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The Effective Coupling in QCD*

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ABSTRACT

The use of an effective coupling in QCD is investigated in the context of a simple class of processes which depend on only one kinematic invariant.

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The coupling $g(\mu^2)$ in QCD depends on the renormalization point μ^2 used to define it. Although the complete rate for any process must be independent of μ^2, it may be that a suitable choice of μ^2 will allow many of the higher order terms in the perturbation series for the rate to be absorbed into the coupling $g(\mu^2)$. In this paper I shall consider primarily processes which depend on only one kinematic invariant. For a particular value of μ^2, all terms in these processes up to $O(g^4)$ may be absorbed into an effective coupling $g(\mu^2)$, but in higher orders, there exist terms which cannot be accounted for in this way. The choice of μ^2 which causes the most terms in the perturbation series to be absorbed into the effective coupling is proportional to the value of the kinematic invariant. However, the constant of proportionality, and hence the magnitude of the effective coupling for a particular value of the invariant, depends sensitively on the details of the process considered. I shall discuss at length processes of the form (G is a gluon)

$$A \to B\,G \quad \lfloor\!\!\to \text{anything} \qquad , \tag{1}$$

and then give a briefer discussion of several other processes. The non-Abelian nature of QCD introduces inessential complications into the discussion, and so I shall ignore it throughout. With this simplification (which is equivalent to considering QED rather than QCD), the square of the amplitude for (1) to order g^6 is given by the classes of diagrams in Fig. 1. I assume that there can be no gluon corrections to the A or B lines (the gluon may be considered to carry conserved quantum numbers possessed by A but not by B). I take the effective ABG vertex to be of the form[F1]

$$F(s) = g\left(1 - \frac{s}{\Delta^2}\right)^{(n/2)} \qquad , \tag{2}$$

where s is the invariant mass of the gluon, and $\Delta = m_A - m_B$.

If the quark mass (m_q) were taken to be non-zero, then the gluon propagator could be renormalized on shell, so that the diagrams giving corrections to single real gluon emission would not contribute. In that case the total rate for (1) is given by (removing a trivial phase space factor for the diagram {2})

$$\Gamma = |F(0)|^2 + \frac{1}{\pi} \int_{4 m_q^2}^{\Delta^2} |F(s)|^2 \, \mathrm{Im}[\Pi(s)] \frac{ds}{s} \quad , \tag{3}$$

where $\Pi(s)$ is the reduced gluon vacuum polarization operator, including improper vacuum polarization diagrams such as {6a}. To lowest order, as usual

$$\mathrm{Im}[\Pi(s)] = \frac{g^2}{12\pi^2} \left(1 + \frac{2 m_q^2}{s}\right) \left(1 - \frac{4 m_q^2}{s}\right)^{1/2} \theta(s - 4 m_q^2) \quad . \tag{4}$$

To order g^4, therefore, taking n = 0, the diagrams {2} and {4a} give

$$\Gamma = g^2 \left\{ 1 + \frac{g^2}{4\pi^2} \left[\frac{1}{3} \log \left(\frac{1 + \sqrt{1 - 4 m_q^2/\Delta^2}}{1 - \sqrt{1 - 4 m_q^2/\Delta^2}} \right) - \right. \right.$$

$$\left. \left. - \frac{\sqrt{1 - 4 m_q^2/\Delta^2}}{9} \left(5 + \frac{4 m_q^2}{s}\right) \right] \right\}$$

$$\simeq g^2 \left\{ 1 + \frac{g^2}{4\pi^2} \left[\frac{1}{3} \log \left(\frac{\Delta^2}{m_q^2} \right) - \frac{5}{9} \right] \right\} \quad , \tag{5}$$

for $\Delta^2 >> m_q^2$. This rate clearly diverges as $m_q^2 \to 0$. For $m_q = 0$, however, one must perform renormalization at $s = \mu^2 \neq 0$. In that case, the diagram {4b} gives a contribution (a $q\bar{q}$ pair with zero invariant mass cannot be distinguished from a real gluon)

