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Compressible Rayleigh-Benard Spectral Simulations: A Useful Reference Solution

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Abstract. In this short note, we describe a steady-state solution of fully compressible convection which could be used for comparisons with those obtained with the lattice gas approach. This solution has been obtained with a pseudospectral code. It consists of two steady rolls at a maximum Mach number of 0.60 and at a maximum Reynolds number of 700. Slip boundary conditions for the velocity have been used. Regarding the temperature, we used a radiative boundary condition at the upper boundary of the fluid layer and fixed temperature at the lower boundary.

1. Introduction

Natural thermal convection in incompressible fluids has been the object of considerable interest in the past ten years. Recently, there has been increased interest in compressible fluids, essentially stellar convection. The anelastic approximation, which consists in filtering out the acoustic waves, has been used in the astrophysical context in order to overcome the Boussinesq approximation. This approximation is only valid for low Mach number. More realistic approaches must handle high frequency acoustic waves correctly. One approach was first carried out by Graham [1].

In order to obtain some insight into the sequence of instabilities leading to the temporal chaos, a pseudo-spectral numerical code has been developed. Such a method has been used because of its high accuracy and its ability to accurately describe nonlinear interactions.

Here, we report on a particular steady-state solution which could be used to test results obtained with the lattice gas approach. We also discuss modifications required for a more detailed comparison.

The outline of the paper is the following. Section 2 defines the problem of compressible convection. Section 3 is devoted to the description of the solution. Section 4 describes required modifications for comparison with lattice gas results.

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2. The physical problem and the equations of the model

The fluid motion takes place in a two-dimensional rectangular cavity of width L_x and height d. The z-axis is directed downward so that the gravity, represented by the vector $\mathbf{g} = (0, 0, g)$, is positive in this direction.

The equations of motion for a compressible, viscous, thermally conducting gas are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0, \tag{2.1}$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_i} - g_i \rho, \tag{2.2}$$

and

$$\frac{\partial(\rho e)}{\partial t} + \frac{\partial(\rho e u_j)}{\partial x_j} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - P\frac{\partial u_l}{\partial x_l} + \frac{\partial(K\frac{\partial T}{\partial x_i})}{\partial x_i} \tag{2.3}$$

where the viscous stress tensor is

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_l}{\partial x_l}\right). \tag{2.4}$$

The coordinates, x_1 and x_2 stand for the x and z coordinates, respectively.

This set of equations is closed by the equation of state for the perfect gas.

$$P = R\rho T \qquad \text{and} \qquad e = C_v\, T. \tag{2.5}$$

P, ρ, T, and e are pressure, density, temperature, and internal energy respectively; the u_i are the components of the velocity. The thermal conductivity and the dynamic viscosity are taken as constants. R is the gas constant and C_v the specific heat at constant volume.

Here, we are interested in slip boundary conditions for the horizontal velocity and radiative boundary conditions, as suggested by Spiegel [2], for the temperature. We impose the heat flux at the upper boundary to be fixed by the radiative energy of a black body

$$K\frac{dT}{dz} = \sigma^{sb}\ T^4 \tag{2.6}$$

where σ^{sb} is the Stefan Boltzmann's constant. Taking into account the fluctuations of the thermal conductivity with respect to the density and the temperature up to the first order and linearizing the equation (2.6) leads to the following relation at the upper boundary:

$$\frac{1}{S_f}\frac{d\Theta}{dz} + \frac{1}{K_0}\left(\frac{\delta K}{\delta \rho}\right)\rho + \frac{1}{K_0}\left(\frac{\delta K}{\delta \Theta}\right)\Theta = 4\Theta, \tag{2.7}$$

where $\delta/\delta\rho$ is the derivative with respect to ρ and the Stefan number $S_f = \sigma^{sb}\ T_0^3 d/K_0$ has been used. If the thermal conductivity K is of the form T^3/ρ [3], the temperature fluctuation, Θ, satisfies an inhomogeneous time-dependent Robins type boundary condition at the top of the layer,

$$\frac{1}{S_f}\frac{d}{dz}\Theta(x,z_0,t) - \Theta(x,z_0,t) = \rho(x,z_0,t), \tag{2.8}$$

and

$$T(z_0+d) = T_1$$

at the bottom of the layer. The boundary conditions for the velocity are

$$u_2 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial z} = 0 \quad \text{at} \quad z = z_0,\ z_0 + d. \tag{2.9}$$

Periodic boundary conditions are used in the horizontal direction for all variables. With these boundary conditions, the steady state is given by

$$T(z) = S_f Z, \tag{2.10}$$

$$\rho(z) = (zZ)^{(m+1)Z/S_f - 1}, \tag{2.11}$$

$$P(z) = S_f z^{(m+1)Z/S_f - 1}\ \ Z^{(m+1)Z/S_f}. \tag{2.12}$$

The coordinate z goes from Z^{-1} to $Z^{-1}+1$, where $Z = d/z_0$. This two parameter formula allows very weak stratification of the density and relatively strong pressure stratification.

