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POSITIVITY CONSTRAINTS ON QUARK AND GLUON DISTRIBUTIONS IN QCD

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Arguments are given that quark and gluon momentum distributions generated by asymptotic freedom formulae from distributions which are positive at $Q^2 = Q_0^2$ remain positive for $Q^2 \geqslant Q_0^2$. For $Q^2 < Q_0^2$ they inevitably become negative. Momentum distributions at $Q^2 = Q_0^2$ should be chosen so that this occurs only for very small Q^2. This constrains the partition of momentum between the quarks and gluons and the shapes of their momentum distributions.

1. Introduction

The asymptotic freedom of quantum chromodynamics (QCD) can be used to predict the behaviour of deep inelastic structure functions for large Q^2. These predictions involve matrix elements of certain local operators, or equivalently, effective quark and gluon momentum distributions (structure functions) at a fixed "starting" Q_0^2. We are at present unable to calculate these distributions: some features may be determined from experiment while others must simply be guessed. It is evident, however, that physically acceptable structure functions obtained from asymptotic freedom formulae must remain positive throughout the range of Q^2 in which the formulae should apply. In this paper we investigate the constraints that this places on the starting momentum distributions.

The asymptotic freedom formulae give directly the Q^2 variation of the moments $G_n(Q^2)$ and $\overset{(-)}{Q}{}^i_n(Q^2)$ of the gluon and (anti)quark momentum distributions of each flavour i, where

$$F_n(Q^2) \equiv \int_0^1 x^{n-2} F(x, Q^2)\, \mathrm{d}x\,. \tag{1}$$

In sect. 2 we collect these predictions, and state Hausdorff's theorem, which provides the conditions that the $F_n(Q^2)$ must satisfy in order that $F(x, Q^2)$ be non-negative (for $0 \leqslant x \leqslant 1$). Equivalent conditions have previously been used by Nachtmann [1] to derive constraints on the n dependence of anomalous dimensions. Nachtmann considered moments which are controlled by a single local operator in the operator product expansion. His analysis therefore applies to the singlet operator with the smallest anomalous dimension, which dominates the singlet structure functions at infinite Q^2, but does not address the question of whether the structure functions remain positive at accessible values of Q^2 (non-singlet combinations of structure functions are controlled by a single local operator but can be negative even at Q_0^2).

In sect. 3 we show that the moments predicted from any set of positive starting distributions are positive for $Q^2 \geqslant Q_0^2$. We argue that the further conditions which must be satisfied if the complete momentum distributions are to be positive also hold. For $Q^2 < Q_0^2$, however, one of any set of structure functions eventually becomes negative. This would be irrelevant if it happened only at very small Q^2, since asymptotic freedom formulae * should not be expected to hold there, but we find that unless the starting distributions are chosen carefully it occurs for Q^2 close to Q_0^2. Advocates of particular quark and gluon distributions should therefore check that their distributions remain positive in the region where the use of asymptotic freedom formulae is not completely unreliable.

In sect. 4 we derive some analytic constraints on the starting distributions. Assuming that as $x \to 1$

$$\sum_{i=1}^{F} [Q^i(x, Q_0^2) + \bar{Q}^i(x, Q_0^2)] \sim (1-x)^v ,$$

$$G(x, Q_0^2) \sim (1-x)^g ,$$

$$\bar{Q}^i(x, Q_0^2) \sim (1-x)^s , \qquad (s \geqslant v) , \tag{2}$$

we show that for $n \to \infty$ the moments become negative at $Q^2 = Q_0^2 - \epsilon$ (where ϵ is an arbitrarily small positive number) unless

$$(v-1) \leqslant (s-1) < g \leqslant (v+1) . \tag{3}$$

In fact, the asymptotic freedom formulae cease to apply as $n \to \infty$ at fixed Q^2, so that eq. (3) need not be obeyed exactly. However, the constraints found numerically for finite n reflect this result. Note that if the formulae of asymptotic freedom are used to generate distributions for $Q^2 > Q_0^2$, their behaviour as $x \to 1$ will necessarily satisfy eq. (3).

We also present analytic results for the $n = 2$ moments (total momenta carried

* The formulae are known in full only to leading order in $\alpha_s(Q^2)/\pi \simeq 12/(33-2F) \log(Q^2/\Lambda^2)$ (F is the number of quark flavours) and in $n\Lambda^2/Q^2$. Experiment indicates that $\Lambda^2 \simeq 0.25\ \mathrm{GeV}^2$.

by each species). We find that the requirement that distributions do not rapidly become negative for $Q^2 < Q_0^2$ places constraints on the division of momentum between quarks, antiquarks and gluons. For example, unless gluons carry at least 20% of the total momentum at $Q^2 = 4\ \mathrm{GeV}^2$, the predicted gluon momentum distribution will become negative at a Q^2 above $1\ \mathrm{GeV}^2$. There is also an upper bound on the gluon momentum, and a bound on the antiquark momenta.