$$\Gamma_{\{4b\}} \simeq - \frac{g^4}{12\pi^2} \log \left(\frac{\mu^2}{m_q^2} \right) \quad , \tag{6}$$

if $\mu^2 >> m_q^2$. Adding (5) and (6) the complete rate for the process (1) to $O(g^4)$ in the limit $m_q^2 \to 0$ becomes

$$\Gamma = g^2(\mu^2)\left\{1 + \frac{g^2(\mu^2)}{4\pi^2}\left[\frac{1}{3}\log\left(\frac{\Delta^2}{\mu^2}\right) - a\right]\right\} \quad , \tag{7}$$

where a = 5/9 if n = 0.

In eq. (7) I have displayed the fact that the couplings depend on the renormalization point μ^2; they obey the renormalization group equation (RGE) $\mu \frac{\partial g}{\partial \mu} = \beta(g)$. Of course, the renormalizability of QCD guarantees that at each order in g^2 the rate Γ cannot depend on μ^2, so that $\mu \frac{\partial \Gamma}{\partial \mu} = \beta(g)\frac{\partial \Gamma}{\partial g}$. Nevertheless, by a suitable choice of μ^2 the expression in braces in (7) may be absorbed into the effective coupling $g(\mu^2)$. Keeping only the $O(g^3)$ term in $\beta(g)$ one has (e.g., [1])

$$g^2(\mu^2) \simeq \frac{g^2(\mu_0^2)}{1 - \frac{g^2(\mu_0^2)}{12\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)}$$

$$= \frac{12\pi^2}{\log(\mu^2/\Lambda^2)} \quad , \tag{8}$$

where Λ is a renormalization group invariant mass[F2] used to set the scale of the effective coupling. If the constant a in eq. (7) were zero, then by choosing $\mu^2 = \Delta^2$ the rate Γ could be rewritten (at least to $O(g^4)$) as $\Gamma = g^2(\Delta^2)$. This expression for the rate could be written in terms of $g(\mu^2)$ again by using the first form in eq. (8), giving

$$\Gamma = g^2(\mu^2)\left[1 + \frac{g^2(\mu^2)}{12\pi^2}\log\left(\frac{\Delta^2}{\mu^2}\right) + \left(\frac{g^2(\mu^2)}{12\pi^2}\log\left(\frac{\Delta^2}{\mu^2}\right)\right)^2 + \ldots\right] \quad , \tag{9}$$

which agrees with (7) (with a = 0) to $O(g^4)$, but contains further higher-order terms. The form (9) is a conventional 'renormalization group improved' estimate for Γ.

For any physical value of n, however, a is non-zero. Hence a re-expansion of $\Gamma = g^2(\Delta^2)$ in terms of $g(\mu^2)$ will not give the correct form (7); the 'constant' term proportional to a will be missing. A simple trick may, however, be used to remedy this deficiency. Instead of choosing $\mu^2 = \Delta^2$, take $\mu^2 = \Delta^2/\eta^2$ and arrange η so that the term in brackets in eq. (7) vanishes (i.e., $\eta = \exp(3a)$). In that case, the 'renormalization group improved' estimate $\Gamma = g^2(\Delta^2/\eta^2)$ will agree with the full perturbation theory result, at least to order g^4. Instead of taking $\mu^2 = \Delta^2/\eta^2$, one could take $\mu^2 = \Delta^2$, but replace the Λ^2 in the eq. (8) for $g^2(\mu^2)$ by an effective $\bar{\Lambda}^2 = \eta^2\Lambda^2$ [F2].