The two-dimensional compressible convection problem is characterized by seven dimensionless parameters: the aspect ratio, A; the Prandtl number, σ; the ratio of specific heats, γ; the normalized layer thickness, Z; the polytropic index, m; the Rayleigh number, R; and the Stefan number, S_f.

The numerical scheme used has already been described in reference 4. It consists of an explicit stage for advective terms, pressure terms, and diffusive terms in the horizontal direction and an iterative method, with spectral preconditioning, for the vertical nonlinear diffusive terms. The code has been checked by computing critical exponents of the velocity and the Nusselt number at the onset of convection.

3. Spectral results

In figures 1 through 4, we have displayed steady state results at a Rayleigh number of 8500, a polytropic index and a stratification parameter equal to 1, a Prandtl number of 0.1, a ratio of specific heats of 1.67, a Stefan number of 1.85, and an aspect ratio of 2.79. With this set of parameters, the Rayleigh number is roughly 25 times the critical Rayleigh number of the onset of convection. The ratio of the density at the top and the density at the bottom is 1.06, but the pressure ratio is 2.12. We have given both

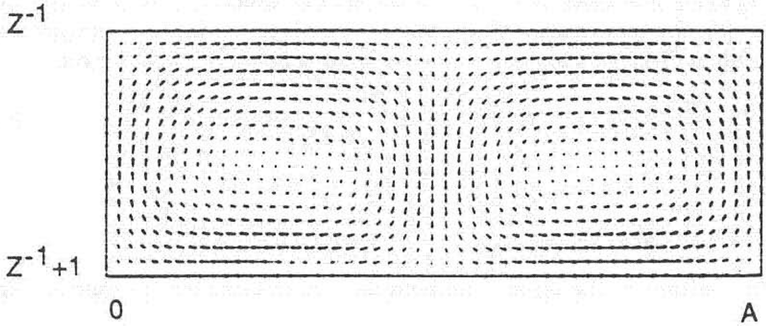

Figure 1: Velocity field of the steady state for the set of parameters described in the text.

local quantities like the maximum of the velocity and the thermodynamical variables and global quantities like the variance of the thermodynamical variables. V^2_{max} is defined as the maximum of the velocity squared

$$u^2(x,z) + v^2(x,z)$$

over the two-dimensional domain. The same definition has been used for the maximum Mach number.

Results are expressed in the following units: d, $d^2\rho\ (z_0+d)/\mu$, $\rho(z_0)$, and $T(z_0)$ for length, time, density, and temperature respectively. The spatial resolution was 32×33 modes, and we have checked that the highest coefficients of the spectral expansion were several orders of magnitude lower that the first ones. In steady state, the heat fluxes are equal at the top and bottom of the layer, as are the Nusselt numbers. Consequently, the integration process was stopped when the relative difference was less than 5×10^{-5}.

The spatial pattern shows a left-right symmetry with respect to a vertical line at the middle of the box. As it is now well known, rolls of convection are deformed by the compressibility. Relative fluctuations of density and pressure are quite different, due to the high Rayleigh number and the relatively low Prandtl number. The temperature profile reflects the Robins boundary condition which leads to strong relative temperature fluctuations at the top. This two-roll solution is stable, even at a Reynolds number of the order of 10^3, where the relative pressure fluctuations can reach 50 percent of the average pressure and the Reynolds and Mach numbers are respectively of the order of 700 and 0.6. Moreover, the transient leading to this steady state reveals that the largest eigenvalue is real. In other words, this configuration is very stable and is far away from the bifurcation point. The relative pressure fluctuations exhibit peaks where the flow changes direction. The largest velocity, 657.4, occurs at $x = 0.785$, $z = 2.0$, below the

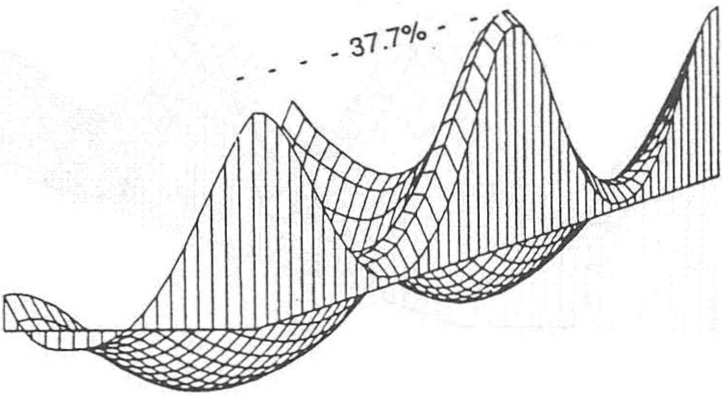

Figure 2: Relative density fluctuations. The maximum fluctuation of 37.7% occurs at $x = 1.396, Z = 1.941$.

