

Cellular Automata as Simple Self-Organizing Systems

Stephen Wolfram*

Physics Department, California Institute of Technology, Pasadena CA 91125

(July 1982; revised November 1982)

ABSTRACT

Cellular automata provide simple discrete deterministic mathematical models for physical, biological and computational systems. Despite their simple construction, cellular automata are shown to be capable of complicated behaviour, and to generate complex patterns with universal features. An outline of their statistical mechanics is given.

* Address from January 1983: Institute for Advanced Study, Princeton NJ 08540.

Introduction

An "elementary" cellular automaton consists of a sequence of sites carrying values 0 or 1 arranged on a line. The value at each site evolves deterministically with time according to a set of definite rules involving the values of its nearest neighbours. In general, the sites of a cellular automaton may be arranged on any regular lattice, and each site may take on any discrete set of values. This article concentrates on the case of "elementary" cellular automata in one dimension with binary values at each site, and shows that despite their simple construction, such systems can exhibit complicated behaviour. Details, extensions and further discussion, together with more extensive references, are given in ref. [1].

Cellular automata were introduced by von Neumann and Ulam as simple models in which to study biological processes such as self-reproduction [2]. Any system with many identical discrete elements undergoing deterministic local interactions may be modelled as a cellular automaton. Non-trivial cellular automata are obtained when the local evolution is non-linear. Physical examples may be found in aggregation phenomena such as snowflake growth (c.f. [3]). Biological examples are found when organisms grow into complicated forms by repeated application of simple local rules (e.g. [4]). Mathematical systems akin to cellular automata are found in number theory [5]. The solitaire computer game of "Life" [6] is an example of a two-dimensional cellular automaton.

Cellular automata may be considered as (parallel-processing) computers, in which the initial configuration encodes the program and input data, and time evolution yields the final output (e.g. [2]). Sufficiently complicated cellular automata (such as the game of "Life" [7]) are found to be "computationally universal" (e.g. [8]), and thus behave as "general-purpose" computers, capable of evaluating any Turing computable function given appropriate input. According to Church's thesis in the formal theory of computation, such cellular automata may thus potentially simulate any possible system.

Figure 1 shows an example of a set of local rules for an elementary cellular automaton. Each of the eight possible sets of values for a site and its nearest neighbours appear on the upper line, while the lower line gives the value to be taken by the central site on the next time step. These rules are applied synchronously to each site at every time step. Thus for example, the sequence 010110110 becomes -0011011- after one time step according to the rule illustrated in fig. 1 (the two end sites depend on unspecified values). Rules may be interpreted as Boolean operations on the values of the three sites in each neighbourhood. Thus, for example, the rule illustrated in fig. 1 may be considered to take the value of a site to be the sum modulo two of the values of its two neighbours on the previous time step. Rules may be denoted by the decimal equivalents of their binary specifications: fig. 1 thus gives rule $01011010_2 = 90$. Since any sequence of eight binary digits corresponds to a (elementary) cellular automaton rule in analogy with fig. 1, there are $2^{(2^3)} = 256$ possible such rules.

Only the 32 rules of the form $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ satisfy reflection symmetry and leave the "quiescent" configuration -000000- unchanged, and are therefore considered "legal".

Simple initial states

Figure 2 shows the "growth" of a pattern from a "seed" consisting of a single site with value 1 (surrounded by value 0 sites) according to each of the 32 possible (legal) elementary cellular automaton rules. The evolution is shown until a particular configuration appears for the second time (a "cycle" is detected) or for at most 20 time steps. The patterns generated by all rules are seen to fall into a few classes. In one class, the initial 1 is either immediately erased (as by rule 0) or maintained unchanged forever (rule 4). A second class of rules (exemplified by 50 or 122) copy the initial 1 to generate a uniform structure analogous to a perfect crystal. Rules such as 18, 22 and 90 form a third class of "complex" rules which generate complicated patterns.

Figure 3 shows two constructions for the pattern generated by rule 90. Since this rule takes each site to be the sum of the previous values of its neighbours modulo two, the pattern it generates is simply Pascal's triangle modulo two (or the binomial coefficients in the expansion of the generating function $(1+x)^T$ modulo two). (Rule 90 may also be interpreted as generating "stunted" binary trees, in which two diagonal branches grow from each nonzero site at each time step, but are inhibited if they would collide.) The final pattern of nonzero sites is obtained as the limit of the recursive geometrical construction shown in fig. 3. This pattern is seen to be "self-similar" or "scale invariant", in that views with different "magnifications" (but the same "resolution") are indistinguishable. Scale invariance is to be expected, since the cellular automaton defines no intrinsic scale (except the size of a single site) in the large time limit.