At $O(g^2)$(in the 'unimproved' perturbation series) the coupling is fixed. Higher-order terms may be interpreted as inducing a dependence of the coupling on μ^2. Since this dependence is governed by the RGE, some of the higher-order terms in the perturbation series for Γ are determined. In particular, the coefficient of the $\log\left(\frac{\Delta^2}{\mu^2}\right)$ in eq. (7) is determined. However, the constant a is not, and depends on the value of n which characterizes the ABG vertex. Hence the choice of $\mu^2 = \Delta^2/\eta^2$ in $\Gamma = g^2(\mu^2)$ which gives the correct non-logarithmic term at $O(g^4)$ depends on n. Alternatively, the effective value of $\bar{\Lambda}^2 = \eta^2\Lambda^2$ if $\Gamma = g^2(\Delta^2)$ is sensitive to constant terms at $O(g^4)$ and hence to n[F3]. Table 1 gives the values of a and η for various n[F4]. Note the large difference of η from one in all cases, and the strong dependence of η on n. Let us temporarily ignore $O(g^6)$ and higher terms. Then eq. (7) gives the exact rate for the process (1) which could, in principle, be measured experimentally. The experimental measurement could be used to determine the effective QCD coupling. Of course, the measured size of the coupling could be fit to the formula of eq. (8) and an effective value of Λ^2 deduced. However, because of the presence of constant terms, this effective Λ^2 will differ from the true Λ^2. It is clear from Table 1 that the difference is rather large, and depends considerably on n. To $O(g^4)$ the rates for all processes which depend on only one invariant will have the form (7), so that the same phenomena will occur. For example, if the measured cross-section for $e^+e^- \to$ hadrons is fit to the naive QCD prediction $(1 + g^2(Q^2)/4\pi^2)\sigma_0$ then the effective Λ^2 deduced will not necessarily be even close to the true value of Λ^2. To obtain a better estimate of the true Λ^2 from the experimental $\sigma(e^+e^- \to \text{hadrons})$, one must compute the constant a in (F is the effective number of quark flavors)

$$\sigma \simeq \sigma_0 \left(1 + \frac{g^2(\mu^2)}{4\pi^2}\left\{1 + \frac{g^2(\mu^2)}{4\pi^2}\left[\frac{(33-2F)}{12}\log\left(\frac{Q^2}{\mu^2}\right) + a\right] + O(g^4)\right\}\right) . \tag{10}$$

Agreement between values of Λ^2 deduced from experimental measurements of various processes without making at least this correction is clearly not a direct prediction of QCD. Rather it will occur only if (unlike the case of changing n) the constant a does not differ significantly between the processes considered.

I return now to the process (1), and discuss the $O(g^6)$ contributions to its rate. Examples of the relevant classes of diagrams are given in Fig. 1. The iterated ('improper') $O(g^2)$ vacuum polarization diagrams ({6a}, {6b}) contribute

$$\frac{g^6}{(4\pi^2)^2}\left[\frac{1}{9}\log^2\left(\frac{\Delta^2}{\mu^2}\right) - 2\,a\log\left(\frac{\Delta^2}{\mu^2}\right) + b_1\right] , \tag{11}$$

to Γ, while the diagrams involving true $O(g^4)$ vacuum polarization contribute

$$\frac{g^6}{(4\pi^2)^2}\left[\frac{1}{4}\log\left(\frac{\Delta^2}{\mu^2}\right) \mp b_2\right] . \tag{12}$$

The coefficient of the $\log^2$ in (11) corresponds to the $O(g^6)$ term in the expansion of the effective coupling (8) deduced from the $O(g^3)$ term in $\beta(g)$, while the coefficient of the log in (12) corresponds to the $O(g^5)$ term [2] in $\beta(g)$[F5]. The coefficient of the log in (11) depends on the constant a at $O(g^4)$.

