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Inhomogeneous Cellular Automata and Statistical Mechanics

Hyman Hartman
Bitter National Magnet Laboratory,
Massachusetts Institute of Technology,
Cambridge, MA 02139, USA

Pablo Tamayo
William Klein
Center for Polymer Studies and Department of Physics,
Boston University,
Boston, MA 02215, USA

Abstract. An inhomogeneous cellular automaton is considered consisting of two two-dimensional planes, one of which contains the transition functions or programs and the other the evolving data. With a two-level hierarchy, basic experiments are performed with quenched and annealed XOR and AND functions. These functions were distributed both at random and with correlations produced by a Glauber algorithm simulating a nearest-neighbor ferromagnet, anti-ferromagnet and spin-glass.

1. Introduction

Invented in 1948 by John von Neumann and Stanislaw Ulam, cellular automata consists of regular arrays of cells with a discrete variable at each cell. With an evolution law in discrete time, a cellular automaton is a fully discrete dynamical system [1,2]. The law is local: the value assumed by any one site at time $t + 1$ is determined by the values taken at time t by the neighboring sites. These systems are homogeneous in the sense that all sites evolve according to the same transition function. If the function varies from cell to cell we have an inhomogeneous cellular automaton (INCA) [3]. The simplest case of an INCA consists in having two different local transition functions. Each cell applies either one of them according to some criterion. In this paper we consider two-dimensional inhomogeneous automata in which the function is either XOR or AND.[1]

[1] For these transition functions the XOR or AND 4-input Boolean function is applied to the nearest neighbors: north, south, east and west, in other words, to the von Neumann neighborhood excluding the center. The motivation of using these functions stems from the fact that they are simple and constitute a universal set.

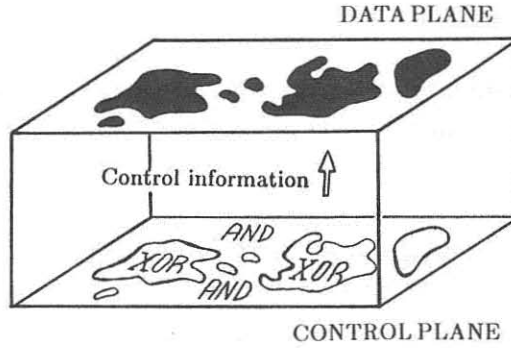

Figure 1: Two level hierarchy between control plane and data plane.

The plane on which the variables (1 or 0) evolve by the application of XOR or AND, specified by the corresponding variables in the control plane, is called the data plane. This structure establishes a simple hierarchy between the two planes (see figure 1).

We study the behavior of the data plane under different evolution dynamics and thermodynamic conditions implemented in the control plane. The evolution dynamics chosen were essentially dynamical Ising models, whose properties (critical points, cluster distributions, etc.) are known. We study the dynamics and structure observed in the data plane when it is riding on top of the control plane.

Generally two species of functions were distributed at random. Interesting behavior however arises if, instead, the functions are distributed with some correlation, such as assigning the XOR function to an up spin and the AND function to a down spin in an Ising model which is allowed to evolve with Glauber dynamics. Then the Ising Hamiltonian

$$-\beta H = \beta \sum_{<ij>} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i \qquad (1.1)$$

controls the distribution of functions on the control plane. The interactions between the spins (functions) can be taken to be ferromagnetic ($J_{ij} = 1$ for all i and j) anti-ferromagnetic ($J_{ij} = -1$ for all i and j) or a mixture such as a spin-glass (the sign of J_{ij} chosen at random).

In a previous paper [4] Vichniac *et al* mapped a probabilistic one-dimensional cellular automaton model proposed by Domany and Kinzel [5] into an inhomogeneous cellular automaton with the Boolean functions XOR and AND as transition functions. Wolfram's classification (in a phenomenological sense) was recovered by varying the ratio of these two simple functions and by quenching or annealing the inhomogeneity.

S. Wolfram has recently considered several one-dimensional quenched inhomogeneous cellular automata as special cases of cellular automata engineering [6].

Figure 2: Quenched case (61% of u's). Growing evolution, from a single 1, of the XOR-AND dynamics. The 1's on the data plane percolate through the fractal formed by XOR-sites.

2. Description and interpretation of the experiments

The simulations were performed on a CAM-5[2]. Periodic boundary conditions were chosen for the 256×256 planes.

For the data plane the XOR function was applied if the corresponding control cell was in an up state (u), and AND if it was in a down state (d). The initial conditions were a single 1 (seed) in a sea of 0's.

