The Acoustic Wave Equation in the Expanding Universe: Sachs–Wolfe Theorem

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This article considers the acoustic field propagating in the radiation-dominated ($p = \epsilon / 3$) universe of arbitrary space curvature ($K = 0, \pm 1$). The field equations are reduced to the d'Alembert equation in an auxiliary static Robertson–Walker spacetime and dispersion relations are discussed.

■ Introduction

The Sachs–Wolfe theorem ([1], pp. 76–77) contains two separate results formulated for two different equations of state: the first (i, p. 76) for pressureless matter ($p = 0$) and the second (ii, p. 77) for an ultrarelativistic gas ($p = \epsilon / 3$). The first result was recently recalculated and discussed in [2]. In this article we concentrate on the latter and call it the *acoustic theorem* to distinguish it from the former.

The acoustic theorem refers to the spatially flat ($K = 0$), hot ($p = \epsilon / 3$) Friedmann–Robertson–Walker universe and the scalar perturbation propagating in it. The theorem states that with the appropriate choice of the perturbation variable, one can express the propagation equation in the form of d'Alembert's equation in Minkowski spacetime. Scalar perturbations in the flat, early universe propagate like electromagnetic or gravitational waves ([1], p. 79; see also [3]).

On the other hand, the wave equation for the scalar field of the dust ($p = 0$) cosmological model can be transformed into the d'Alembert equation in the static Robertson–Walker spacetime, regardless of the universe's space curvature (see [4]). Therefore, we can suppose that the flatness assumption in the Sachs–Wolfe theorem is not needed and that the theorem is true in the general case. The proof of this fact, formulated as a symbolic computation, is the subject of this article.

We have found the synchronous coordinate system the most convenient for algorithmic purposes. The code is concise and does not involve concepts other than those covered in

classical papers and textbooks [1, 5, 6]. The perturbation equations can also be reduced to d'Alembert's equation in other gauge-invariant formalisms [7]. The level of difficulty depends on the formalism, but in each case difficulties are computational rather than conceptual.

Originally, the Sachs–Wolfe theorem was published without a detailed proof. The theorem contradicts the heuristic, but popular Jeans instability criterion (instability is not the best word; gravitational and electromagnetic waves show the same kind of behavior [1, p. 79]). Probably these two circumstances are responsible for the Jeans instability criterion's still playing a marginal role in explaining cosmic microwave background radiation fluctuations [8] and cosmological structure genesis.

This article consists of a brief description of the problem and a short verification using *Mathematica*. Throughout, $c = 1$ and $8 \pi G = 1$.

■ Scalar Perturbations on the Friedmann–Robertson–Walker Background

We adopt Robertson–Walker metrics in spherical coordinates $x^\sigma = \{\eta, \chi, \vartheta, \phi\}$:

$$g_{(\mathcal{RW})} = a^2(\eta) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{\sin^2\!\left(\sqrt{K}\,\chi\right)}{K} & 0 \\ 0 & 0 & 0 & \dfrac{\sin^2\!\left(\sqrt{K}\,\chi\right)\sin^2(\vartheta)}{K} \end{pmatrix} \tag{1}$$

with the scale factor $a(\eta)$ appropriate for the equation of state $p = \epsilon / 3$,

$$a(\eta) = \frac{\sin\!\left(\sqrt{K}\,\chi\right)}{\sqrt{K}}. \tag{2}$$

Consider the scalar perturbations in the synchronous system of reference. The metric tensor correction is then determined by two scalar functions $\mathcal{E}$ and C [9]:

$$\delta g_{\mu 0} = 0, \tag{3}$$

$$\delta g_{mn} = a^2 \left(\nabla_m \nabla_n \mathcal{E} + \frac{1}{3}\,(C - \triangle \mathcal{E})\, g_{mn} \right). \tag{4}$$

Here $\triangle$ stands for the Beltrami–Laplace operator on the $\eta = $ constant hypersurface. Eventually, the metric takes the form

$$g_{\mu\nu} = g_{(\mathcal{RW})\,\mu\nu} + \delta g_{\mu\nu}. \tag{5}$$

To first order in the perturbation expansion, the Einstein equations with the metric tensor (5) reduce to

$$\frac{\partial^2}{\partial \eta^2} \lambda(x^\sigma) = -2 \frac{a'(\eta)}{a(\eta)} \frac{\partial}{\partial \eta} \lambda(x^\sigma) - \frac{1}{3} \triangle (\lambda(x^\sigma) + \mu(x^\sigma)), \tag{6}$$

$$\frac{\partial^2}{\partial \eta^2} \mu(x^\sigma) =$$

$$-\left(2 + 3 c^2(\eta)\right) \frac{a'(\eta)}{a(\eta)} \frac{\partial}{\partial \eta} \mu(x^\sigma) + \left(\frac{1}{3} + c^2(\eta)\right) (3 K + \triangle) (\lambda(x^\sigma) + \mu(x^\sigma)), \tag{7}$$