In sect. 5, we analyse the numerical consequences of our bounds for various forms of the starting distributions. A typical result is that if the $\overset{(-)}{Q}{}^i(x, Q_0^2 = 4\ \mathrm{GeV}^2)$ are those deduced from studies of present deep inelastic scattering data, then $G(x, Q_0^2)$ must fall less steeply with x than about $(1-x)^9$.

2. Basic formulae

QCD predicts that in the asymptotically free limit ($Q^2 \to \infty$, n fixed), if there are F flavours of quarks,

$$\overset{(-)}{Q}{}^i_n(Q^2) = \overset{(-)}{Q}{}^i_n(Q_0^2)\, T^{d_n^{\mathrm{NS}}} + \frac{1}{2F}\{M_n^+(Q_0^2)(T^{d_n^+} - T^{d_n^{\mathrm{NS}}}) + M_n^-(Q_0^2)(T^{d_n^-} - T^{d_n^{\mathrm{NS}}})\}\,, \tag{4}$$

$$G_n(Q^2) = E_n^+ M_n^+(Q_0^2)\, T^{d_n^+} + E_n^- M_n^-(Q_0^2)\, T^{d_n^-}\,, \tag{5}$$

where

$$M_n^+(Q^2) = \frac{1}{d_n^+ - d_n^-}\{(d_n^{\mathrm{NS}} - d_n^-)\, Q_n(Q^2) + d_n^{\mathrm{QG}} G_n(Q^2)\}\,,$$

$$M_n^-(Q^2) = \frac{1}{d_n^+ - d_n^-}\{(d_n^+ - d_n^{\mathrm{NS}})\, Q_n(Q^2) - d_n^{\mathrm{QG}} G_n(Q^2)\}\,,$$

$$Q_n(Q^2) = \sum_{i=1}^{F} (Q_n^i(Q^2) + \bar{Q}_n^i(Q^2)) = M_n^+(Q^2) + M_n^-(Q^2)\,,$$

$$G_n(Q^2) = E_n^+ M_n^+(Q^2) + E_n^- M_n^-(Q^2)\,,$$

$$T(Q^2) = \frac{\log(Q_0^2/\Lambda^2)}{\log(Q^2/\Lambda^2)}\,,$$

$$d_n^{\mathrm{QQ}} \equiv d_n^{\mathrm{NS}} = \frac{4}{33-2F}\left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n} \frac{1}{j}\right],$$

$$d_n^{\mathrm{GG}} = \frac{2F}{33-2F} + \frac{9}{33-2F}\left[\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_{j=2}^{n}\frac{1}{j}\right],$$

$$d_n^{\mathrm{GQ}} = -\frac{8(n^2+n+2)}{(33-2F)\,n(n^2-1)},$$

$$d_n^{\mathrm{QG}} = -\frac{6F(n^2+n+2)}{(33-2F)\,n(n+1)(n+2)},$$

$$2d_n^{\pm} = d_n^{\mathrm{QQ}} + d_n^{\mathrm{GG}} \pm \sqrt{(d_n^{\mathrm{QQ}} - d_n^{\mathrm{GG}})^2 + 4d_n^{\mathrm{QG}}\,d_n^{\mathrm{GQ}}},$$

$$E_n^{\pm} = \frac{d_n^{\pm} - d_n^{\mathrm{QQ}}}{d_n^{\mathrm{QG}}}. \tag{6}$$

To derive (4) recall that the non-singlet combinations of structure functions are

$$\left[(Q^i + \bar{Q}^i) - \frac{1}{F}\sum_k (Q^k + \bar{Q}^k)\right], \qquad (Q^i - \bar{Q}^i).$$

Writing

$$(Q^i + \bar{Q}^i) = \left[(Q^i + \bar{Q}^i) - \frac{1}{F}\sum_k (Q^k + \bar{Q}^k)\right] + \frac{1}{F}\sum_k (Q^k + \bar{Q}^k),$$

one obtains

$$[Q_n^i(Q^2) + \bar{Q}_n^i(Q^2)] = \left[(Q_n^i(Q_0^2) + \bar{Q}_n^i(Q_0^2)) - \frac{1}{F}(M_n^+(Q_0^2) + M_n^-(Q_0^2))\right] T^{d_n^{\mathrm{NS}}}$$

$$+ \frac{1}{F}(M_n^+(Q_0^2)\,T^{d_n^+} + M_n^-(Q_0^2)\,T^{d_n^-}),$$

$$[Q_n^i(Q^2) - \bar{Q}_n^i(Q^2)] = [Q_n^i(Q_0^2) - \bar{Q}_n^i(Q_0^2)]\,T^{d_n^{\mathrm{NS}}},$$

from which (4) follows.