The pattern shown in fig. 3 contains many congruent (inverted) triangles with base lengths of 2^k sites. At large times, the number of triangles of size $n=2^k$ is given by $T(n/2)=3T(n)$ so that $T(n)\sim n^{-\log_2 3}\sim n^{-1.59}$. The "fractal" dimension [9] of the pattern is thus ≈ 1.59 . A uniform pattern, as generated by rule 50, has fractal dimension two, while a line, as generated by rule 4, has fractal dimension one.

Figure 2 shows that all "complex" cellular automaton rules yield asymptotically self-similar patterns. All give the same fractal dimension $\log_2 3 \approx 1.59$ except for rule 150 which gives a fractal dimension $1+\log_2 \varphi \approx 1.69$ where $\varphi=(1+\sqrt{5})/2 \approx 1.618$ is the "golden ratio".

Patterns generated by cellular automaton evolution from "seeds" containing not one but several nonzero sites are found to be similar to those of fig. 2, at least on scales much larger than the region of nonzero initial sites, and exhibit the same fractal dimensions. The generation of self-similar patterns is thus a generic feature of cellular automaton evolution from simple "seeds". This result

may provide some explanation for the widespread occurrence of self similarity in natural systems [9], and suggests the appearance of fractal dimensions ≈ 1.59 and ≈ 1.69 .

Random initial states

Having considered "ordered" initial states with only a few nonzero sites, we now turn to "disordered" initial states, in which each site is chosen independently to have value 1 with probability p_0 , giving an initial density $\rho_0 = p_0$ of nonzero sites.

Figure 4 shows the density of nonzero sites as a function of time in evolution according to three "complex" cellular automaton rules from a disordered initial state with $\rho_0 = 0.2$. For almost all initial densities, the density tends rapidly to a fixed "equilibrium" limit (although in the case of rule 90, large fluctuations are apparent even at large times). In a "mean field" approximation, the evolution of the density could be estimated by a master equation: however, the presence of non-Markovian effects associated with feedback renders this approach inaccurate in practice. Other methods nevertheless provide exact results for the density in a few cases. One of these cases is rule 90, which exhibits the simplifying feature of "additive superposition". Patterns generated from arbitrary initial configurations according to this rule may be obtained by appropriate superpositions (addition modulo two) of displaced copies of the pattern generated from a single initial nonzero site in fig. 3. The number of nonzero sites in the pattern of fig. 3 after τ time steps is equal to the number of odd binomial coefficients in row τ of Pascal's triangle, and is found by a geometrical method to be (c.f. [10]) $2^{\#_1(\tau)}$. Here $\#_1(n)$ denotes the number of occurrences of the digit 1 in the binary decomposition of the integer n (thus $\#_1(0)=0$, $\#_1(1)=1$, $\#_1(2)=1$, $\#_1(3)=2$, and so on), and has a very irregular form as a function of n . Whenever $\tau = 2^k$, $\#_1(\tau)=1$ and there are only two nonzero sites in the pattern; the maximum of τ nonzero sites is achieved when $\tau = 2^k - 1$. Superposing an initial density ρ_0 of these patterns yields an analytical form for the result shown in fig. 4: $\rho_\tau = \frac{1}{2}[1 - (1 - 2\rho_0)^{2^{\#_1(\tau)}}]$. The additive superposition property of rule 90 is shared by rule 150 (but by no other "complex" rules). The density for rule 150 is found to be $\rho_\tau = \frac{1}{2}[1 - (1 - 2\rho_0)^{N_\tau}]$ where now N_τ is a product of factors $\chi(j)$ associated with each sequence of j ones (delimited by zeroes) in the binary decomposition of the integer τ . $\chi(j)$ is given by the recurrence relation $\chi(j) = (2j \pm 1)\chi(j-1)$ where the upper (lower) sign is taken for j odd (even), and $\chi(1)=3$ (so that $\chi(2)=5$, $\chi(3)=11$ and so on).