where we set $\mu = C$, $\lambda = -\triangle \mathcal{E}$ (in agreement with [5, 6]); $c(\eta)$ stands for sound velocity. The density contrast $\delta = \delta \epsilon / \epsilon$ reads

$$\delta(x^\sigma) = \frac{1}{3 \epsilon(\eta) a^2(\eta)} \left(3 \frac{a'(\eta)}{a(\eta)} \frac{\partial}{\partial \eta} \mu(x^\sigma) - (3 K + \triangle) (\lambda(x^\sigma) + \mu(x^\sigma))\right). \tag{8}$$

Let us define a new perturbation variable Ψ with the help of the second-order differential transformation of the density contrast δ,

$$\Psi(x^\sigma) = \frac{1}{\cos\left(\sqrt{K} \, \chi\right)} \frac{\partial}{\partial \eta} \left(\frac{K}{\tan^2\left(\sqrt{K} \, \chi\right)} \frac{\partial}{\partial \eta} \left(\frac{\tan^2\left(\sqrt{K} \, \chi\right)}{K} \cos\left(\sqrt{K} \, \chi\right) \delta(x^\sigma)\right)\right). \tag{9}$$

The function $\Psi(x^\sigma)$ is the solution of the d'Alembert equation

$$\frac{\partial^2}{\partial \eta^2} \Psi(x^\sigma) - \frac{1}{3} \triangle \Psi(x^\sigma) = 0 \tag{10}$$

with the Beltrami–Laplace operator $\triangle = {}^{(3)}g_{mn} \nabla^m \nabla^n$ acting in this space,

$$^{(3)}g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sin^2\left(\sqrt{K} \, \chi\right)}{K} & 0 \\ 0 & 0 & \frac{\sin^2\left(\sqrt{K} \, \chi\right) \sin^2(\vartheta)}{K} \end{pmatrix}. \tag{11}$$

Sachs–Wolfe Acoustic Theorem for $K = 0, \pm 1$

Scalar perturbations in the hot ($p = \epsilon / 3$) Friedmann–Robertson–Walker universe of arbitrary space curvature ($K = 0, \pm 1$) expressed in terms of the perturbation variable (9) *obey the wave equation* (10) *in the static Robertson–Walker spacetime $g = \text{diag}\left(-1, {}^{(3)}g\right)$.*

Proof

We perform this calculation in *Mathematica*. The commands are reproduced below and also can be downloaded from drac.oa.uj.edu.pl/usr/woszcz/kody/acoucticRWcode.nb.

☐ *Mathematica* Verification

▪ *Coordinates*

```
Clear[K, a, c, ε, H, Lap, Ψ];
X = {η, χ, θ, φ}; XX = {η_, χ_, θ_, φ_};
x = Sequence @@ X;
xx = Sequence @@ XX;
```

▪ *Friedmann–Robertson–Walker Background for $\epsilon = 3\,p$*

$$c[\eta\_] = \sqrt{\frac{1}{3}}\ ;$$

$$a[\eta\_] = \text{Limit}\left[\frac{\text{Sin}\left[\sqrt{\varkappa}\ \eta\right]}{\sqrt{\varkappa}}, \varkappa \to K\right];$$

$$H[\eta\_] = \frac{a'[\eta]}{a[\eta]^2}\ \text{// Simplify};$$

$$\epsilon[\eta\_] = \frac{3\,K}{a[\eta]^2} + 3\,H[\eta]^2\ \text{// Simplify};$$

▪ *Lifshitz–Khalatnikov Perturbation Equations (Partial Differential Form)*

$$\lambda^{(2,0,0,0)}[xx] = -\frac{2\,a'[\eta]}{a[\eta]}\,\partial_\eta\lambda[x] - \frac{1}{3}\,(\text{Lap}[\lambda][x] + \text{Lap}[\mu][x]);$$

$$\mu^{(2,0,0,0)}[xx] = -\left(2 + 3\,c[\eta]^2\right)\frac{a'[\eta]}{a[\eta]}\,\partial_\eta\mu[x] +$$

$$\frac{1}{3}\left(1 + 3\,c[\eta]^2\right)$$

$$((\text{Lap}[\lambda][x] + 3\,K\,\lambda[x]) + (\text{Lap}[\mu][x] + 3\,K\,\mu[x]));$$

■ *The Density Contrast*

```
δ[xx] =
       1
   ──────────
   3 ϵ[η] a[η]²

       ( - ((Lap[λ][x] + 3 K λ[x]) + (Lap[μ][x] + 3 K μ[x])) +

            a'[η]
         3 ──── ∂_{η,1} μ[x] ) // Simplify;
            a[η]
```