If we define

$$f_n \equiv \int_0^1 x^n f(x)\,\mathrm{d}x$$

then the necessary and sufficient conditions on the f_n for $f(x)$ to be non-negative on

[0,1] are

$$\Delta^m f_n \equiv \sum_{k=0}^{m} (-1)^k \binom{m}{k} f_{n+k}$$

$$= \int_0^1 x^n (1-x)^m f(x)\, dx \geqslant 0\,, \qquad n, m = 0, 1, 2, \ldots\,, \tag{7}$$

which is Hausdorff's theorem [3]. The necessity of these conditions is obvious since $x^n(1-x)^k \geqslant 0$ $(0 \leqslant x \leqslant 1)$.

For $f(x) \geqslant 0$ $(0 \leqslant x \leqslant 1)$ it is necessary and sufficient that $\int_0^1 R(x) f(x)\, dx \geqslant 0$ for all polynomials $R(x) \geqslant 0$ $(0 \leqslant x \leqslant 1)$. We can use the fact that

$$x^n = \lim_{N\to\infty} \left\{ \left[\left(\frac{1}{N} p \frac{\partial}{\partial p} \right)^n (p+q)^N \right]_{\substack{p=x \\ q=(1-x)}} \right\}$$

$$= \lim_{N\to\infty} \left\{ \left[\left(\frac{1}{N} p \frac{\partial}{\partial p} \right)^n \sum_{k=0}^{N} \binom{N}{k} p^k q^{N-k} \right]_{\substack{p=x \\ q=(1-x)}} \right\}$$

$$= \lim_{N\to\infty} \left\{ \sum_{k=0}^{N} \binom{N}{k} \left(\frac{k}{N} \right)^n x^k (1-x)^{N-k} \right\}$$

to write

$$R(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \binom{N}{R}^R \left(\frac{k}{N} \right) x^k (1-x)^{N-k}\,,$$

from which the sufficiency of (7) follows.

3. Features of the positivity constraints

We now consider the constraints which arise from demanding that $\overset{(-)}{Q}{}^i(x, Q^2)$ and $G(x, Q^2)$ be non-negative *. We begin with the necessary condition that their moments $\overset{(-)}{Q}{}^i_n(Q^2)$ and $G_n(Q^2)$ be non-negative. If the starting distributions are positive

* It is easy to construct cross sections involving hypothetical but conceivable currents which are proportional to the quark and antiquark distributions of each flavour, and to the glue distribution. These quantities, and the corresponding local operator matrix elements, cannot therefore be negative.

it is easy to show that $M_n^-(Q_0^2) > 0$ (eqs. (5), (6)), but one must consider two possibilities for $M_n^+(Q_0^2)$.

(i) $M_n^+(Q_0^2) \leqslant 0$

In this case $G_n(Q^2)$ is positive for all Q^2 since $E_n^+ < 0$ and $E_n^- > 0$. $\overset{(-)}{Q}{}_n^i(Q^2)$ is positive for $Q^2 \geqslant Q_0^2$ ($T \leqslant 1$) since $d_n^+ > d_n^{\rm NS} \geqslant d_n^-$. As $T \to \infty$, however, $\overset{(-)}{Q}{}_n^i(Q^2)$ inevitably becomes negative and we must choose the starting distributions so that this does not occur in a region where the asymptotic freedom formulae apply.

(ii) $M_n^+(Q_0^2) > 0$

The condition $G_n(Q^2) \geqslant 0$ now becomes

$$\left[\frac{-E_n^- M_n^-(Q_0^2)}{E_n^+ M_n^+(Q_0^2)}\right]^{1/(d_n^+ - d_n^-)} \geqslant T\,. \tag{8}$$

If the starting distributions are positive, this holds for $T = 1$, and since $d_n^+ - d_n^- > 0$ it continues to hold for $T < 1$ ($Q^2 > Q_0^2$). For $T > 1$ ($Q^2 < Q_0^2$), however, it will eventually fail.