Figure 5 shows the evolution of a disordered initial state with density $\frac{1}{2}$ according to each of the 32 possible elementary cellular automaton rules. As in fig. 3, several classes of behaviour are evident. Rules such as 250 evolve rapidly to uniform states. Rules such as 94 or 132 evolve after a few time steps to stable patterns, sometimes involving several independent sections executing short-

period cycles. This behaviour is analogous to that found in dynamical systems with limit points or limit cycles. "Complex" rules, such as 18 and 90, yield complicated patterns, analogous to "strange attractors" in dynamical systems (e.g. [11]). Although the values of initial sites are statistically uncorrelated, the cellular automaton evolution is seen to generate definite structures. One characteristic of this simple "self-organization" is the appearance of long correlated sequences of sites, giving rise to the inverted triangles visible in fig. 5. Triangles of all sizes are found, but their spectrum is exponentially damped. Denoting as before the density of triangles of length n by $T(n)$, one finds that for large n , $T(n)$ takes on the universal form λ^n for all complex rules, and (almost) all initial states. The value of λ is found to be ≈ 0.75 for all (complex) rules except the "additive" ones 90 and 150, which instead give $\lambda=0.5$. Once again, therefore, the statistical properties of the patterns generated by cellular automaton evolution are found to exhibit universality, and to be independent of the details of the cellular automaton rule or the initial state.

Global properties

One may restrict a cellular automaton to contain a finite number N of sites (arranged, for example, on a circle). Then the total number of possible configurations in the "phase space" of the system is 2^N . Cellular automaton evolution consists in the iteration of a mapping between these configurations, with each configuration tracing out a trajectory in phase space. For complex cellular automata one finds that initially nearby trajectories (corresponding to configurations differing at only a few sites) diverge exponentially with time (so that the number of differing sites increases linearly), and the mapping from initial to final configurations becomes apparently "random" after a few time steps (although deviations from a uniform random mapping remain).

Cellular automaton rules have the important property that they may map several initial configurations to the same final configuration, so that the corresponding trajectories merge, and microscopically irreversible time evolution occurs*. Starting from an initial ensemble in which all 2^N possible configurations occur with equal probabilities (corresponding to complete "disorder"), cellular automaton evolution thus irreversibly "concentrates" the probabilities in the ensemble into a small fraction of all the possible configurations, thereby reducing the entropy of the ensemble. The properties of these few "attractor" configurations (or cycles) then dominate ensemble averages, leading to statistically similar results independent of the initial state (c.f. [12]).

* Reversible cellular automata may nevertheless be constructed, for example by allowing a configuration to be determined by two previous configurations, in analogy with the finite difference form of a second-order differential equation in time (E.Fredkin and N.Margolus, private communication).

In a cellular automaton of finite size, every trajectory must eventually be periodic, or must merge with a periodic trajectory. In practice, the length of the "transient" before a trajectory merges is typically $\lesssim N$. The lengths of periodic cycles could in principle be $\sim 2^N$, but in practice are typically $\lesssim 2^{N/2}$. For rules 90 and 150 period lengths may be obtained by algebraic methods [13], and depend on number theoretical properties of N .

The cellular automata considered here are entirely deterministic. However, simulation of some natural processes requires introduction of "noise" into the local rules, yielding non-deterministic cellular automata (a simple example is the Ising model at finite temperature). An arbitrarily small amount of noise prevents stable cyclic behaviour, and causes all 2^N possible states to be visited in evolution from any initial state. However, the statistical properties of structures generated in the evolution appear to degrade continuously as the intensity of noise is increased.

Conclusions

Cellular automata may be used as mathematical models for physical, biological and computational systems. They are simple in construction, and thus potentially amenable to precise mathematical analysis, yet are capable of complicated behaviour. This article has outlined a first step in the analysis of the (non-equilibrium) statistical mechanics of cellular automata, and has described some generic features of their behaviour, such as the formation of particular self-similar patterns. Further investigations along these lines may reveal general universal features of irreversible and non-equilibrium statistical systems analogous to those found in equilibrium reversible systems.

