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On the Periods of Some Graph Transformations

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Abstract. For any undirected graph with arbitrary integer values attached to the vertices, simultaneous updates are performed on these values, in which the value of a vertex is moved by one in the direction of the average of the values of the neighboring vertices. (A special rule applies when the value of a vertex equals this average.) It is shown that these transformations always reach a cycle of length one or two. This proves a generalization of a conjecture made by Ghiglia and Mastin in connection with their work on a "phase-unwrapping" algorithm.

1. Introduction

Let G be an undirected graph with vertices labelled $1, \ldots, n$, and suppose that for each i, an integer $x_i(0)$ is initially assigned to vertex i. We perform a sequence of synchronous updates on these values. If $x_i(t)$ is the value of vertex i at time t, then

$$x = \begin{cases} x_i(t) - 1 & \text{if } \sum_{j \in J_i} x_j(t) < d_i x_i(t), \\ x_i(t) & \text{if } x_j(t) = x_i(t) \text{ for all } j \in J_i, \\ x_i(t) + 1 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} J_i &= \{j : \text{ vertex } j \text{ is connected to vertex } i\}, \\ d_i &= |J_i| = \text{ degree of vertex } i. \end{aligned} \tag{1.1}$$

Less formally, the value $x_i(t)$ assigned to vertex i moves by one in the direction of the average of the values assigned to the neighbors of vertex i, but a special rule applies when $x_i(t)$ equals this average; if the values of all the neighbors of vertex i equal $x_i(t)$, then the value of vertex i stays the same, whereas if the neighbors' values are not all equal, then the value of vertex i increases by 1. Since $\max_i x_i(t)$ does not increase and $\min_i x_i(t)$ does not decrease as t varies, the iteration described above eventually reaches a cycle, so that for some minimal $p > 1$, $x_i(t+p) = x_i(t)$ for all i and all $t > t_0$. For example, when G is a simple path of length 5 (i.e., there are 5 vertices numbered 1 through 5, and vertex i is connected to vertex j if and only if $|i-j| = 1$), and the initial assignment is $(x_1(0), ..., x_5(0)) = (0, 2, 1, 10, 4)$, then the iteration is given by the array below, in which the i-th row presents the values of $x_1(i-1), ..., x_5(i-1)$:

0	1	1	2	3	4	5	6	7
2	1	2	3	4	5	6	7	6
1	2	3	4	5	6	7	6	7
10	9	8	7	6	7	6	7	6
4	5	6	7	7	6	7	6	7

The last two states above form a cycle which repeats from then on. Our main result is that this case is not unusual, and the length of the cycle is 1 or 2 in all cases of the iteration.

Theorem 1. *For any undirected graph G and any initial assignment of integers $x_1(0), ..., x_n(0)$ to the vertices of G, there is a t_0 such that the above iteration satisfies $x_i(t+2) = x_i(t)$ for all i and all $t > t_0$.*

The problem of determining the cycle length of the above iteration arose in the work of D. Ghiglia and G. Mastin [1]. They considered such iterations for the cases of G being (a) a simple path and (b) a $k \times m$ rectangular grid of lattice points, with edges between points that are horizontal or vertical neighbors. The rules described above were constructed as part of an algorithm for "phase unwrapping"; i.e., determining the argument of a complex function given the principal value of the argument, so as to eliminate the jump discontinuities by integer multiples of 2π. The Ghiglia and Mastin paper [1] contains several pictures presenting their algorithm in operation.

The "phase unwrapping" origin of the transformation accounts for the lack of symmetry in the rules which prescribe that if the average of the values of a site's neighbors equals the value at that site, but not all the neighbors have values equal to that of the given site, then the value of the site should be incremented by 1. As it turns out, even if this condition is changed so that the value of a site stays constant when that value equals the average of the values of the neighbors, the length of the cycle is still at most 2. The proof of this is similar to that of our main theorem, and will be sketched at the end of Section 2.

Ghiglia and Mastin found by extensive simulations that iterations of the transformation always led to cycles of length 1 or 2. They conjectured that this is always the case, and their "phase unwrapping" algorithm is based on the assumption that this conjecture is true. Our theorem, which proves this conjecture, guarantees that the Ghiglia-Mastin algorithm will always terminate.