Eq. (3) shows that $T^{-d_n^{\rm NS}} \overset{(-)}{Q}{}_n^i(Q^2)$ is non-negative as $T \to 0$ or $T \to \infty$ since it is dominated by the positive terms $M_n^-(Q_0^2)\, T^{(d_n^- - d_n^{\rm NS})}$ or $M_n^+(Q_0^2)\, T^{(d_n^+ - d_n^{\rm NS})}$ in these limits, but it has one minimum at

$$T = \overline{T} = \left[\frac{-(d_n^+ - d_n^{\rm NS})\, M_n^+(Q_0^2)}{(d_n^- - d_n^{\rm NS}) M_n^-(Q_0^2)}\right]^{1/(d_n^- - d_n^+)}$$

$$= \left[\frac{-E_n^+ M_n^+(Q_0^2)}{E_n^- M_n^-(Q_0^2)}\right]^{1/(d_n^- - d_n^+)} \tag{9}$$

The validity of eq. (8) at $T = 1$ implies that $\overline{T} > 1$. Hence $T^{-d_n^{\rm NS}} \overset{(-)}{Q}{}_n^i(Q^2)$ decreases between 0 and 1 but $\overset{(-)}{Q}{}_n^i(Q^2)$ is positive throughout this region. For $T > 1$, however, $\overset{(-)}{Q}{}_n^i(Q^2)$ may become negative, depending on the starting distributions.

We have now shown that the moments of the (anti)quark and gluon distributions remain positive for $Q^2 > Q_0^2$ if they are positive at $Q^2 = Q_0^2$. In order that the distributions themselves remain positive, their moments must satisfy eq. (7) for all m. For the large class of starting distributions whose moments behave like inverse powers of n for large n, the asymptotic freedom formulae predict that their moments continue to behave like inverse powers of n for $Q^2 > Q_0^2$ (apart from log n factors which will not affect our argument). In this case $\Delta^m \overset{(-)}{Q}{}_n^i$ and $\Delta^m G_n$ will be positive for large n since $\Delta^m(1/n^p) \geqslant 0$ for $p \geqslant 0$ (where Δ^m is the mth-order finite difference operator defined in eq. (7)). A detailed numerical investigation of $\Delta^m \overset{(-)}{Q}{}_n^i(Q^2)$ and $\Delta^m G_n(Q^2)$ with a wide variety of positive starting distributions suggests that in all cases they indeed remain positive for $Q^2 \geqslant Q_0^2$ for all n.

We showed above that for $Q^2 < Q_0^2$ some of the moments inevitably become negative. This is also true of their finite differences. For "reasonable" starting distributions, however, we find (sect. 5) that the $m \neq 0$ conditions only very rarely fail before the $m = 0$ conditions as Q^2 is decreased below Q_0^2. This is to be expected since for a large class of distributions (including all those which decrease monotonically with x) the finite differences of the moments cannot be negative unless the moments themselves are negative.

4. Analytic results

It is expected [4] that the effects of operators occurring in the operator product expansion with twists greater than two become very important for $n \gtrsim Q^2/\Lambda^2$, corresponding to $x \gtrsim 1 - O(\Lambda^2/Q^2)$ (terms of higher order in $\alpha_s(Q^2)$ also diverge in the limit $n \to \infty$ at fixed Q^2, but this is only significant at much larger values of n). Starting distributions need only be rejected, therefore, if they lead to negative distributions for $n \lesssim 4Q^2$ (GeV^2). Nevertheless, as explained in sect. 1, it is interesting to consider the limit $n \to \infty$, where we find

$$Q_n(Q^2) \simeq Q_n(Q_0^2)\left(T^{d_n^-} + \frac{3FT^{d_n^+}}{(5n\log n)^2}\right) + \frac{3FG_n(Q_0^2)}{10n\log n}\left(T^{d_n^-} - T^{d_n^+}\right),$$

$$G_n(Q^2) \simeq G_n(Q_0^2)\left(T^{d_n^+} + \frac{3FT^{d_n^-}}{(5n\log n)^2}\right) + \frac{2Q_n(Q_0^2)}{5n\log n}\left(T^{d_n^-} - T^{d_n^+}\right), \tag{10a}$$

where

$$d_n^+ \simeq \frac{36\log n}{33-2F}, \qquad d_n^- \simeq \frac{16\log n}{33-2F}. \tag{10b}$$