For the control plane the state u was interpreted as spin up and d as spin down. An identity function was applied for the quenched case, and a Monte Carlo algorithm with an external random number generator (to simulate temperature) was applied for the annealed case.

2.1 Quenched case, XOR-AND dynamics and percolation

The control plane is initialized with a random distribution of u's and d's, the concentration of u's (called p) is kept fixed in time.

The distribution of u's in this plane (XOR regions) form a fractal structure for the data plane. A single 1 on a background of 0's is seeded in the data plane and the simulation is run until the configuration of 1's attains its maximum size (see figure 2). For different values of p the 1's percolate on the XOR sites forming clusters of increasing size as p increases. We have collected statistics which indicate there is a critical concentration $p_c \simeq 0.61$ at which the growth of the cluster of 1's is very dendritic and may span the whole system. These clusters are subsets of the standard site percolation

[2]CAM-5 is a dedicated hardware simulator developed by Tom Toffoli and Norman Margolus [7].

clusters which can be obtained by changing the XOR function to the OR function[3]

At $p_c = 0.61$ we measured the fractal dimension d_f of the clusters as defined by the sites available to the 1's percolating on the XOR regions. We found $d_f \simeq 1.88$. We interpret this to mean that the fractal dimension is the same as for site percolation clusters, which is $d_f = 1.89$ [8]. We conclude that the fractal dimension of the XOR-AND clusters is the same as site percolation. The threshold may be changed but our data is, at present, insufficient to draw that conclusion. The difference between XOR-AND percolation and OR-AND percolation (site-percolation) is basically due to a "necking phenomena" which occurs in certain areas where the functions define peculiar regions that behave as dynamical bottlenecks. The 1's evolving in an XOR region try to pass through an articulation point one cell thick surrounded by a sea of ANDs, but when they arrive with the same phase relation a destructive interference occurs making impossible the transit through the neck. A close look at those regions reveals that two or three paths converge at the neck and the phase relation determines if the neck will be crossed or not. For example, if for a given initial condition two 1's arrive at the same time, the XOR function will always give 0 stopping the propagation at that place. On the other hand, if one arrives first, the XOR function will give 1 allowing the spreading of 1's through the neck.

This model implements a simple and natural method to build random nearest neighbor boolean nets (like those studied by Kauffman [9]). Their size can be controlled by p.

2.2 Annealed case, ferromagnetic Ising-Glauber and infection process

In the Glauber model the system evolves according to the following rule: at every active lattice site the spin is flipped[4]. If the energy of the system is lowered the flip is accepted. If, however, the energy of the system is raised by the spin flip the number

$$\Delta = \exp(-\beta \Delta E) \tag{2.1}$$

is compared with a number between 0 and 1 generated with a random number generator. Here ΔE is the energy change generated by the spin flip. If Δ is less than the random number the flip is rejected. If, however, Δ is greater than the random number the flip is accepted. It should be clear from the above description that neither the energy or the magnetization (which is the order parameter in this model) is conserved during this evolution.

[3] Formally the AND function should be changed too, in this case to a ZERO function which always gives 0 as the next state. However in practice for p not close to 1 the AND behaves effectively as a ZERO function.

[4] The two sublattices of the square lattice alternate in time as active lattices, then the updating occurs in a checkerboard fashion to avoid undesired oscillations (see Vichniac's paper in [1]).

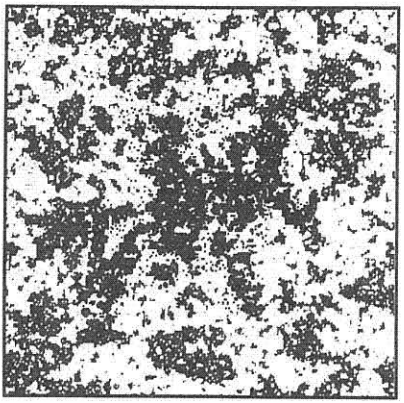

Figure 3: Ferromagnetic Ising-Glauber. Trace of the XOR-AND dynamics on the data plane.

Conservation of the magnetization can be achieved using the Kawasaki dynamics.

For this simulation the Ising-Glauber dynamics is implemented in the control plane. The temperature is obtained from an external hardware-implemented random number generator. A single 1 is seeded in the data plane, then, as the simulation runs the 1's spread over the Ising clusters which become "islands" in where the XOR 1's can percolate (for a review on correlated percolation and Ising models see [10]).