References

1. S.Wolfram, "Statistical Mechanics of Cellular Automata", Caltech preprint CALT-68-915 (May 1982).
2. J.von Neumann, "The General and Logical Theory of Automata", in J.von Neumann, "Collected Works" (ed. A.H.Taub), Vol. 5, p.288; J.von Neumann, "Theory of Self-Reproducing Automata", (ed. A.W.Burks), Univ. of Illinois press (1966); ed. A.W.Burks, "Essays on Cellular Automata", Univ. of Illinois press (1970).
3. J.S.Langer, "Instabilities and Pattern Formation in Crystal Growth", Rev.Mod.Phys. 52 (1980) 1.
4. D'A. W. Thompson, "On Growth and Form", (abridged edition ed. J.T.Bonner), Cambridge Univ. Press. 1961; P.S.Stevens, "Patterns in Nature", Little, Brown & co. (Boston) (1974), chaps. 5 and 6; R.M.Baer and H.M.Martinez, "Automata and Biology", Ann. Rev. Biophys. (1974) 255.

5. J.C.P.Miller, "Periodic Forests of Stunted Trees", Phil. Trans. Roy. Soc. Lond. A266 (1970) 63; A293 (1980) 48.
6. M.Gardner, "Mathematical Games", Sci. Amer. 224 (1971) 112.
7. E.R.Berlekamp, J.H.Conway and R.K.Guy, "Winning Ways, for Your Mathematical Plays", vol. 2, Academic Press (1982), chap. 25; ed. R.T.Wainwright, "Lifeline", nos. 1 (March 1971) through 11 (Sept. 1973); D.J.Buckingham, "Some Facts of Life", Byte 3 (1978) 54; R.T.Wainwright, "Life is Universal!", Proc. Winter Simul. Conf., Washington D.C., ACM (1974).
8. M.L.Minsky, "Computation: Finite and Infinite Machines", Prentice-Hall (1967); M.A.Arbib, "Theories of Abstract Automata", Prentice-Hall (1969); Z.Manna, "Mathematical Theory of Computation", McGraw-Hill (1974).
9. B.Mandelbrot, "Fractals: Form, Chance and Dimension", Freeman (1977).
10. J.W.L.Glaisher, "On the Residue of a Binomial-Theorem Coefficient with Respect to a Prime Modulus", Quart. J. Math. 30 (1899) 150; S.H.Kimball et al., "Odd Binomial Coefficients", Amer. Math. Mon. 65 (1958) 368; J.B.Roberts, "On Binomial Coefficient Residues", Canad. J. Math. 9 (1957) 363; R.Honsberger, "Mathematical Gems II", Dolciani Math. Expositions, Math. Assoc. Amer. (1976), p.1.
11. E.Ott, "Strange Attractors and Chaotic Motions of Dynamical Systems", Rev. Mod. Phys. 53 (1981) 655.
12. G.Nicolis and I.Prigogine, "Self-Organization in Nonequilibrium Systems", Wiley (1977); I.Prigogine, "From Being to Becoming", Freeman (1980).
13. O.Martin, A.Odlyzko and S.Wolfram, "Algebraic Properties of Cellular Automata", Bell Laboratories preprint, in preparation.

Figure captions

Figure 1: Example of a set of local rules for an elementary cellular automaton.

Figure 2: Growth of patterns from simple seeds according to each of the 32 possible legal elementary cellular automaton rules. Configurations at successive time steps are shown on successive lines. Sites with value 1 are represented by stars, and those with value 0 by blanks. The evolution is shown until a particular configuration appears for the second time, or for at most 20 time steps.

Figure 3: Algebraic and geometrical constructions for the pattern generated by evolution according to the modulo two rule 90 from a "seed" consisting of a single nonzero site.

Figure 4: Time evolution of the density of nonzero sites obtained with a disordered initial state of density 0.2.

Figure 5: Evolution of a "disordered" initial state with density 0.5 according to each of the 32 possible legal elementary cellular automaton rules. (Periodic boundary conditions are assumed, but are inessential.) The evolution is shown until a particular configuration appears for the second time, or for at most 30 time steps.

$$\begin{array}{cccccccc}
 \frac{111}{0} & \frac{110}{1} & \frac{101}{0} & \frac{100}{1} & \frac{011}{1} & \frac{010}{0} & \frac{001}{1} & \frac{000}{0}
 \end{array}$$

Figure 1: Example of a set of local rules for an elementary cellular automaton.