For $Q^2 < Q_0^2$ $(T > 1)$, the factor $(T^{d_n^-} - T^{d_n^+})$ is negative. The distributions therefore become negative unless the first terms in eq. (10a) dominate as $n \to \infty$. Making the assumption of eq. (2) for the behaviour of the starting distributions as $x \to 1$, $Q_n(Q_0^2)$ and $G_n(Q_0^2)$ behave like $n^{-(v+1)}$ and $n^{-(g+1)}$, respectively, for large n. Eq. (3) is then the condition that $Q_n(Q^2)$ and $G_n(Q^2)$ remain positive for $T = 1 + \delta$ $(Q^2 = Q_0^2 - \epsilon)$. If the inequalities of eq. (3) are satisfied, $G_n(Q^2)$ remains positive for all $Q^2 < Q_0^2$ but $Q_n(Q^2)$ still becomes negative for Q^2 sufficiently small. For $n \to \infty$ this occurs at

$$T = \exp\left(\frac{33-2F}{20}\,[g - v + 1]\right).$$

As stated in sect. 1, it is clear that distributions for $Q^2 > Q_0^2$ generated from any starting distributions must obey eq. (3). When $Q^2 \to \infty$ at fixed n,

$$\frac{(5 \log n)^2 \, T^{d_n^+}}{3F T^{d_n^-}} \to 0 \,,$$

so that eq. (10a) shows that $g \simeq v + 1$: but in general this will only become true at extremely high Q^2 when the structure functions will all be infinitesimal for x close to one. To make more useful statements about the form of the structure functions for $Q^2 > Q_0^2$, we must assume a particular form for the starting distributions. If, for example, following ref. [5], we take $G_n(Q_0^2) = \overline{Q}_n^i(Q_0^2) = 0$ for $Q_0^2 \simeq \Lambda^2$, then, ignoring $\log n$ terms, eqs. (10) and (4) give $g = v + 1$, $s = v + 2$ for $Q^2 > Q_0^2$.

Next we consider the $n = 2$ moments. The energy-momentum sum rule implies $G_2(Q^2) + Q_2(Q^2) = 1$, so that for the singlet structure functions

$$Q_2(Q^2) = \frac{3F}{3F + 16} + \left(\frac{16}{3F + 16} - G_2(Q_0^2)\right) T^{d_2^+} \,,$$

$$G_2(Q^2) = \frac{16}{3F + 16} - \left(\frac{16}{3F + 16} - G_2(Q_0^2)\right) T^{d_2^+} \,. \tag{11}$$

Hence $G_2(Q^2)$ will inevitably become negative as $Q^2 \to 0$ ($T \to \infty$), unless $G_2(Q_0^2) > 16/(3F + 16)$, in which case $Q_2(Q^2)$ will become negative. In fig. 1 we have plotted the value of Q^2 below which $G_2(Q^2)$ or $Q_2(Q^2)$ becomes negative (which we call Q_-^2) as a function of $G_2(Q_0^2 = 4 \text{ GeV}^2)$ for $\Lambda^2 = 0.25 \text{ GeV}^2$ and $F = 3$ (the results are very insensitive to the number of flavours). The very reasonable demand that $Q_-^2 \lesssim 1 \text{ GeV}^2$ requires $0.87 \gtrsim G_2(Q_0^2 = 4 \text{ GeV}^2) \gtrsim 0.2$, (experimentally $G_2(Q_0^2 =$