The temperature of the model controls the degree of correlation of the functions. At $T = \infty$ we recover the random model with uncorrelated functions. For high temperatures the clusters are small and very fragmentary and the percolating 1's remain localized or disappear completely. At low temperatures the percolation of 1's becomes an infection process in which clusters "infected by 1's" eventually make contact with other clusters spreading the disease. Figure 3 shows a trace[5] of the growing evolution, from a single one, of the XOR-AND dynamics being driven by an Ising-Glauber ferromagnetic dynamics on the control plane (at a finite temperature).

2.3 Annealed case, antiferromagnetic Ising-Glauber

This simulation is basically the same as the previous case, but now the interaction is anti-ferromagnetic. In this case the dominant phases are checkerboards of alternating spins up and down. This configuration makes the percolation of XOR 1's impossible at $T = 0$ because in a checkerboard

[5]By trace we mean the set of sites which took the value 1 at least once in the past.

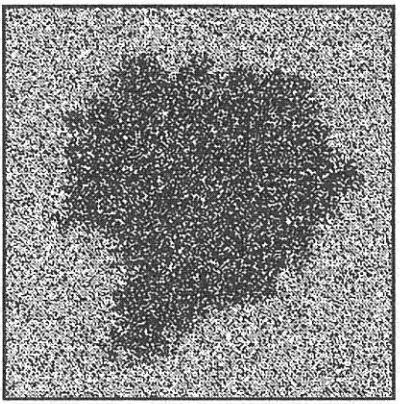

Figure 4: Anti-ferromagnetic Ising-Glauber at high temperature. Trace of the XOR-AND dynamics on the data plane.

all nearest neighbors for an XOR-site are AND-sites.

At high temperatures the checkerboard clusters are extremely fragmentary and the 1's manage to percolate and form an almost isotropic growing cluster. The speed of growth can be controlled by the temperature. Figure 4 shows a trace of the growing evolution, from a single one, of the XOR-AND dynamics being driven by an Ising-Glauber anti-ferromagnetic dynamics on the control plane (at a finite temperature).

At low temperatures the "percolating" 1's are mostly confined to the domain borders between checkerboard phases, because in the interface the checkerboard pattern is disrupted. The domain walls are followed by the "surviving 1's" in the data plane as they move and oscillate. Eventually some domain borders merge and the 1's travel and spread along the new domain border. This process allows us to study the dynamics of domain borders and keep track of the interactions between them. Figure 5 shows the u's checkerboard phases (very close to $T = 0$) and the 1's on the data plane percolating along the domain borders.

2.4 Interaction with random bonds (spin-glass)

Perhaps the richest of the models we discuss in this paper is the spin glass. The two parameters of interest are the temperature and the density of anti-ferromagnetic bonds. At $T = 0$ there exists a critical density of anti-ferromagnetic bonds at which the system develops an infinite number of ground states. The cause of this unusual behavior is known to be the so-called frustration generated by the mixture of bonds.

For example, the elemental plaquette in a square lattice has three ferromagnetic bonds and one anti-ferromagnetic bond. With this arrangement

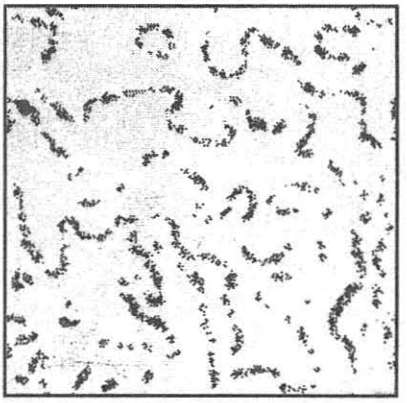

Figure 5: Anti-ferromagnetic Ising-Glauber at $T \simeq 0$. Trace of XOR-AND dynamics on data plane. The u's on the control plane are also shown.

of bonds there exists no configuration of the Ising spins with all the bonds satisfied. A satisfied bond is one where the spins at each end of the bond generate an energy of K as opposed to $-K$. The minimal energy possible with the bond configuration of a frustrated plaquette is $2K$. What is more important is that this "ground state" is not unique. It is the frustration which gives rise to the infinite number of ground states in the thermodynamic limit.

There is some controversy about the nature of the phases at finite temperature, however, the impact on the automata evolution of the frustration caused by the mixture of the anti-ferromagnetic and ferromagnetic bonds is substantial.