E. Brickell and M. Purtill were the first to consider the general transformation as we defined it above. When all the $x_i(0)$ are 0 or 1, they showed by the following very elegant combinatorial argument that the cycle length is at most two. At any time t, divide the vertices of G into four classes as follows:

$$\begin{aligned} C_1 &= \{i : x_i(t) &= 0 \text{ and } x_j(t) &= 0 \text{ for all } j \in J_i\}, \\ C_2 &= \{i : x_i(t) &= 1 \text{ and } x_j(t) &= 1 \text{ for all } j \in J_i\}, \\ C_3 &= \{i : x_i(t) &= 0 \text{ and there exists a } j \in J_i \text{ with } x_j(t) &= 1\}, \\ C_4 &= \{i : x_i(t) &= 1 \text{ and there exists a } j \in J_i \text{ with } x_j(t) &= 0\} \end{aligned} \quad (1.2)$$

Any site in C_1 at time t will be in C_1 or C_3 at time $t+1$ since the value will remain 0, but we cannot predict what will happen to its neighbors. Similarly, any site which falls in C_2 at time t will be in C_2 or C_4 at time $t+1$. Anything in C_3 will move to C_4 at time $t+1$, and all members of C_4 will move to C_3. Therefore, eventually all elements will either stay in C_1 or in C_2 or will continue switching between C_3 and C_4, and so the cycle length will be 1 or 2.

When the $x_i(0)$ are not all 0 or 1 (or u and $u+1$, more generally), the iteration is much more complicated and no simple combinatorial argument has been found to prove the theorem. For example, even when G is a simple path, differences between values of adjacent vertices can be arbitrarily large on a cycle (as large as a constant times n for a path of length n). This can be seen by generalizing the construction of a simple path of length 11 with initial assignments $(x_1(0), ..., x_{11}(0)) = (0, 1, 1, 4, 6, 11, 15, 22, 25, 28, 27)$.

The proof we will give for the theorem is based on a modification of the proof used by Goles-Chacc, Fogelman-Soulie, and Pellegrin [2] to prove that cycle lengths are at most two in certain threshold networks. Their theorems imply the Brickell-Purtill result, but do not seem to directly cover the general case of our iteration. However, their concept of decreasing energy is a key ingredient in our proof. Furthermore, after reading an early version of this paper, E. Goles found a way to encode the iteration studied here into an iteration similar to those of [2]. This not only gives another proof of our theorem, but also leads to an explicit bound for the length of the non-periodic part of the iteration.

Other cases where iterations on graphs produce cycles of length at most 2 occurs in the work of Poljak and Sura [3] and Poljak and Turzik [4-6]. The paper [5] proves that cycle lengths are at most 2 for some very general classes of discrete iterations, and provides another possible approach to our problem.

2. Proof of theorem

It clearly suffices to prove the theorem when G is connected, and so we will assume this from now on.

Lemma 1. *If the period of the cycle is not 1, then for all large t and for all i,*

$$x_i(t) \neq x_i(t+1). \tag{2.1}$$

Proof of lemma. Suppose there exist i', t such that $x_{i'}(t) = x_{i'}(t+1)$ and that the t-th iteration is in the cycle. We know there exists j' such that $x_{j'}(t) \neq x_{j'}(t+1)$ since the period of the cycle is not 1. Hence we can find vertices i and j that are connected such that

$$\begin{aligned} x_i(t) &= x_j(t), \\ x_i(t) &= x_i(t+1), \\ x_j(t) &\neq x_j(t+1), \end{aligned} \tag{2.2}$$

and so

$$x_j(t+1) = x_j(t) \pm 1 . \tag{2.3}$$

Hence

$$x_i(t) - x_j(t) \equiv 0 \pmod 2 \tag{2.4}$$

and

$$x_i(t+1) - x_j(t+1) \equiv 1 \pmod 2 . \tag{2.5}$$

But if $x_i(t+k) - x_j(t+k) \equiv 1 (\mathrm{mod}\ 2)$, then $x_i(t+k) \neq x_j(t+k)$, hence $x_i(t+k+1) = x_i(t+k) \pm 1$ and $x_j(t+k+1) = x_j(t+k) \pm 1$, so $x_i(t+k+1) - x_j(t+k+1) \equiv 1 (\mathrm{mod}\ 2)$. Since this is true for all k, there does not exist any $k' > 0$ such that $x_i(t+k') \equiv x_j(t+k') (\mathrm{mod}\ 2)$, which means that $x_i(t)$ cannot be in the cycle and we have reached a contradiction, which proves the lemma. ∎

Proof of theorem. Using the idea of a decreasing "energy" function utilized by Goles-Chacc et al. [2], we define:

$$E(t) = -\sum_{i,j=1}^{n} a_{ij} x_i(t) x_j(t-1), \tag{2.6}$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ but } j \in J_i; \\ -d_i & \text{if } i = j; \\ 0 & \text{if } i \neq j \text{ and } j \in J_i. \end{cases}$$

Note that E is bounded below since the maximal element at any stage never increases with time.

