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Further Evidence for Randomness in π

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Abstract. It has been generally conjectured, based on empirical evidence, that the digits of π represent a random sequence. Some recent experiments appear to challenge this conjecture. This note describes a more extensive set of experiments concerning these reported anomalies, which betray no significant evidence of non-randomness. Results for e and $\sqrt{2}$ are also given.

1. Introduction

The question of whether the digit sequence of π and other transcendental numbers are random sequences has a unique appeal to those interested in number theory. A real number is *normal in base* b [1] if for $m \geq 1$, all b^m m-strings occur equally often. A number is *normal* if is normal in all bases. It has been conjectured that many transcendental numbers, including π, e, and $\sqrt{2}$ are normal. As a result of the difficulty in proving normality, much effort has gone into empirical testing. In fact, testing for normality is one of the two common excuses for calculating further digits of π, the other being computer hardware verification. Normality is related to the question of *randomness*, whether a representation of a number simulates the behavior of an unpredictable process.

Bailey [2] calculated the first 29,360,000 decimal digits of π and subjected them to normality testing for up to 6-strings and tests for subsequence repetition and run lengths. He found no deviations from randomness, continuing a long line of similar results dating back at least to ENIAC [3]. However, Wolfram [4], as part of a study on random number generation, conducted more rigorous tests on a variety of pseudo-random sequences and reported results which question whether π is truly random. This note describes a series of experiments which expand on these tests, which show that π must still be considered a random sequence.

2. Methodology

This work is based on the randomness testing techniques of Knuth [5] who gives a variety of empirical tests designed to uncover local and global non-randomness. Three of these tests gave results indicating non-randomness significant to at least the 0.05 level: the coupon collector's test, the permutation test, and the gap length test. Each of these three tests have been repeated and expanded.

The length of the sequences used in these experiments is 27,398,104 bits for π, 9,699,336 bits for e, and 47,579,136 bits for $\sqrt{2}$.

2.1 Coupon Collector's Test

The block accumulation or coupon collector's test measures how many successive samples of m bits it takes for all 2^m outcomes to occur. A cutoff of k samples is used to ensure that the test terminates. The expected distribution of sample sizes for a random sequence is calculated and compared to the observed distribution. We use $m = 3$ and $k = 40$ for our experiment.

Following Knuth, each test was repeated $5/p_i$ times, where p_i is the probability of the least likely outcome, after which the χ^2 test was used to determine the probability that the observed distribution was based on the experimental one. By repeating this procedure as many times as possible with the given number of bits, we obtain a distribution of χ values which should be uniform over the interval (0,1). The distribution of these values were subjected to the Kolmogorov-Smirnov test, which identifies the largest deviation from the uniform distribution and assesses its significance.

The number of samples were doubled over [4] by considering both the first and second three bits of each 8-bit byte of the expansion. The other two bits per byte were unused. The cumulative distribution is shown in Figure 1.

With 151 χ values, $k^+ = 0.671$ for a $p = 0.608$ and $k^- = 0.773$ for a $p = 0.710$. Neither of these values is near enough to 0 or 1 to be significant, so the coupon test presents no evidence of non-randomness for π.

2.2 Permutation Test

The permutation test measures the frequency of the $q!$ distinct permutations of q n-bit blocks. Excluding samples where two or more of the q blocks have identical values, for a truly random sequence the probability of each occurrence $p_i = 1/q!$. We use $n = 8$.

Additional results can be obtained by considering permutations of a different number of bytes and shifting the starting point for each sequence. These results are not completely independent, since subsequences of frequently occurring permutations should occur more often with shifting and resizing. However, this is adequate to show the negative result.

For each of our three transcendental bit sequences, we vary q between 3 and 6 and the offset between 0 and $q - 1$. The results are summarized in

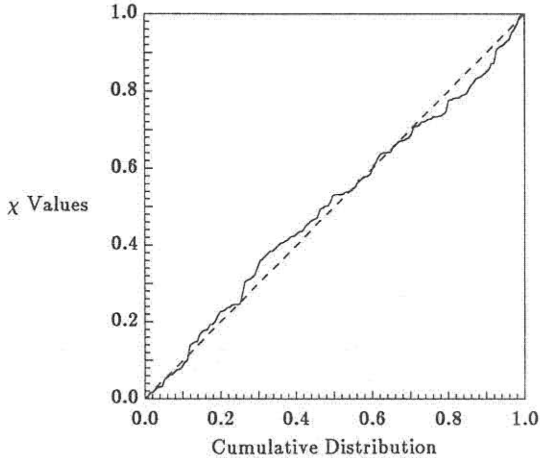

Figure 1: The distribution of χ values for the coupon collector's test for π. A random sequence would be expected to distribute χ values uniformly over the interval $(0, 1)$, resulting in the line $y = x$.

Table 1.

The p values in the table come from the Kolmogorov-Smirnov test, as described in the previous section. The totals in Table 1 are the results of running the Kolmogorov-Smirnov test on the distribution of these p values, and show no significant deviations from randomness.

2.3 Gap Length Test

The gap length test measures the length of runs of m-bit samples such that they do not fall in the gap between two values α and β. All runs of length $\geq k$ are lumped together. We use $m = 8$. and a variety of gaps with $\beta - \alpha = 50$ and $\beta - \alpha = 100$, covering the interval between 0 and $2^8 - 1$. Values 8, 16, and 24 are used for k.