If the effective coupling is computed from $\mu\frac{\partial g}{\partial\mu} = \beta(g)$ keeping up to $O(g^5)$ in $\beta(g)$[F4], and then this coupling, evaluated at $\mu^2 = \Lambda^2$, is used to estimate the rate for (1) using $\Gamma = g^2(\Lambda^2)$, then not only will the constant term at $O(g^4)$ be missed, but also a part of the log at $O(g^6)$. Of course, the same choice of η which accounts for the constant at $O(g^4)$ will also account for the log at $O(g^6)$. However, the constant terms $b = b_1 + b_2$ at $O(g^6)$ remain unaccounted for even when $\Gamma = g^2(\Lambda^2/\eta^2)$. If $n = 0$, then $b_1 = \frac{25}{81} - \pi^2/27$ and the full form of Γ to $O(g^6)$ is [3]

$$\Gamma = g^2(\mu^2)\left\{1 + \frac{g^2(\mu^2)}{4\pi^2}\left[\frac{1}{3}\log\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{5}{9}\right]\right.$$
$$\left. + \left(\frac{g^2(\mu^2)}{4\pi^2}\right)^2\left[\frac{1}{9}\log^2\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{13}{108}\log\left(\frac{\Lambda^2}{\mu^2}\right) + \left(\zeta(3) - \frac{\pi^2}{27} + \frac{65}{648}\right)\right]\right\} . \tag{13}$$

I showed above that to $O(g^4)$, a suitable choice of η allows the rate for (1) to be written simply as $g^2(\Lambda^2/\eta^2)$. The value of η is chosen so that the expansion of the 'renormalization group improved' effective coupling $g^2(\Lambda^2/\eta^2)$ (obtained from $\mu\frac{\partial g}{\partial\mu} = \beta(g)$) in terms of $g^2(\mu_0^2)$ agrees with the explicit perturbation theory result (7) for Γ to $O(g^4(\mu_0^2))$. However, if $g^2(\Lambda^2/\eta^2)$ is expanded to $O(g^6(\mu_0^2))$ there is no longer a choice of η such that $\Gamma(\Lambda^2/\eta^2) = g^2(\Lambda^2/\eta^2)$ to $O(g^6(\mu_0^2))$. This is because the equation to determine log (η) both depends on $g^2(\mu_0^2)$ and in general has no real roots (if only the $O(g^6)$ terms are considered, the quadratic equation for log (η) has no real roots). Hence the trick of changing the renormalization point or of modifying the effective Λ^2 to account for the constant terms at $O(g^4)$ cannot be used at $O(g^6)$, and the naive prescription for including higher-order effects by replacing the g^2 encountered in a lowest-order calculation by $g^2(\mu^2)$ has failed. Nevertheless, whereas the effective value of Λ^2 is very sensitive to the $O(g^4)$ terms in Γ, it is not particularly affected by the $0(g^6)$ terms, at least for g^2 sufficiently small that perturbation theory should be reliable. The change in $\bar{\Lambda}^2$ due to $0(g^6)$ terms is given very approximately by $(\bar{\Lambda}^2)'/(\bar{\Lambda}^2) \simeq \exp(3g^2 b/4\pi^2)$, where b is the $0(g^6)$ constant term in Γ. For reasonable values of g^2, the change is less than about 25%.

I shall now discuss briefly the effective coupling in quark-quark scattering through one-gluon exchange, again ignoring non-Abelian effects. The amplitude for this process is proportional to the real part of the improper gluon vacuum polarization, which is related to Im[Π] by a once-subtracted dispersion relation (assuming $m_q \neq 0$, and taking $\mu = 0$):

$$\mathrm{Re}[\Pi(t)] = \frac{1}{\pi} \int_{4m_q^2}^{\infty} \left(\frac{1}{s-t} - \frac{1}{s}\right) \mathrm{Im}[\Pi(s)]ds \quad . \tag{14}$$