■ *Higher-Order Derivatives*

```
λ^(k_Integer/;k>2,0,0,0) [xx] := D[λ^(2,0,0,0) [x], {η, k - 2}]
μ^(k_Integer/;k>2,0,0,0) [xx] := D[μ^(2,0,0,0) [x], {η, k - 2}]
λ^(2,d1_,d2_,d3_) [xx] = D[λ^(2,0,0,0) [x], {χ, d1}, {ϑ, d2}, {φ, d3}];
μ^(2,d1_,d2_,d3_) [xx] = D[μ^(2,0,0,0) [x], {χ, d1}, {ϑ, d2}, {φ, d3}];
```

■ *Laplacian in the Maximally Symmetric Three-Dimensional Curved Space*

```
w[K_, χ_] = Limit[2 √κ Cot[√κ χ], κ → K];
Lap[f_][xx] =
        w[K, χ]²
    ──────────────── (Csc[ϑ]² ∂_{φ,2} f[x] + Cot[ϑ] ∂_ϑ f[x] + ∂_{ϑ,2} f[x]) +
    4 Cos[√K χ]²

      w[K, χ] ∂_χ f[x] + ∂_{χ,2} f[x];
```

■ *The Gauge-Invariant Variable* Ψ

```
                   Tan[√κ η]²
τ[K_, η_] = Limit[ ──────────, κ → K];
                       κ

Φ[xx] =
        (    1
         ────────── ∂_{η,1} (    1
         Cos[√K η]              ───────── ∂_η (τ[K, η] Cos[√K η] δ[x]))));
                                τ[K, η]
```

■ *The Wave Equation*

```
∂{η,2} Ψ[x] - 1/3 Lap[Ψ][x] == 0;
```

```
Timing[Simplify[%, TimeConstraint → 5000]]
```

```
{559.997, True}
```

■ Dispersion Relations

Equation (10) reduces the problem of cosmological density perturbations to a field theory on a curved background (see [10]). Acoustic waves are dispersed by the space curvature. The dispersion relation for equation (10) takes the form ([10] equation (5.27))

$$\omega = \sqrt{\frac{1}{3}\left(k^2 - K\right)}\,, \tag{12}$$

and the group velocity becomes

$$v_g = \frac{\partial}{\partial k}\,\omega = \frac{k}{\sqrt{3}\,\sqrt{k^2 - K}}\,. \tag{13}$$

In flat space ($K = 0$), the group velocity is constant and equal to $1/\sqrt{3}$. For $K \neq 0$, the dispersion relation is nonlinear. In a space of negative curvature ($K = -1$), the waves behave as a scalar field with mass $m = 1$. The group velocity is a function of k and decreases to zero in the limit as $k \to 0$,

$$\lim_{k \to 0} v_g = 0, \tag{14}$$

while the limit frequency is still positive

$$\lim_{k \to 0} \omega = \frac{1}{\sqrt{3}}\,. \tag{15}$$

Acoustic behavior does not extend to the solutions for imaginary k such that $k^2 \in (-1, 0)$ (supplementary series [11, 12], supercurvature modes [13]).

In a closed universe ($K = 1$), the wave numbers are integers and $1 < k$. The general solution is a countable combination of hyperspherical functions.

Using the basis functions $\Psi(x^\sigma)$, one can reconstruct the solution for the density fraction $\delta(x^\sigma)$ by means of the reciprocal relation (9). Compare the analogous procedure for the scalar field [4]. Reconstructing $\delta(x^\sigma)$ makes it necessary to restore the less desirable gauge ambiguity.

■ Conclusion

A generic solution to equation (10) forms a random wave field in some auxiliary static Robertson–Walker spacetime. The space curvature K is the only quantity that affects the wave dispersion. Neither the Jeans scale nor the horizon size appears in equation (10); consequently, the dispersion relation (12) does not indicate them as critical values. Critical scales of the Jeans type, which emerge in some Newtonian approaches, disappear in a fully relativistic treatment. Equation (10) does not confirm the existence of *frozen modes* [14]. The waves, whether inside or outside the horizon, propagate in the same manner (for $K = 0$ with the same phase and group velocity), and do not change at the horizon crossing. Frozen modes (implemented in cosmological software [15]) involved in deciphering cosmological parameters [8] are not solutions consistent with general relativity.

A further consequence of the Sachs–Wolfe acoustic theorem is that the canonical formalism is adequate to define the constants of motion for a scalar perturbations field. For $K = 0$, the construction of conserved quantities (energy, momentum, and angular momentum) is fairly obvious and reproduces that from other scalar fields in Minkowski space [10]. For $K < 0$, the hyperbolic momentum must be introduced and the Fourier expansion limited to the principal series of the Laplacian eigenfunctions.

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