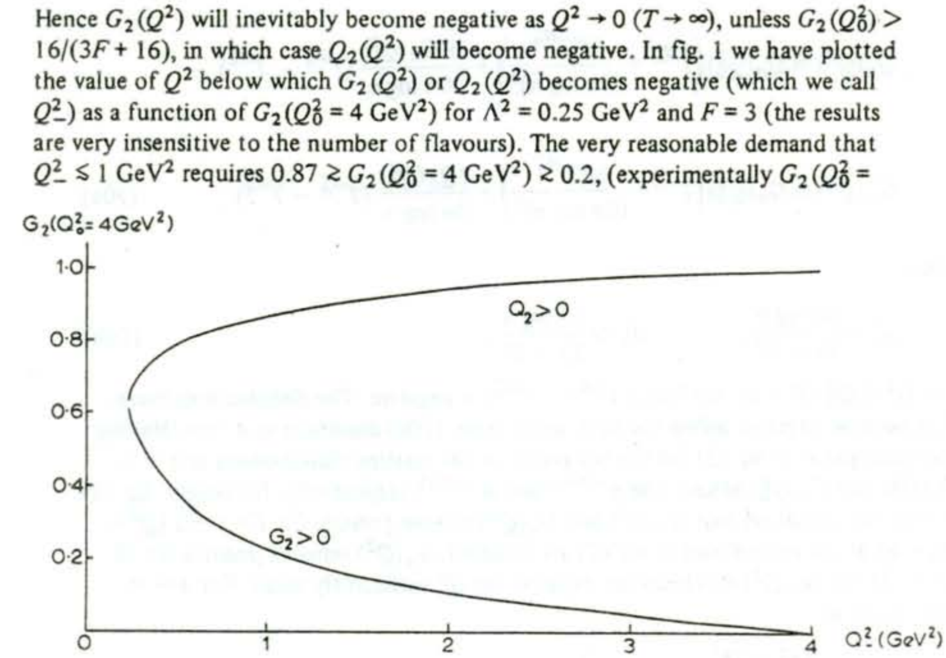

Fig. 1. The value Q_-^2 of Q^2 below which the predicted total momenta carried by the gluons (lower line) or quarks (upper line) becomes negative, as a function of the momentum carried by the gluons at $Q^2 = Q_0^2 = 4 \text{ GeV}^2$ ($\Lambda^2 = 0.25 \text{ GeV}^2$ and $F = 3$).

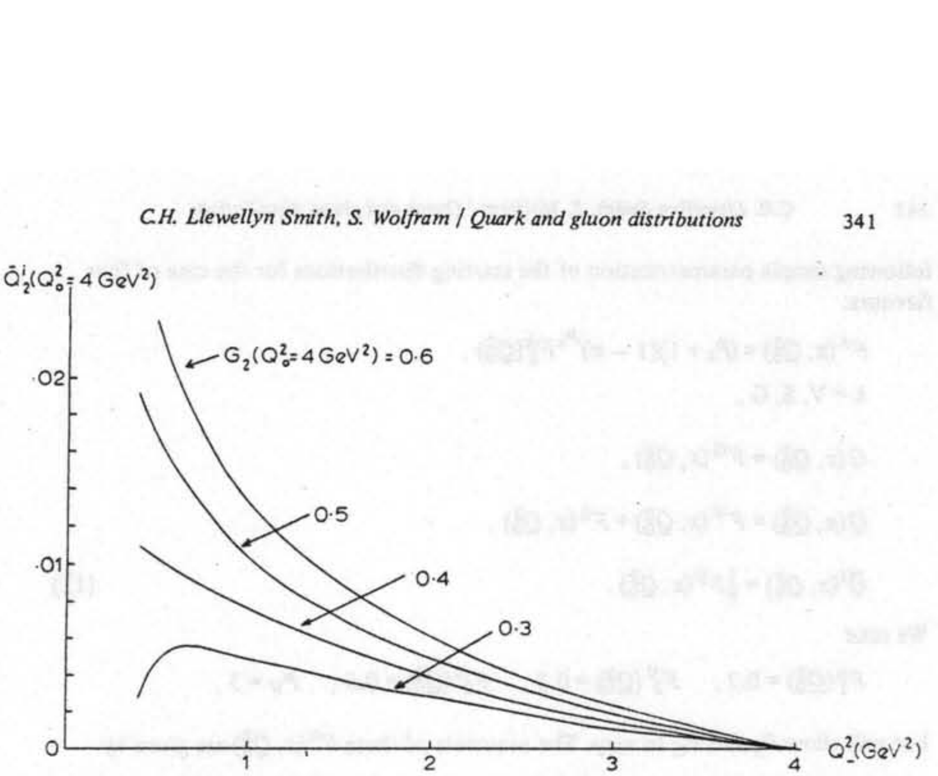

Fig. 2. The value Q_-^2 of Q^2 below which the predicted total momentum carried by one flavour of antiquark ($\overline{Q}_2^i(Q^2)$) becomes negative, as a function of the momentum it carried at $Q^2 = Q_0^2 = 4\ \mathrm{GeV}^2$ for various starting gluon momenta, $G_2(Q_0^2 = 4\ \mathrm{GeV}^2)$ ($\Lambda^2 = 0.25\ \mathrm{GeV}^2$ and $F = 3$).

$4\ \mathrm{GeV}^2) \simeq 0.5$) and if $Q_-^2 \lesssim 0.5\ \mathrm{GeV}^2$ (by which point higher-twist operators should be becoming significant for $n = 2$), $0.79 \gtrsim G_2(Q_0^2 = 4\ \mathrm{GeV}^2) \gtrsim 0.37$.

