The Non-wandering Set of a CA Map

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Abstract. An example is given of a cellular automaton map F implementing a binary counter whose non-wandering set, $\Omega(F)$, is strictly larger than the closure, $\Pi(F)$, of the set of points periodic under F.

1. Introduction

Cellular automata (see [2]) are shift-commuting maps on the full shift on k symbols. Therefore, they may be analyzed with the tools of dynamical systems theory. See for example [4], [6]. In [3] the non-wandering set, $\Omega(F)$, and the closure, $\Pi(F)$, of the set of points periodic under F are calculated for a number of cellular automaton rules. In all the examples illustrated there for which both $\Pi(F)$ and $\Omega(F)$ were calculated, they were equal. This is not true in general, however. This paper gives an example of a cellular automaton map F_1 for which $\Omega(F_1) \neq \Pi(F_1)$.

2. Definitions

If S is a finite set with $|S| = k$, the full shift on k symbols, $S^{\mathbf{Z}}$, is the set of functions from the integers to S. Equivalently $S^{\mathbf{Z}}$ may be viewed as the set of doubly infinite words in the symbols from S. The full shift can be topologized by giving S the discrete topology and $S^{\mathbf{Z}}$ the product topology, or equivalently by defining a metric:

$$d(x, y) = \sum_{i=-\infty}^{\infty} \delta(x_i, y_i)/2^{|i|} \tag{2.1}$$

where $\delta(a, b) = 0$ if $a = b$ and 1 otherwise.

There is a natural (left) shift map σ acting on $S^{\mathbf{Z}}$ defined by $\sigma(x)_i = x_{i+1}$. A closed, shift-invariant subset of $S^{\mathbf{Z}}$ is referred to as a *subshift*.

Definition 1. *A cellular automaton map is a continuous function $F : S^{\mathbf{Z}} \to S^{\mathbf{Z}}$ which commutes with σ.*

One way of constructing cellular automaton maps is to specify an arbitrary function

$$f : S^{2r+1} \to S \tag{2.2}$$

and allow $F : S^{\mathbf{Z}} \to S^{\mathbf{Z}}$ to be defined by the rule

$$F(x)_i = f(x_{i-r}, \ldots, x_{i+r}) \tag{2.3}$$

By the Hedlund-Curtis-Lyndon theorem (see [1]) all cellular automaton maps arise in this fashion.

Definition 2. *Given a continuous map F on a compact metric space, X, the periodic set of F, is denoted Per(F).*

$$\text{Per}(F) = \{x \in X \text{ such that } F^p(x) = x \text{ for some } p > 0\} \tag{2.4}$$

The closure of this set is denoted $\Pi(F)$.

$$\Pi(F) = \overline{\text{Per}(F)} \tag{2.5}$$

If X is a compact metric space, and $F : X \to X$ is a continuous function, the non-wandering set (see [7]) consists of all points $x \in X$ for which the inverse images of any neighborhood of x are not all disjoint.

Definition 3. *If $F : X \to X$ is a continuous map of a compact metric space, the non-wandering set of F, denoted $\Omega(F)$, consists of all points $x \in X$ such that for every neighborhood U of x, there exists an integer $n \geq 1$ such that $F^{-n}(U) \cap U \neq \emptyset$.*

Evidently $\Omega(F) \supseteq \Pi(F)$ for all F.

If $X = S^{\mathbf{Z}}$, and $F : X \to X$ commutes with σ, it is clear from the definitions that $\Pi(F)$ and $\Omega(F)$ are closed and shift-invariant (i.e., subshifts).

3.   A CA rule F_1 for which $\Omega(F_1) \neq \Pi(F_1)$

The rule F_1 implements a binary counter. Due to the local nature of cellular automata evolution, this counter does not produce a new number at every time step, but instead takes time increasing linearly with the number of consecutive ones which must be converted to zeros. Thus incrementing $2n$ to $2n + 1$ takes a single time step, whereas to get from $2^n - 1$ to 2^n requires $2n - 1$ steps ($n \geq 1$). This is due to the fact that the carry bit travels with unit speed.

The set S has five elements:

$$S = \{W, 0, 1, \vec{0}, c\} \tag{3.1}$$

The symbol "W" is a boundary marker. The symbols 0 and 1 represent digits. The symbol $\vec{0}$ represents a zero carrying a right-traveling signal. This signal has the function of carrying the information that all necessary carry operations have been executed. The c represents a 1 which is about to be incremented to a 0 (converting the site to the left to a 1 or a c, dependent

d_1	d_2	d_3	$\rightarrow$	d_2	digits remain fixed
d_1	d_2	W	$\rightarrow$	d_2	
d_1	d_2	$\vec{0}$	$\rightarrow$	d_2	
d_1	$\vec{0}$	W	$\rightarrow$	c	incrementing rule
$\vec{0}$	0	d_3	$\rightarrow$	$\vec{0}$	right-moving signal
$\vec{0}$	0	W	$\rightarrow$	$\vec{0}$	
d_1	$\vec{0}$	0	$\rightarrow$	0	
0	c	d_3	$\rightarrow$	$\vec{0}$	carry signal
0	c	W	$\rightarrow$	$\vec{0}$	
d_1	0	c	$\rightarrow$	1	
1	c	d_3	$\rightarrow$	0	
1	c	W	$\rightarrow$	0	
d_1	1	c	$\rightarrow$	c	
otherwise			$\rightarrow$	W	error condition
$d_i \in \{0, 1\}$					

Table 1: The function $f_1 : S^3 \rightarrow S$ generating F_1.

upon its current value). The carry travels to the left until it hits a zero to turn into a 1. A signal then travels to the right indicating that the operation has been completed.