This is very closely related to the formula (3) for Γ when $n = 0$. In the limit $m_q \to 0$, $\mathrm{Re}[\Pi(t)] = -\Gamma$ to $0(g^4)$, and in fact, for all terms in $\mathrm{Im}[\Pi(s)]$ of the form $(m^2/s)^p$ $(p \geq 0)$ the principal part of the $1/(s-t)$ integral in (14) vanishes as $m_q \to 0$, so that the result holds. At $0(g^6)$, $\mathrm{Im}[\Pi(s)]$ contains a term $\log(s/m^2)$. In that case the results for Γ and $-\mathrm{Re}[\Pi(t)]$ differ by $\frac{2}{9}\zeta(2)$. The result for the qq scattering amplitude (which may be deduced directly from the calculation of Källén and Sabry [4][F6]) is

$$\begin{aligned} A(t) = A_0(t) g^2(\mu^2) \Bigg\{ 1 &+ \frac{g^2(\mu^2)}{4\pi^2} \left[\frac{1}{3} \log\left(\frac{t}{\mu^2}\right) - \frac{5}{9}\right] \\ &+ \left(\frac{g^2(\mu^2)}{4\pi^2}\right)^2 \left[\frac{1}{9} \log^2\left(\frac{t}{\mu^2}\right) - \frac{13}{108} \log\left(\frac{t}{\mu^2}\right) \right. \\ &+ \left. \left(\zeta(3) + \frac{65}{648}\right)\right] + 0(g^6) \Bigg\} \quad , \end{aligned} \tag{15}$$

where t is the invariant mass transferred along the gluon. It is clear that the conclusions reached above for the process (1) also apply here.

I now give several further examples of the phenomena discussed above. The first is the anomalous magnetic moment (κ) of a charged lepton (ℓ) in QED. Using the lowest-order result, $\kappa = e^2(\mu^2)/8\pi^2$, the RGE gives ($\mu^2 \gg m_e^2$) [6]

$$\kappa = \frac{e^2(\mu^2)}{8\pi^2}\left\{1 + \frac{e^2(\mu^2)}{4\pi^2}\left[\frac{1}{3}\log\left(\frac{m_\ell^2}{\mu^2}\right) - a\right]\right.$$
$$+\left(\frac{e^2(\mu^2)}{4\pi^2}\right)^2\left[\frac{1}{9}\log^2\left(\frac{m_\ell^2}{\mu^2}\right) + \left(\frac{1}{4} - 2a\right)\log\left(\frac{m_\ell^2}{\mu^2}\right) - b\right]$$
$$\left. + \ldots\right\} \quad . \tag{16a}$$

Explicit perturbation theory calculations show that (e.g., [7])

$$a = -\left(\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\log 2 + \frac{3}{4}\zeta(3)\right) \simeq 0.328,$$
$$b \simeq -\ 1.2 \quad . \tag{16b}$$

If one makes the conventional choice $\mu^2 = 0$ then (see [F4]) the logarithms in (16) become $\log(m_\ell^2/m_e^2)$. If $\ell = e$, they therefore do not contribute. However, if $\ell = \mu$, they will contribute and generate a large part of the difference between κ_μ and κ_e. In practice there are also terms arising from the presence of new diagrams in the μ case, containing both e and μ loops, and in addition terms of order m_e^2/m_μ^2, which are not amenable to a RGE analysis. Nevertheless the coefficients of the logarithmic terms at $O(e^6)$ are determined by the RGE [6]. From eq. (16) one may deduce the value of η which would allow the complete $O(e^4)$ term to be absorbed into the effective coupling. The result is $\eta \simeq 2.7$.

The second example is the decay of positronium. The rate for the decay of the $^1S_0$ state is given by [8]

$$\Gamma = \Gamma_0\, e^4(\mu^2)\left\{1 + \frac{e^2(\mu^2)}{4\pi^2}\left[\frac{2}{3}\log\left(\frac{m_e^2}{\mu^2}\right) - a\right] + \ldots\right\} \tag{17}$$

$$a = \left(5 - \frac{\pi^2}{4}\right) \simeq 2.53$$

yielding $\eta \simeq 6.7$. For the ${}^3S_1$ state, [9]