To allow the possibility of different J_{ij} additional planes (J_{ij} planes) should be employed. These planes contain the information about the nature of the interaction (i.e. an f would mean ferromagnetic and an a would mean anti-ferromagnetic). In this way a third level in the hierarchy is introduced; the spins on the control plane look to the J_{ij} planes to evaluate the J_{ij} in the calculation of $\sum J_{ij}\sigma_i\sigma_j$. The J_{ij} planes are initialized with some random distribution of f's and a's, and kept fixed.

With this model we can, of course, reproduce the ferromagnetic and anti-ferromagnetic simulations already described, but we can also interpolate between them by selecting a concentration of bonds between the two extremes.

At high temperatures the growth of a single seed in the data plane resembles the growth observed in the case of ferromagnet but now it is more dendritic, caused presumably by the additional disorder introduced by the random bonds, and proceeds at a faster rate. Figure 6 shows a

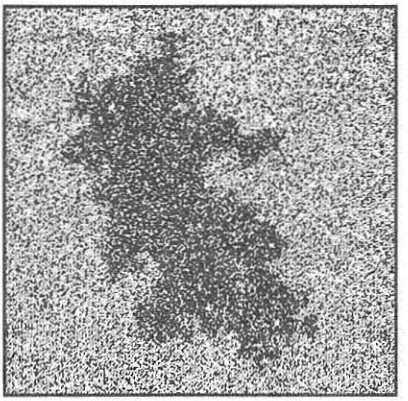

Figure 6: Spin-glass at high temperature. Trace of the XOR-AND dynamics on the data plane.

trace of the growing evolution, from a single one, of the XOR-AND dynamics being driven by an Ising-Glauber Spin-glass dynamics (random bonds, 30% ferromagnetic) on the control plane (at a finite temperature).

Although the most interesting case is $T = 0$; starting with 50% of spins up and 50% of spins down on the control plane, we run the simulation for 2000 steps to let the system relax to local energy minimum states. Then we performed two sets of experiments.

In the first case the spin plane is "frozen" (with an identity function) and a single 1 is seeded in the data plane. Then the 1's percolate over the static mask of the spin clusters. The experiment is repeated for various distributions of f's and a's.

In the second case the spin plane is "unfrozen" and the experiment is performed as before. The difference is that flipping spins localized in the border of clusters (frustrated spins) will eventually open new paths facilitating the percolation of 1's in the data plane. Figure 7 shows on the left three local energy minimum states for spin glasess with 5%, 35%, and 50% of anti-ferromagnetic bonds. On the right the trace of the XOR-AND evolution is shown, dark for the percolation over frozen spins (control plane is inactive), light for the percolation over frustrated spins (control plane is active).

3. Plans for future research

The study of INCA can be extended in several directions. In particular the following topics deserve attention.