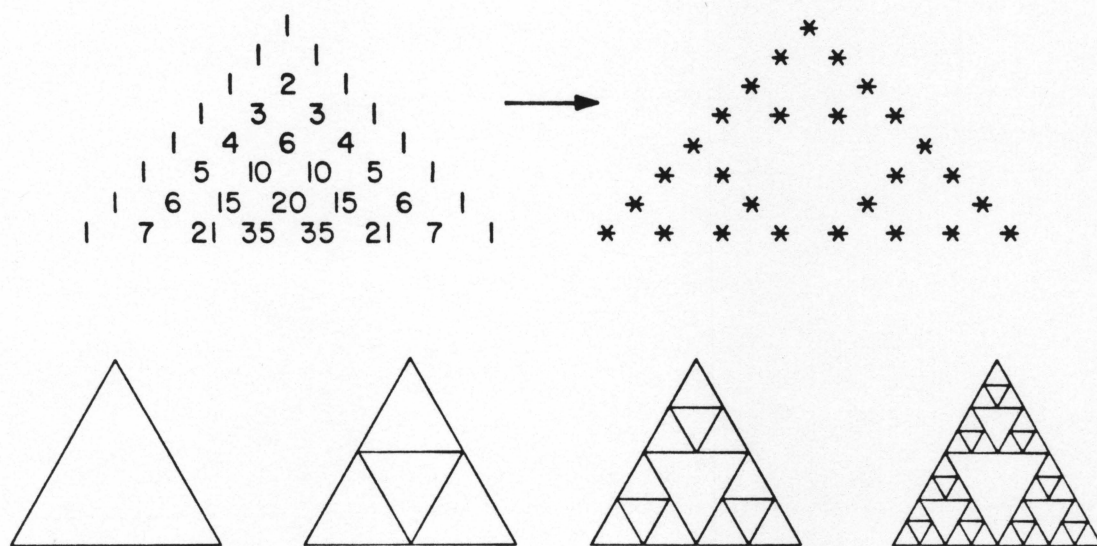


Figure 3: Algebraic and geometrical constructions for the pattern generated by evolution according to the modulo two rule 90 from a "seed" consisting of a single nonzero site.