$$\Gamma \simeq \Gamma_0\, e^6(\mu^2) \left\{1 + \frac{e^2(\mu^2)}{4\pi^2}\left[\log\left(\frac{m_e^2}{\mu^2}\right) - a\right] + \ldots\right\} \quad ,$$

$$a \simeq 10.3 \quad ,$$

so that $\eta \simeq 172$. It is clear from this example that the effective value of Λ^2 deduced from $Q\bar{Q} \to GGG$ decays may differ considerably from the true Λ^2. The complete cross-section (averaged over the initial spins) for e^+e^- annihilation in the nonrelativistic limit is of the form (17), but with [10] $a = (17 - 19\pi^2/12)$, corresponding to $\eta \simeq 2.8$.

Finally it should be pointed out that in QED there exist low-energy theorems which show that the cross-sections for certain processes (for example, $\gamma e \to \gamma e$ [11]) in the nonrelativistic limit are proportional simply to $e^2(\mu^2)$, and contain no constant terms. In QCD, however, analogous low-energy theorems are rendered useless by the strong coupling of the theory in the low-energy domain.

In this paper I have considered only processes which depend on one kinematic invariant. For processes which involve two invariants (e.g., the total cross-section for deep inelastic lepton-hadron scattering) similar results should hold, but the value of η which causes all $O(g^4)$ terms to be absorbed into the effective coupling will then consist of a dimensionless combination of the invariants.

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Footnotes

F1: Gauge invariance formally requires $n > 0$, but the limit $n \to 0$ may be taken without damage. n is governed by, for example, the spins of A and B. By changing n one essentially changes the process under consideration.

F2: Note that the construction for Λ^2 works only to $O(g^4)$. In higher orders, it is necessary to specify the coupling constant by giving its size at $\mu^2 = \mu_0^2$, where μ_0 is a fixed reference mass. This procedure is many cases more satisfactory in general, since it avoids the exponential sensitivity of Λ^2.

F3: The sensitivity is to be expected because

$$\frac{g^2(Q^2;\ \Lambda_1^2)}{g^2(Q^2;\ \Lambda_2^2)} = 1 - \frac{g^2(Q^2;\ \Lambda_1^2)}{12\pi^2} \log\left(\frac{\Lambda_1^2}{\Lambda_2^2}\right) + O(g^4).$$

F4: $n = 3$ corresponds to the well-known case $\pi^o \to \gamma\gamma^*$. It is conventional in QED to define e^2 from the low-energy limit of $\gamma e \to \gamma e$, that is, at $\mu_0^2 = 0$. In this case the first formula in (8) fails; the assumption $\mu_0^2 >> m_e^2$ is incorrect. When the exact formula is used, the logarithm for $\mu_0^2 = 0$ becomes $\log\,(\mu^2/m_e^2)$ [2].

F5: To $O(g^6)$ the inclusion of the $O(g^5)$ term in $\beta(g)$ results simply in the addition of $-\frac{1}{4}\left(\frac{g^2(\mu_0^2)}{4\pi^2}\right)^2 \log\left(\frac{\mu^2}{\mu_0^2}\right)$ to the denominator in eq. (8). To obtain the exact result one must solve a complicated transcendental equation for $\log\left(\frac{\mu^2}{\mu_0^2}\right)$ in terms of g.

F6: The results for some of the necessary integrals are given in [5]. For $t > 0$, $-\pi^2/9$ is added to the $O(g^6)$ constant term.

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n	a	η
0	5/9	2.3
1	8/9	3.8
2	19/18	4.9
3	7/6	5.8
4	5/4	6.5
10	11581/7560	10.0

Table 1: The corrections to the effective value of Λ (η) resulting from the inclusion of constant terms (a) at $0(g^4)$ for various forms of $A \to BG$ transition.

Figure Caption

Fig. 1: Typical diagrams contributing to $A \to BG$ ↳ anything to order g^6.

