52	15
17	24	24	17	54	54	62	62	54	17
20	20	20	20	58	58	58	58	58	20

Table 7. Equivalence classes for the 74 number-conserving rules.

♯	$f(X, Y, Z, x, y, z)$
1	z
2	$yz + x - xy$
3	$y - Zy + Zyz + Yyz - Yy + Yx - Yxy + YZy - YZyz + Xx - Xxy - XYx + XYxy$
4	$yz + x - xy + Zy - Zyz - Yx + Yxy - YZy + YZyz + XYx - XYxy$
5	$y - Yy + Yyz + Xx - Xxy$
6	$yz + x - xy + Yy - Yyz - YZy + YZyz - Xx + Xxy + XYx - XYxy$
7	$y - Zy + Zyz + Yx - Yxy$
8	$yz + x - xy + Zy - Zyz + Yy - Yyz - Yx + Yxy - 2YZy + 2YZyz - Xx + Xxy + 2XYx - 2XYxy$
9	$y - YZy + YZyz + XYx - XYxy$
10	$z - yz + xy - Zz + Zyz - Yz + Yy + Yyz - Yxy + YZz - YZy + Xy - Xxy - XYy + XYx$
12	$y - Zy + Zyz - Yy + Yyz + Yx - Yxy + 2YZy - 2YZyz + Xx - Xxy - 2XYx + 2XYxy$
14	$y - Yy + Yyz + YZy - YZyz + Xx - Xxy - XYx + XYxy$
15	$y + Yz - Yy - YZz + YZy - Xy + Xx + XYy - XYx$
17	$y - Zy + Zyz + Yx - Yxy + YZy - YZyz - XYx + XYxy$
20	y

Table 8. Polynomials $f(X, Y, Z, x, y, z)$ for the 15 representative number-conserving rules.

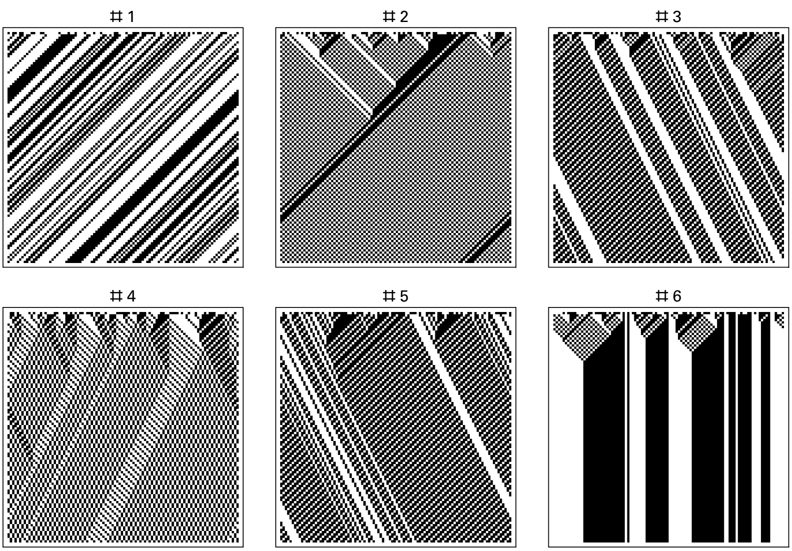

Figure 4. (*continues*).

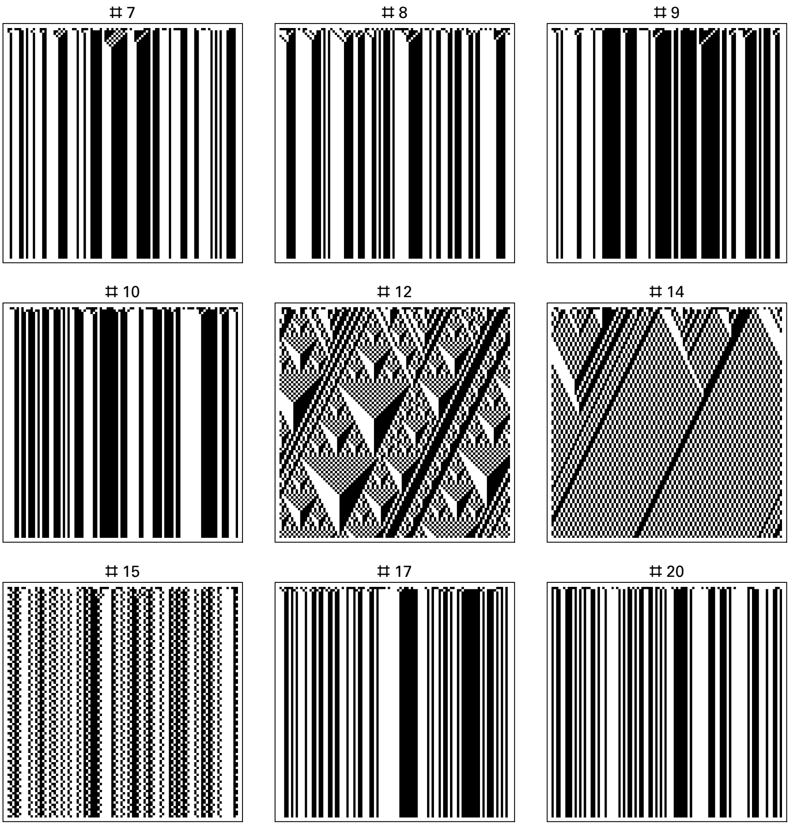

Figure 4. Spatiotemporal diagrams for the 15 representative number-conserving rules.

6. Concluding Remarks

In this paper, we derived a formula for directly determining the coefficient c_j of each term in the polynomial representation of equation (3) of the local rule of the second-order three-neighbor binary cellular automaton (SOCA). Moreover, the 74 number-conserving rules are derived and enumerated from the 2^{64} SOCA rules based on the polynomial representation. This procedure of enumerating number-conserving rules in a cellular automaton (CA) can theoretically be applied to any CA with an arbitrary state, neighbor and/or order. Furthermore, the necessary and sufficient condition for the SOCA rules to be number conserving is derived. The 74 number-conserving rules can be classified into 30 equivalence classes, and the polynomial representations for the 15 representative rules are given in Table 8.

Spatiotemporal diagrams of these rules allow us to observe a fractal pattern or traffic flow phenomena. Detailed analysis of these rules is a subject for future work.

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