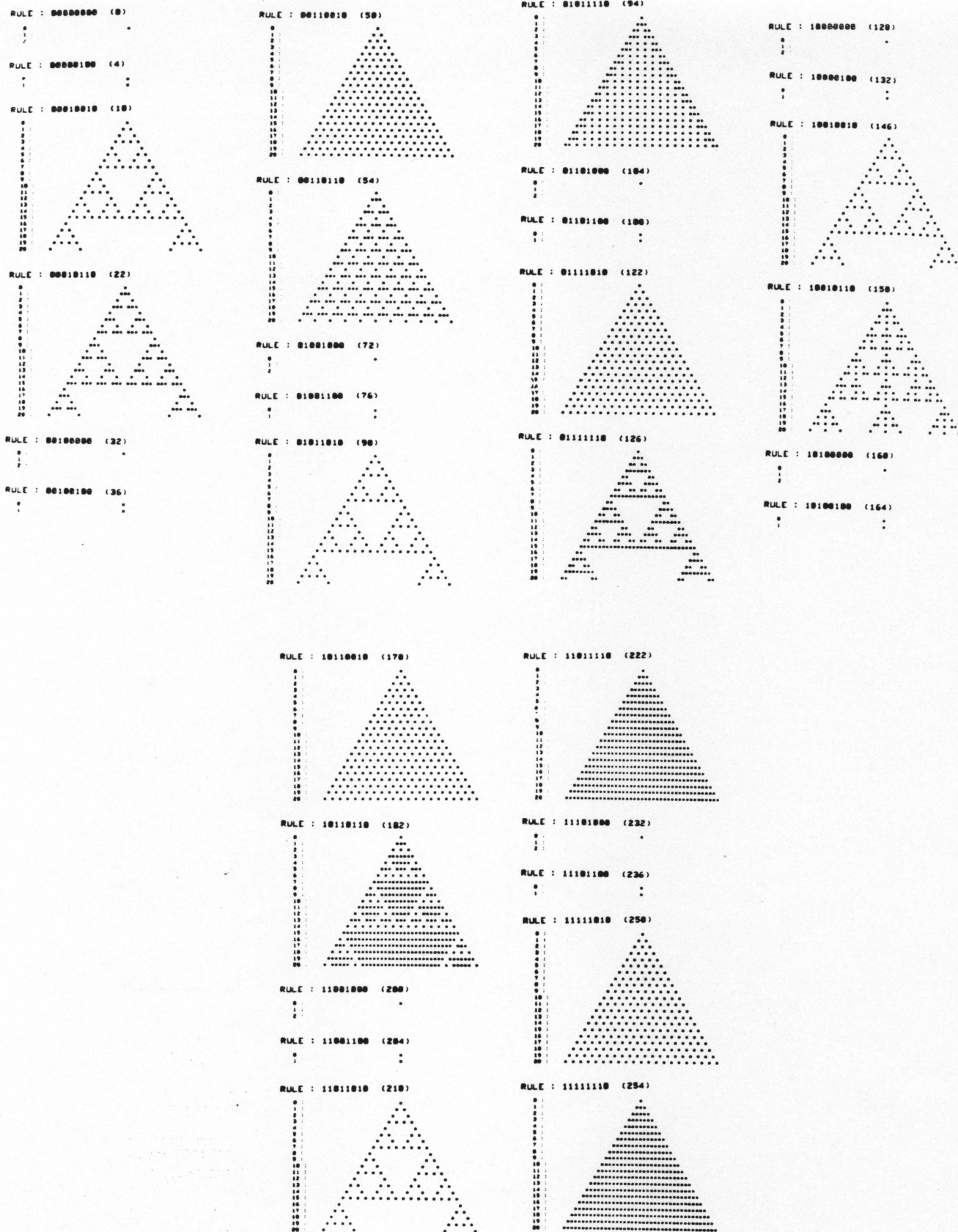


Figure 2: Growth of patterns from simple seeds according to each of the 32 possible legal elementary cellular automaton rules. Configurations at successive time steps are shown on successive lines. Sites with value 1 are represented by stars, and those with value 0 by blanks. The evolution is shown until a particular configuration appears for the second time, or for at most 20 time steps.

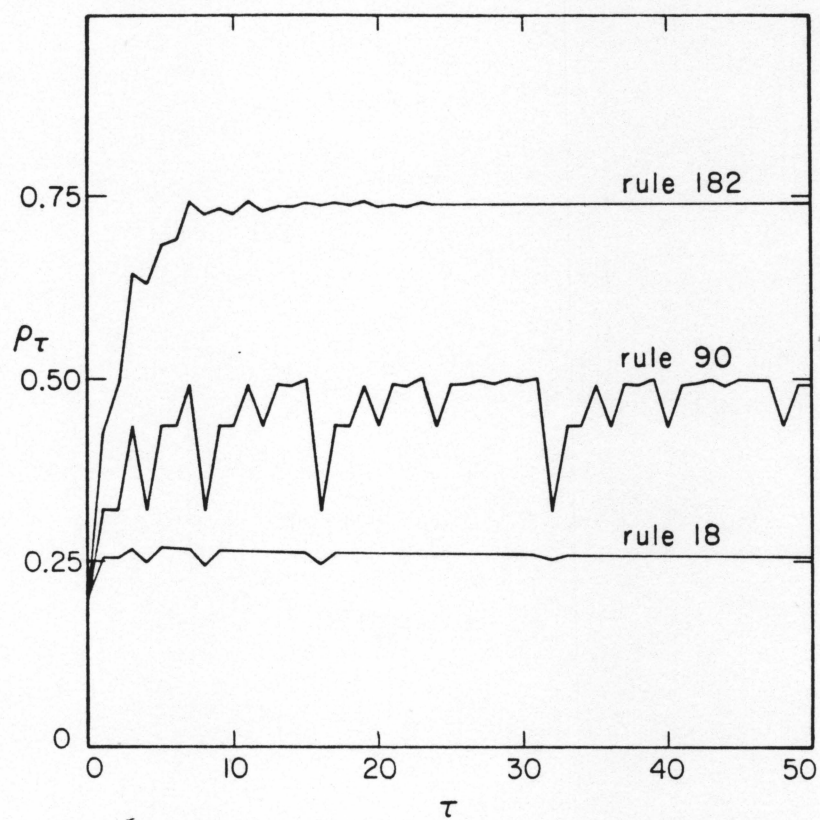


Figure 4: Time evolution of the density of nonzero sites obtained with a disordered initial state of density 0.2.