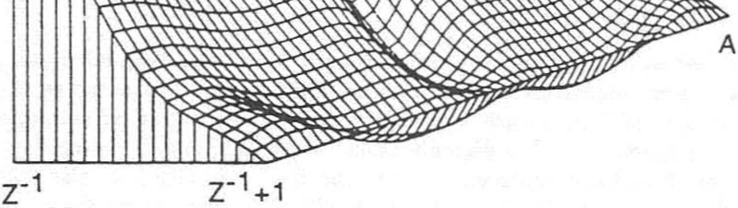

Figure 3: Relative temperature fluctuations. The maximum fluctuation of 46.1% occurs at $x = 0.0, z = 1.059$.

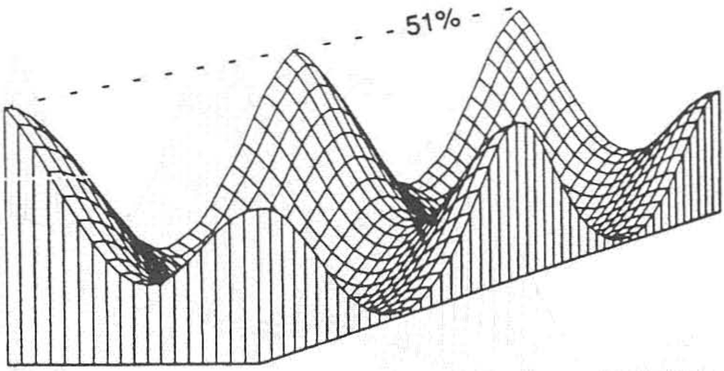

Figure 4: Relative pressure fluctuations. The maximum fluctuation occurs at $x = 1.396, z = 1.0$.

center of the rolls, on the lower boundary. The largest momentum, 708.0, occurs at $x = 0.785$, $z = 1.978$ very close to this boundary.

We have used the following definition of the Nusselt number [1]:

$$Nu = \frac{(F_t - F_a)}{(F_c - F_a)}. \tag{3.1}$$

This number coincides with the classic definition in Boussinesq theory for vanishing values of the stratification of the parameter, Z. $F_a = gK/C_p$ and $F_c = K(T_l - T_u)/d$ are respectively the flux of the adiabatic gradient and the conductive flux. The Nusselt number at both boundaries is 2.071.

4. Requirements for comparisons between lattice gas and spectral calculations

In order to make accurate comparisons between spectral calculations and lattice gas calculations, it is essential that both methods solve identical problems. Although such comparisons could be done in principle, no one has yet completed this difficult task. Rayleigh-Benard lattice gas calculations have been reported [5], but the implementation of the boundary conditions and gravitational forcing is different from standard spectral calculations.

One serious problem in comparison calculations is that the lattice gas equation-of-state is velocity dependent [6]. This velocity dependence vanishes as the velocity goes to zero, but that is not a useful limit for lattice gas calculations. A possible solution is to provide the spectral calculation with the same equation-of-state as used by lattice gas models. This could be done, but this equation-of-state has only been derived in the low velocity limit. It is possible to obtain the lattice gas equation-of-state from lattice gas calculations, but no one has performed these calculations.

Another serious problem exists with the lattice gas viscosity: It is density and velocity dependent in a complicated way which could be determined through extensive lattice gas calculations. (The analytic formulas for viscosity derived in the low-velocity limit differ slightly from calculated viscosities.) Again, in order to make accurate comparisons with spectral calculations, this complicated viscosity would have to be inserted into the spectral codes.

Another problem in making accurate comparisons is the fact that spectral methods and lattice gas methods have different higher-order "correction" terms to the Navier-Stokes equations. These terms depend on choice of time step and lattice size or smallest wavelength. Which of the two methods has the better correction terms is a question which has not been resolved and is probably problem dependent.

The last problem we cite is the existence of considerable noise in the lattice gas calculations. Much time averaging of a steady state result will be required to obtain densities and velocities accurate to three significant figures.

Given the above difficulties, it appears that considerable time will elapse before accurate comparisons can be made between the two methods.

5. Conclusion

A steady-state solution of two-dimensional compressible convection has been presented. This solution could be compared with results obtained within the lattice gas approach. A significant amount of work remains to be done before such comparisons can be done accurately.

Acknowledgments

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