The local function $f_1 : S^3 \rightarrow S$ generating $F_1 : S^{\mathbf{Z}} \rightarrow S^{\mathbf{Z}}$ is given in Table 1. The evolution of F_1 from the configuration representing the value 0 is shown in Table 2.

Definition 4. *The point z is defined by $z = \ldots 000\vec{0}WWWW\ldots$, or more precisely $z_i = 0$ if $i < 0$, $z_0 = \vec{0}$ and $z_i = W$ for $i > 0$.*

Lemma 1. $z \in \Omega(F_1)$.

Proof. In fact z satisfies a stronger condition. Given a neighborhood U of z, there always exists $n > 0$ such that $F_1^n(z) \in U$. To show this we define a series of neighborhoods of z.

Definition 5. *Let $B_n = \{x \in S^{\mathbf{Z}}$ such that $x_i = 0$ for $-n \leq i \leq -1$, $x_0 = \vec{0}$, and $x_i = W$ for $1 \leq i \leq n\}$.*
Equivalently B_n is the ball of radius $1/2^n$ about z.

Every neighborhood of z contains all but finitely many B_n and while the exact time it takes the orbit to return to a given B_n is somewhat complicated, it satisfies the inequality:

t	configuration						n
0	0.	0	0	0	$\vec{0}$	W	0
1	0	0	0	0	c	W	1
2	0	0	0	1	$\vec{0}$	W	2
3	0	0	0	1	c	W	3
4	0	0	0	c	0	W	
5	0	0	1	$\vec{0}$	0	W	
6	0	0	1	0	$\vec{0}$	W	4
7	0	0	1	0	c	W	5
8	0	0	1	1	$\vec{0}$	W	6
9	0	0	1	1	c	W	7

Table 2: The Evolution of Rule F_1 from the configuration z.

$$F^i(z) \in B_n \text{ for some } i \leq 2n\left[\log_2 n + 1\right] \tag{3.2}$$

∎

To complete the proof of the theorem we need to show that $z \notin \Pi(F_1)$ or equivalently that there is a neighborhood of z disjoint from $\text{Per}(F_1)$.

Lemma 2. *If* $x \in B_1$ *then* $x \notin \text{Per}(F_1)$.

Proof. It is useful to introduce a definition.

Definition 6. *A configuration* $x \in S^{\mathbf{Z}}$ *is numerical if* $x_i = W$ *for all* $i > 0$, *and there exists* $n \leq 0$ *such that* $x_n \in \{\vec{0}, c\}$, $x_i = 0$ *for* $i > n$ *and* $i \leq 0$, *and* $x_i \in \{0, 1\}$ *for* $i < n$.

The image of a numerical configuration is numerical. Assume that x is a numerical periodic point. The orbit of x must maintain a bounded distance from any given non-numerical point. The rule, however, has been set up so that the orbit of any numerical point approaches the non-numerical configuration $z' = \dots 000WWW \dots$ arbitrarily closely. Thus it follows that no numerical point is periodic.

If $x \in B_1$ is not numerical, there are two cases. Either there is a $\vec{0}$ to the left of a 1, c, or $\vec{0}$. In that case the extraneous $\vec{0}$ would have to hit the 1, c, or $\vec{0}$ to the left of site 0, causing a new W to appear which would then propagate to the right contradicting the assumption that the orbit returns to B_1 infinitely often. The other possibility is that there is a c to the left of a 1, c, or $\vec{0}$. Either the c continues to the left forever, or it will turn into a $\vec{0}$ reducing it to the previous case. ∎

These lemmas combine to give the result:

Theorem 1. $\Omega(F_1) \neq \Pi(F_1)$.

4. Further Discussion

With a little more work one can write down $\Pi(F_1)$ and $\Omega(F_1)$. First the periodic points form two classes. The fixed points consist of all configurations consisting entirely of digits 0 and 1, or with 0's and 1's followed by W's, or the configuration of all W's.

Definition 7.

$$K_1 = \{x \in S^{\mathbf{Z}} \text{ such that there exists } -\infty \leq n \leq \infty \text{ with}$$
$$x_i \in \{0,1\} \text{ for all } i < n \text{ and } x_i = W \text{ for } i \geq n\}$$

The second type of periodic configuration arises from spatially periodic words in the symbols $\{0, \vec{0}\}$ for which no two of three consecutive sites have value $\vec{0}$. On this subset the rule acts like the right shift, and there are periodic points of all periods except two. The closure of this set of periodic points is the entire subshift of finite type described as follows.

Definition 8.

$$K_2 = \{x \in S^{\mathbf{Z}} \text{ such that } x_i \in \{0,\vec{0}\} \text{ and } x_j = x_k = \vec{0} \Rightarrow |j - k| > 2\}$$

These cases exhaust $\Pi(F_1)$. The remaining case for the non-wandering set, is the case in which $\sigma^i(x)$ is numerical for some $i \in \mathbf{Z}$. If a numerical configuration c has infinitely many 0's, then its orbit will eventually return to any given neighborhood of c. The other possibility is that a numerical configuration has finitely many 0's in which case it will eventually evolve to a configuration f the form $\ldots 111c0^i WWW \ldots$ for some $i \geq 0$. All such configurations, may be realized as limit points of configurations with infinitely many 0's and thus are in the non-wandering set by virtue of its being a closed set.

Definition 9.

$$K_3 = \{x \in S^{\mathbf{Z}} \text{ such that } \sigma^i(x) \text{ is numerical for some } i \in \mathbf{Z}\}$$

The set K_3 is not closed, but its closure consists of $K_1 \cup K_3$.

Theorem 2.

$$\Pi(F_1) = K_1 \cup K_2$$
$$\Omega(F_1) = K_1 \cup K_2 \cup K_3$$

Acknowledgements

References

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