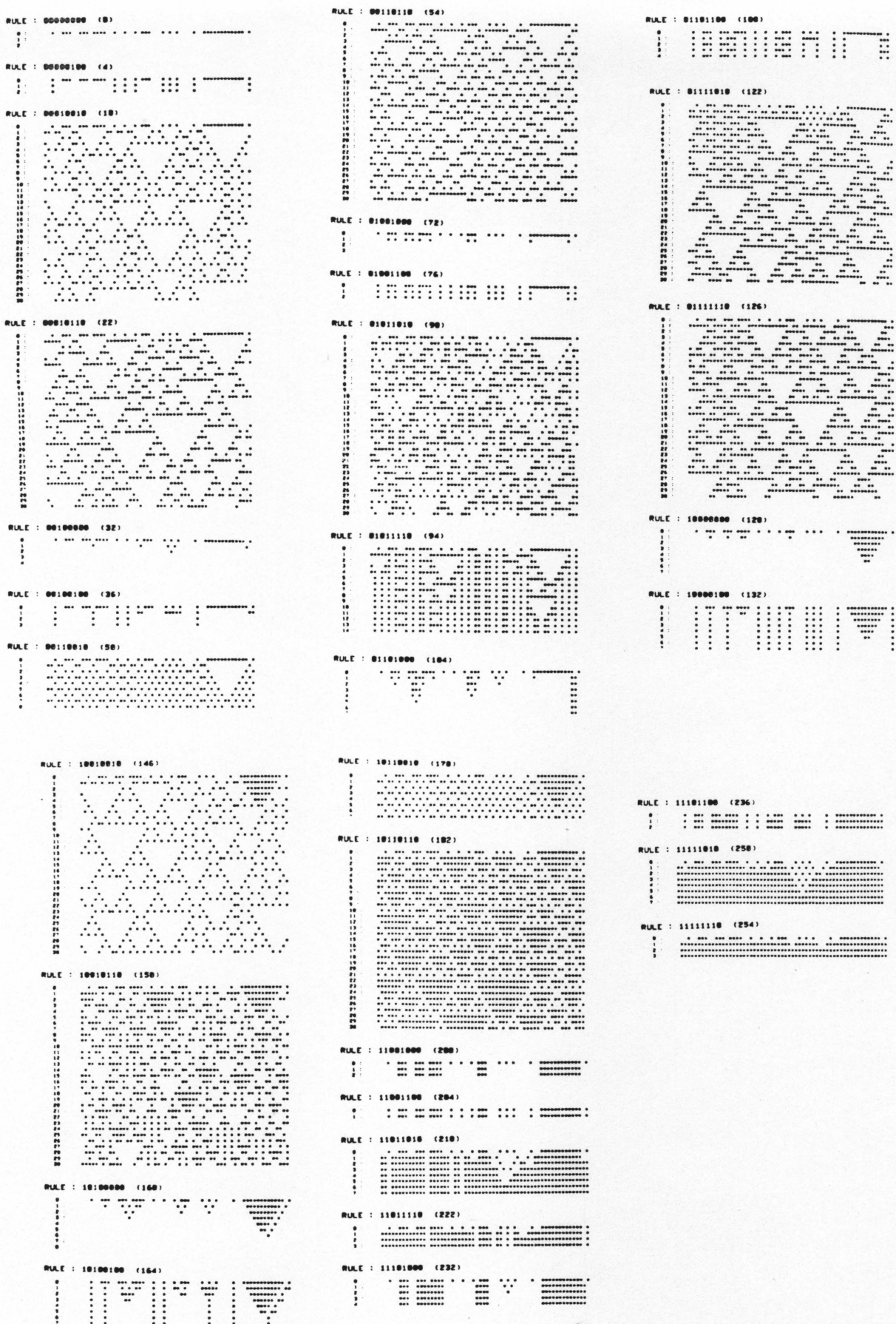


Figure 5: Evolution of a "disordered" initial state with density 0.5 according to each of the 32 possible legal elementary cellular automaton rules. (Periodic boundary conditions are assumed, but are inessential.) The evolution is shown until a particular configuration appears for the second time, or for at most 30 time steps.