We now consider the change in energy during iterations of the transformation:

$$\begin{aligned}\Delta E(t) &= E(t+1) - E(t) = -\sum_{i,j=1}^{n} a_{ij}x_i(t+1)x_j(t) - \sum_{i,j=1}^{n} a_{ij}x_i(t)x_j(t-1)\\ &= -\sum_{i=1}^{n}(x_i(t+1) - x_i(t-1))\sum_{j=1}^{n} a_{ij}x_j(t), \quad (2.7)\end{aligned}$$

since $a_{ij} = a_{ji}$ for all i, j. For each i, if $\sum_{j=1}^{n} a_{ij}x_j(t) < 0$, then

$$d_i x_i(t) > \sum_{j \in J_i} x_j(t), \tag{2.8}$$

so

$$x_i(t+1) < x_i(t), \tag{2.9}$$

$$x_i(t+1) - x_i(t-1) \leq 0, \tag{2.10}$$

and

$$-(x_i(t+1) - x_i(t-1))\sum_{j=1}^{n} a_{ij}x_j(t) \leq 0. \tag{2.11}$$

If $\sum_{j=1}^{n} a_{ij}x_j(t) > 0$, then

$$\begin{gathered} d_i x_i(t) < \sum_{j \in J_i} x_j(t), \\ x_i(t+1) > x_i(t), \\ x_i(t+1) - x_i(t-1) \geq 0, \end{gathered} \tag{2.12}$$

and

$$-(x_i(t+1) - x_i(t-1))\sum_{j=1}^{n} a_{ij}x_j(t) \leq 0. \tag{2.13}$$

Finally, if $\sum_{j=1}^{n} a_{ij}x_j(t) = 0$, then

$$-(x_i(t+1) - x_i(t-1))\sum_{j=1}^{n} a_{ij}x_j(t) = 0. \tag{2.14}$$

Thus in all cases $\Delta E(t) \leq 0$ and each term in the sum on i on the right side of (1) is ≤ 0. Since E is bounded below, we must have $\Delta E(t) = 0$ for all large $t \geq t_0$ and, moreover, for all $t \geq t_0$ and all i,

$$(x_i(t+1) - x_i(t-1))\sum_{j=1}^{n} a_{ij}x_j(t) = 0. \tag{2.15}$$

We can take t_0 so large that $t_0 - 1$ is already in the cycle. Now suppose there exists an i such that $x_i(t+1) \neq x_i(t-1)$ for some $t > t_0$. Then there must exist a $t > t_0$ with $x_i(t-1) > x_i(t+1)$. We must then have

$$\sum_{j \in J_i} x_j(t) = d_i x_i(t), \tag{2.16}$$

and so

$$x_i(t+1) \geq x_i(t), \tag{2.17}$$

and hence by the Lemma, $x_i(t+1) > x_i(t)$, which implies that $x_i(t+1) \geq x_i(t-1)$, which is a contradiction.

Therefore, $x_i(t+1) = x_i(t-1)$ whenever $x_i(t-1)$ is in the cycle, so the length of the cycle is, at most, 2. ∎

It is not always true that $\Delta E(t) < 0$ for t not in the cycle. For example, when G consists of a simple path of length 5, with $(x_1(0), ..., x_5(0)) = (0,2,2,3,5)$, then $E(1) = E(2) = -1$, but the cycle starts only at $t = 2$.

When the definition of the iteration is changed so that $x_i(t)$ does not change when it equals the average of the $x_j(t)$ for $j \in J_i$, the proof of the Theorem becomes somewhat easier. In this case, the Lemma is false. However, the expansion of Eq. (1) still holds, and we again find that Eq. (2) holds for all $t \geq t_0$ and all i. But that means that for any $t \geq t_0$ and any i, either $x_i(t+1) = x_i(t-1)$ or else $x_i(t+1) = x_i(t)$. If it ever happens that $x_i(u+1) = x_i(u)$ any i and some $u \geq t_0$, then by the above observations we must have $x_i(v) = x_i(u)$ for all $v > u$. This proves that the cycle length is 1 or 2.

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