The value of Q^2 at which $\overset{(-)}{Q}{}_2^i(Q^2)$ becomes negative is easily calculated from eq. (4) as a function of $G_2(Q_0^2)$ and $\overset{(-)}{Q}{}_2^i(Q_0^2)$ using the fact that $M_2^-(Q^2) = 3F/(3F + 16)$ and $M_2^+(Q_0^2) + M_2^-(Q_0^2) = Q_2(Q_0^2) = 1 - G_2(Q_0^2)$ (which has already been used in deriving eq. (11)). The results are shown in fig. 2 for $\Lambda^2 = 0.25\ \mathrm{GeV}^2$ and $F = 3$ (they are insensitive to F). The reasonable demand that $Q_-^2 \lesssim 1\ \mathrm{GeV}^2$ gives $\overline{Q}_2^i(Q_0^2 = 4\ \mathrm{GeV}^2) \gtrsim 0.01$ for $G_2(Q_0^2 = 4\ \mathrm{GeV}^2) = 0.5$ and $Q_-^2 \lesssim 0.5\ \mathrm{GeV}^2$ implies $\overline{Q}_2^i(Q_0^2 = 4\ \mathrm{GeV}^2) \gtrsim 0.017$. Neutrino data suggest that $\overline{Q}_2^u(4\ \mathrm{GeV}^2) + \overline{Q}_2^d(4\ \mathrm{GeV}^2) \simeq 0.04$ not greatly in excess of this bound. For strange quarks the bound probably fails with $Q_-^2 \simeq = 1\ \mathrm{GeV}^2$ but larger Q^2 is needed for it to be credible in this case. However, our bounds show that the distributions of new flavours must be non-zero in regions of Q^2 where the application of asymptotic freedom formulae to them is at all reasonable.

Figs. 1 and 2 show that the observed values of $G_2(4)$ and $\overline{Q}_2^u(4\ \mathrm{GeV}^2) + \overline{Q}_2^d(4\ \mathrm{GeV}^2)$ are obtained if G_2 and $\overline{Q}_2^i$ vanish at $Q^2 = Q_-^2 = 0.4\ \mathrm{GeV}^2$. This was originally pointed out in ref. [5].

5. Numerical results

It is necessary to resort to numerical techniques to make a complete investigation of the positivity of distributions for $Q^2 < Q_0^2$. As an example, we consider first the following simple parametrization of the starting distributions for the case of four flavours:

$$F^k(x, Q_0^2) = (P_k + 1)(1 - x)^{P_k} F_2^k(Q_0^2) ,$$
$$k = \mathrm{V, S, G} ,$$
$$G(x, Q_0^2) = F^{\mathrm{G}}(x, Q_0^2) ,$$
$$Q(x, Q_0^2) = F^{\mathrm{V}}(x, Q_0^2) + F^{\mathrm{S}}(x, Q_0^2) ,$$
$$\overline{Q}^i(x, Q_0^2) = \tfrac{1}{8} F^{\mathrm{S}}(x, Q_0^2) . \tag{12}$$

We take

$$F_2^{\mathrm{S}}(Q_0^2) = 0.2 , \quad F_2^{\mathrm{V}}(Q_0^2) = 0.3 , \quad F_2^{\mathrm{G}}(Q_0^2) = 0.5 , \quad P_{\mathrm{V}} = 3 ,$$

but will allow P_{S} and P_{G} to vary. The moments of these $F^k(x, Q_0^2)$ are given by

$$F_n^k(Q_0^2) = (P_k + 1)\, B(n - 1, P_k + 1)\, F_2^k(Q_0^2) ,$$

where B is the Beta function.

We consider the $\overset{(-)}{Q}{}_n^i(Q_T^2)$ and $G_n(Q^2)$ generated from the starting distributions of eq. (12) at $T = 1.2$, which corresponds to $Q^2 = 2.5\ \mathrm{GeV}^2$ for $Q_0^2 = 4\ \mathrm{GeV}^2$ and $\Lambda^2 = 0.25\ \mathrm{GeV}^2$. We know that these quantities are positive for $n = 2$ (see figs. 1 and 2) but they become negative for large n unless P_{S} and P_{G} satisfy eq. (3). Table 1 gives the minimum n (which we call n_-) at which this occurs for various P_{S} and P_{G}. The contributions of higher-twist operators probably invalidate the use of the asymptotic freedom formulae for $n \gtrsim 4Q^2\ (\mathrm{GeV}^2)$, or $n \gtrsim 10$ in this case. We can therefore only exclude values of P_{S} and P_{G} which give $n_- \lesssim 10$. The outlined central area in table 1 (in which $n_- \geqslant 10$) corresponds to the allowed values of P_{S} and P_{G}; it is considerably larger than the region allowed by the $n \to \infty$ bound of eq. (3) but has similar features. For $P_{\mathrm{G}} \gtrsim 0.6\, P_{\mathrm{S}}$, it is $G_n(Q^2)$ which becomes negative for $n \geqslant n_-$ while for $0.6\, P_{\mathrm{S}} \lesssim P_{\mathrm{G}}$ it is $\overline{Q}{}_n^i(Q^2)$.