Kawasaki Dynamics. This algorithm proceeds essentially the same as

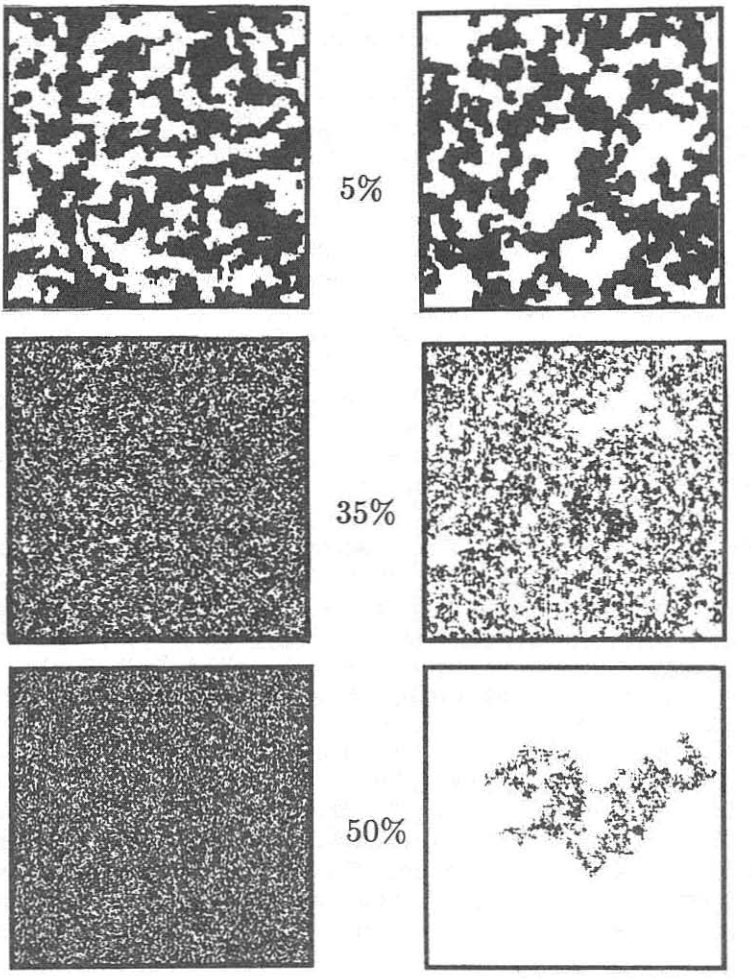

Figure 7: Spin-glass local minimum states. In the left appear the spins local energy minimum states and in the right the trace of XOR dynamics: dark over frozen control plane and light over unfrozen control plane.

previously mentioned, except that spins are not flipped by themselves but are flipped in up-down pairs. That is, one exchanges two spins of opposite sign. If, as before, the energy is lowered by the spin exchange then the move is accepted. If, however, the energy is raised the same method as described in the Glauber algorithm is employed to decide whether the exchange is accepted or not. This algorithm conserves the order parameter but not the energy. This can be done with the Creutz algorithm.

Creutz Dynamics. The Creutz algorithm [11] is similar to the Glauber with one essential difference. Creutz defines a site demon which during the spin flip process can receive or donate energy. The demons can carry energy only up to a certain maximum amount (which is to some extent arbitrary) and Creutz requires that the total energy of the system and the demons be conserved. If the maximum demon's energy is small compared to the energy of the system, which is the condition in most simulations and is rigorously true in the thermodynamic limit, the system energy is conserved to a good approximation. The case with zero energy demons is equivalent to a transformed version of the Q2R rule.

Time scales. This problem comes from the fact that there are two dynamical phenomena occurring simultaneously. On the control plane the Ising spins evolve with a characteristic time scale set by the dynamical model employed. The data plane evolves according to the automata rules. The free parameter we have at our disposal is the relation between these two time scales. In the language appropriate to simulation we must decide how many Monte Carlo time steps we will have on the control plane between each update of the data plane. We can clearly specify two extremes. The annealed case is the situation in which we update the data plane after each Monte Carlo time step. The quenched case is where the control plane is allowed to evolve to the desired degree and then frozen so that the data plane then evolves on a fixed set of rules. Clearly one can interpolate between these two extremes. This type of study will be particularly interesting in the case where the control plane is not in equilibrium but is in either a metastable or unstable state.

Staggered field. For the anti-ferromagnet it is also of interest to consider the effect of the addition of a staggered field. Restricting our consideration to the square lattice the Hamiltonian becomes

$$-\beta H = K \sum_{<ij>} \sigma_i \sigma_j + h \sum_i \sigma_i + \sum_j \tilde{h}_j \sigma_j \tag{3.1}$$

where $\tilde{h}$ is the staggered field.This field points in the positive direction on one of the two sublattices of the square lattice and in the negative direction in the other. The staggered field in the anti-ferromagnet plays the role of the applied field in the ferromagnet. The field h in the anti-ferromagnet

can now be varied at zero staggered field. A line of critical points will result in the h, K plane. At a critical value of the field h the line of critical points will terminate at a tri-critical point. The behavior at this point of the automaton is of great interest since our understanding of the interplay between the correlations on the control plane and the evolution of the data plane is not well understood.

Spin-glasses. The major problem in cellular automata is how to program (for general purpose) such automaton. With inhomogeneous cellular automata, the distribution of functions on the control plane is the program. If one adds an Ising Hamiltonian we are studying the Statistical Mechanics of programs. Ground states of Spin-glass have recently been used to model the origin of life [12] and to solve combinatorial optimization problems [13]. In particular the ground states of the two-dimensional $\pm J$ Ising Spin-glass has been shown to be equivalent to the Chinese Postman problem [14]. Therefore, we can if we choose some cost function use the spin-glass ground states to optimize a program using techniques from Statistical Mechanics. One such cost function is to interact the ones on the data plane with the spins on the control plane using a Creutz-like dynamics.

Spin-glasses (for a review see [15]) are now being used to model neural networks and content addressable memories [16,17]. Ground states can be used to store patterns, with the INCA implementation that patterns can play an active role as templates defining computation for the data plane. In this way we carry out one step further the computational capabilities of this models.

Cross coupled interaction between planes. We expect much more complicated and interesting phenomena if we couple the planes in such a way that the data plane can have an influence on the control plane. In other words, the data plane can alter the program on the control plane. There are many possibilities that can be explored with this scheme.

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