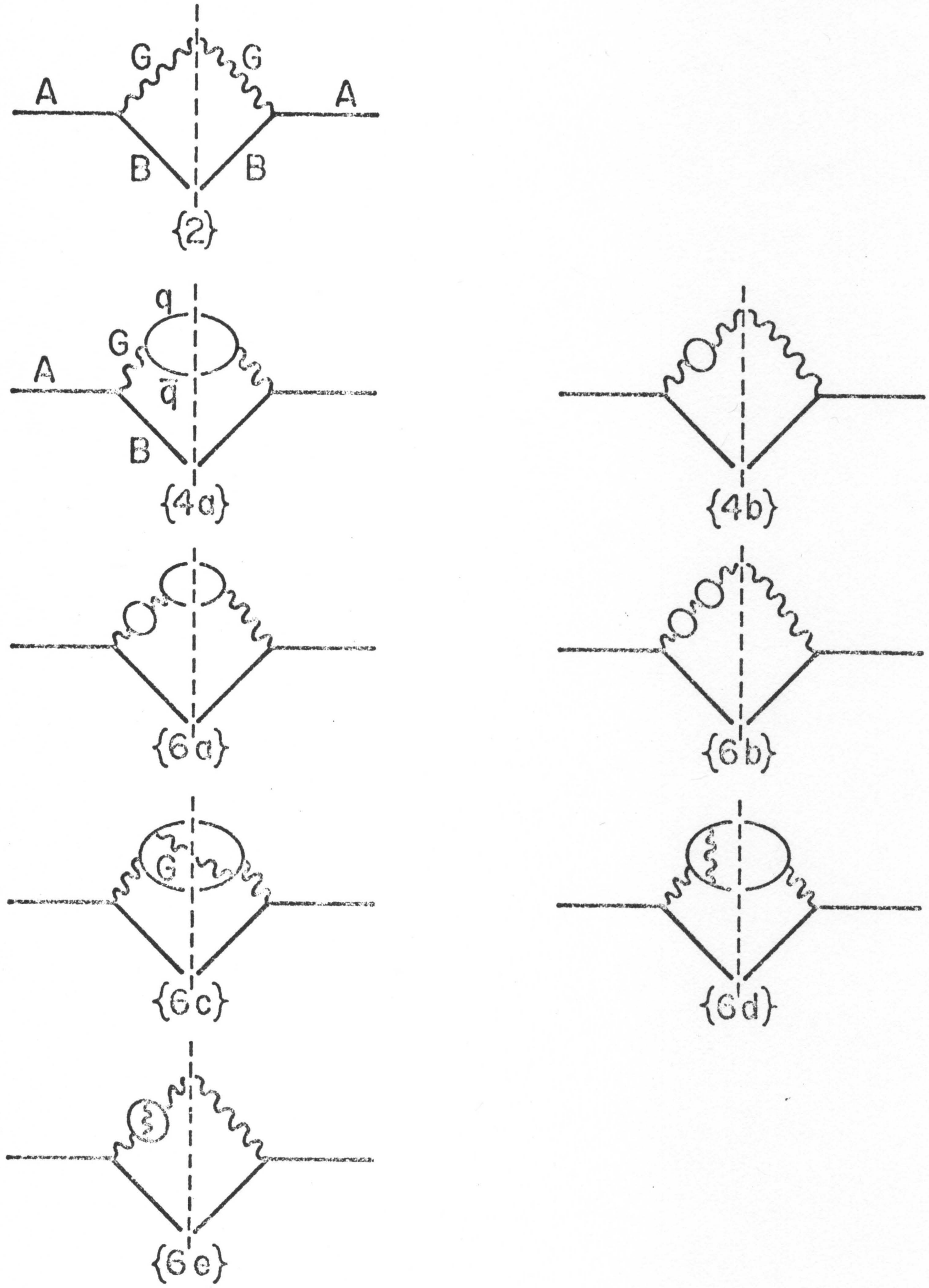

FIGURE 1