In this case, it turns out that the $m = 0$ positivity constraints are at least as strong as the $m \neq 0$ ones for a given n. If we consider values of $T = T_-$ larger than 1.2, the maximum allowed value of P_{G} tends to grow very slowly with T_-, and the $m \neq 0$ constraints begin to become stricter than the $m = 0$ ones for small P_{G} and P_{S} (although this never happens in a region which is not seriously hampered by higher-twist effects). An increase (decrease) of $G_2(Q_0^2)$ increases (decreases) n_- throughout the domain in which $G_n(Q^2)$ becomes negative for $n > n_-$ (this domain, which was originally $P_{\mathrm{G}} \gtrsim 0.6\, P_{\mathrm{S}}$ is itself decreased (increased)). Likewise, an increase (decrease) of $\overline{Q}{}_2^i(Q_0^2)$ increases (decreases) n_- in the region where $\overline{Q}{}_n^i(Q^2)$ becomes negative.

Of course, actual quark and gluon momentum distributions will not be of the simple form (12). Some idea of the results of using more complicated forms may be

Table 1
The minimum values of $n = n_-$ for which the moment of one of the momentum distributions becomes negative at $Q^2 = 2.5\ \mathrm{GeV}^2$ starting from distributions at $Q_0^2 = 4\ \mathrm{GeV}^2$ (with $\Lambda^2 = 0.25\ \mathrm{GeV}^2$)

p_G \ p_S	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
15	6	6	7	7	7	8	8	8	8	8	8	8	8	8	8
14	6	7	7	8	8	8	8	8	8	8	8	8	8	8	8
13	6	7	8	9	8	9	9	9	9	9	9	9	9	9	9
12	7	8	9	9	9	10	10	10	10	10	10	10	10	10	10
11	7	8	10	10	11	11	11	11	11	11	11	11	11	11	11
10	8	9	11	12	13	13	13	13	13	13	13	13	13	13	13
9	8	11	13	15	15	15	15	15	15	15	15	15	15	15	15
8	10	13	17	19	21	21	21	21	21	21	21	21	16	12	10
7	12	18	27	32	33	33	33	34	33	33	20	13	11	7	8
6	15	29							29	16	12	10	8	9	7
5	23							24	14	11	9	7	7	6	6
4				>40			19	8	9	8	7	6	6	5	5
3						16	10	8	7	6	6	5	5	5	5
2				35	12	8	7	6	5	5	5	5	5	4	4
1			23	10	7	6	5	5	5	4	4	4	4	4	4

The starting gluon and antiquark momentum distributions are taken to behave like $(1-x)^{p_G}$ and $(1-x)^{p_S}$, respectively (and the valence quark distributions like $(1-x)^3$). The lines enclose the region in which $n_- \geqslant 10$ where higher-twist operators are important.

obtained by admixing a small $(1-x)^3$ component into the $G(x, Q_0^2)$ and $F^S(x, Q_0^2)$ of eq. (12). Very small contamination of $F^S(x, Q_0^2)$ is sufficient to prevent it from becoming negative for values of x and Q^2 at which asymptotic freedom formulae should apply (ignoring higher-twist operators). It would also, however, ruin the agreement between theory and experiment for the Drell-Yan process. On the other hand, even 10% admixtures into $G(x, Q_0^2)$ (changing it drastically for large x) do not appreciably change the point at which it becomes negative (although for small P_S this occurs at larger $x(n)$).

If we use the phenomenological fits of Field and Feynman [6] or Barger and Phillips [7] as the starting quark distributions, then we find that positivity is only respected for Q^2 a small distance below Q_0^2 if $G(x, Q_0^2)$ falls less rapidly with x than about $(1-x)^9$. Steeper starting gluon distributions (e.g. [8]) should therefore probably be abandoned.

Note added in proof

Table 1 gives the values of $n = n_-$ for which the predicted moments become negative. The finite differences formed from the moments with $n \leqslant n_-$ do not in this case yield stronger information. However, M.R. Pennington has pointed out to us that if one considers all moments but eliminates those with $n > N$ which are not determined by the theory, then the constraints obtained are stronger than those resulting from the positivity of the first N moments and their finite differences alone [9]. The additional conditions hardly affect the lower boundary of the allowed region in table 1 but the upper boundary is reduced to $P_G \leqslant 9$ for all P_S [10].

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