Again, these values are not totally independent, but suffice to show a negative result. As with both the previous tests, the χ^2 test was run on samples of size $5/p_i$, where $p_i = (1 - \beta - \alpha)^{-k}$ or the smallest probability of any sample. The p values in the table again are from the Kolmogorov-Smirnov test the uniformity of the χ distribution. The results are summarized in Table 2. Note that the number of bits of e is not sufficient for significance when $t = 24$ and $\beta - \alpha = 100$. The total p values come from the Kolmogorov-Smirnov test and betray no evidence of non-randomness.

3. Conclusions

We have shown that all three transcendental numbers, π, e, and $\sqrt{2}$ pass our extended randomness tests. Thus, the evidence supports the hypothesis

		π		e		$\sqrt{2}$	
size	offset	$p(k^+)$	$p(k^-)$	$p(k^+)$	$p(k^-)$	$p(k^+)$	$p(k^-)$
3	0	0.253	0.566	0.095	0.951	0.270	0.547
	1	0.604	0.245	0.413	0.662	0.330	0.454
	2	0.704	0.706	0.181	0.538	0.258	0.839
4	0	0.996	0.060	0.515	0.286	0.062	0.660
	1	0.069	0.752	0.184	0.713	0.157	0.956
	2	0.110	0.962	0.067	0.753	0.531	0.484
	3	0.087	0.922	0.108	0.921	0.201	0.717
5	0	0.961	0.204	0.869	0.040	0.225	0.894
	1	0.577	0.557	0.337	0.791	0.598	0.287
	2	0.501	0.802	0.222	0.497	0.027	0.986
	3	0.443	0.473	0.338	0.356	0.096	0.961
	4	0.864	0.071	0.984	0.161	0.722	0.539
6	0	0.485	0.274	0.521	0.414	0.319	0.321
	1	0.257	0.443	0.886	0.532	0.923	0.551
	2	0.920	0.015	0.828	0.061	0.880	0.013
	3	0.813	0.076	0.764	0.319	0.508	0.283
	4	0.930	0.090	0.047	0.924	0.378	0.202
	5	0.424	0.183	0.244	0.513	0.565	0.220
total	$p(k^+)$	0.143	0.865	0.844	0.215	0.885	0.157
	$p(k^-)$	0.583	0.060	0.117	0.449	0.043	0.583

Table 1: Permutation test results for π, e, and $\sqrt{2}$. Note that, although certain individual tests seem provocative (for example, size=4, offset=0 for π), the anomalies do not appear with different parameter values and are not significant.

			π		e		$\sqrt{2}$	
α	β	t	$p(k^+)$	$p(k^-)$	$p(k^+)$	$p(k^-)$	$p(k^+)$	$p(k^-)$
0	100	8	0.355	0.966	0.272	0.602	0.519	0.363
		16	0.605	0.269	0.123	0.874	0.865	0.116
		24	0.933	0.055	-	-	0.851	0.213
50	150	8	0.826	0.408	0.483	0.371	0.045	0.791
		16	0.928	0.653	0.050	0.968	0.951	0.155
		24	0.871	0.198	-	-	0.833	0.283
100	200	8	0.231	0.810	0.942	0.085	0.128	0.942
		16	0.991	0.002	0.415	0.821	0.034	0.831
		24	0.145	0.895	-	-	0.211	0.712
155	255	8	0.214	0.448	0.328	0.200	0.199	0.950
		16	0.178	0.843	0.434	0.747	0.203	0.883
		24	0.935	0.049	-	-	0.527	0.546
0	50	8	0.074	0.971	0.040	0.998	0.363	0.534
		16	0.048	0.817	0.367	0.344	0.393	0.483
		24	0.040	0.822	0.197	0.833	0.238	0.350
50	100	8	0.544	0.566	0.882	0.078	0.224	0.592
		16	0.677	0.176	0.933	0.007	0.148	0.700
		24	0.847	0.123	0.997	0.045	0.005	0.872
100	150	8	0.181	0.813	0.406	0.661	0.058	0.922
		16	0.279	0.356	0.201	0.612	0.130	0.532
		24	0.345	0.750	0.335	0.625	0.496	0.213
150	200	8	0.129	0.981	0.854	0.518	0.497	0.410
		16	0.116	0.968	0.760	0.886	0.589	0.368
		24	0.101	0.934	0.796	0.077	0.888	0.317
200	250	8	0.271	0.421	0.948	0.166	0.260	0.930
		16	0.239	0.657	0.648	0.773	0.078	0.530
		24	0.409	0.412	0.396	0.651	0.579	0.637
205	255	8	0.113	0.619	0.592	0.234	0.487	0.535
		16	0.455	0.228	0.519	0.315	0.930	0.087
		24	0.722	0.162	0.906	0.487	0.982	0.026
total	$p(k^+)$		0.954	0.141	0.468	0.558	0.934	0.191
	$p(k^-)$		0.235	0.864	0.446	0.664	0.266	0.466

Table 2: Gap length test results for π, e, and $\sqrt{2}$. Although certain parameter combinations lead to high or low p-values, these are not confirmed over different gaps (α and β) and sequence lengths t.

of randomness.

If a sequence is truly random, certain portions of it must fail any randomness test. For these results to be accepted as significant, closely related tests, such as variations on sample length or starting position should lead to similar results. Our experiments show that these tests do not give any evidence of deviations from randomness.

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