Complex Systems **7** (1993) 171–197

Time Series of Rational Partitions and Complexity of One-dimensional Processes

Alberto Albrigi
Mario Casartelli
Dipartimento di Fisica,
Università degli Studi di Parma,
Viale delle Scienze, 43100 Parma, Italy

Abstract. Time series based on couples of partitions, and a related reduction algorithm, are used to develop indicators of complexity for general one-dimensional processes with discretizable states. After introducing the calculation scheme, we provide algorithms for some typical examples (cellular automata and iterated maps). Experiments show the sensitivity of these indicators—to complexity in the intuitive sense, and to hidden features distinguishing complexity from ordinary randomness.

1. Rational partitions

The concept of a rational partition (r-partition) was introduced in [1], with the purpose of estimating the complexity of objects or situations endowed, in a broad sense, with a dynamics (cellular automata (CAs), mappings, shifts, patterns depending on a parameter, and so on). The idea illustrated there may be summarized in the following main points.

- Inasmuch as finite measurable partitions in probability spaces give a frame to estimate (e.g., by Shannon entropy) the uncertainty about an experiment, *couples* of partitions may be adapted to estimate the *nonsimilarity* between two experiments.
- In order to stress such nonsimilarity, it is important to erase the common factors of partitions. This operation, which presents analogies with the reduction to minimal terms of rational numbers, is called the *reduction process.*
- If coupled partitions are dynamically related, the estimate of nonsimilarity is also a measure of the emergence of novelty in the dynamical process, which is more sensitive to local features than global indicators (such as the Kolmogorov-Sinai entropy). Moreover, the associated algorithmic problems are computable, in principle.

It is therefore plausible that a quantitative characterization of complexity may be based on the r-partitions formalism. Such an approach is applied here to one-dimensional discretizable systems. Through experiments on CAs and quadratic maps, in particular, r-partitions are shown to provide simple and efficient tools to study several nontrivial features of dynamical processes.

General considerations about the nature of complexity, and about the relations between complexity and partitions, may be found in [1–5], and will be briefly discussed in the last section of this paper, with reference to actual results. Only main notations and definitions shall be recalled now, in view of applications. In doing so, clarifications on some combinatorial aspects of r-partitions, which are interesting in themselves, will be explicated.

Let $(\mathbf{M}, \mathcal{M}, \mu)$ be a probability space, and $\alpha \equiv \{A_1, \ldots, A_n\}$ a *finite measurable partition* of $\mathbf{M}$ into "atoms" $A_1, \ldots, A_n$. The set $\mathcal{Z}$ of all such partitions constitutes a metric space. The distance ρ is given by

$$\rho(\alpha, \beta) = H(\alpha \mid \beta) + H(\beta \mid \alpha), \tag{1.1}$$

where $H(\alpha \mid \beta)$ is the conditional entropy of α with respect to β. Symbols $\vee$ and $\wedge$ denote usual operations of product and intersection in $\mathcal{Z}$ (we shall also use $\alpha\beta$ as equivalent to $\alpha \vee \beta$). (For general information on ergodic theory see, for example, [6–8]).

The partial order $\alpha \leq \beta$ means that β refines α or, equivalently, that the atoms of α are made up with atoms of β: in such case α is called a "factor" of β. In particular, α' is a "dichotomic factor" of α if $\alpha' \leq \alpha$ and α' has two atoms. Let $D(\alpha)$ denote the totality of dichotomic factors of α. If there are n atoms in α, then the nontrivial dichotomic factors in $D(\alpha)$ are $2^{n-1} - 1$. Indeed, each atom $A_1^{(k)}$ of the factor α_k in $D(\alpha)$ may be labelled by a string of n binary digits, taking 1 in the jth place when $A_j \in A_1^{(k)}$ (the complementary string corresponds to the second atom $A_2^{(k)}$). Thus, there are 2^n strings or 2^{n-1} dichotomic partitions, but the homogeneous strings of 1s or 0s correspond to the "unit" partition ν, with the single atom $\mathbf{M}$.

Obviously, $D(\alpha)$ generates α, that is, $\vee\alpha_k = \alpha$ for $\alpha_k \in D(\alpha)$; but there is a redundancy, in the sense that not all of these factors are necessary to reproduce α. With some criterion to be specified, we want to select a subclass $E(\alpha) \subseteq D(\alpha)$ of dichotomic partitions, which will be called "elementary factors." $E(\alpha)$ has to be sufficiently large to generate α, but also essentially restricted in order to reduce the redundancy of $D(\alpha)$. Precisely, we say that a criterion $\mathcal{P}$ extracting $E(\alpha)$ from $D(\alpha)$ is "good" if the following four conditions are satisfied.

1. Universality: $\mathcal{P}$ works for every $\alpha \in \mathcal{Z}$, defining $E(\alpha)$.
2. Completeness: selected factors generate α, that is,

$$\vee_k \alpha_k = \alpha, \quad \alpha_k \in E(\alpha) \tag{1.2}$$

If $F(\alpha)$ denotes another *arbitrary* subcollection of $E(\alpha)$, for $\alpha_k \in F(\alpha)$ define $\alpha' = \vee\alpha_k$. From condition 1, $E(\alpha')$ is also well defined. Of

course, since α' is generated by $F(\alpha)$, every $\alpha_k \in F(\alpha)$ is also in $D(\alpha')$. The third condition then follows.

3. "Self-compatibility": $E(\alpha') \supseteq F(\alpha)$.

$D(\alpha)$ itself trivially satisfies conditions 1–3, therefore a further condition is required for an optimal reduction of redundancy.

4. Effectiveness: there are no more factors in $E(\alpha)$ than atoms in α, or $\text{card}(E(\alpha)) \leq n$.

Note 1. There always exists at least one good class of elementary factors. Take a single atom A_k and its complement $\mathbf{M} - A_k$; then, for $k = 1, \ldots, N$ we obtain the family $S(\alpha)$ of factors (called "simple partitions" in [1]). It is evident that $S(\alpha)$ satisfies all conditions for a good criterion $\mathcal{P}$. This choice seems to be natural for abstract probability spaces, that is, in the absence of further characterizations.

Note 2. On the contrary, conditions 1–4 do not define a class of elementary factors uniquely. Actually, such a freedom may be used to fit the features of the particular model of interest.

Note 3. *Optimal reduction* does not mean *complete elimination* of redundancy. In the example of simple factors shown in Note 1, for instance, $n - 1$ factors are sufficient to generate α. But this would introduce artificial asymmetry and useless complexity in the definition of $S(\alpha)$.

Note 4. Condition 4 has been stated with partitions of one dimensional objects in mind. It could be too restrictive, in general, for d-dimensional spaces, where interesting classes with more than n dichotomic partitions cannot be excluded. (On the contrary, however, special examples easily may be built with *less* than n factors). A growth rate $\text{card}(E(\alpha))$ proportional to n^{1+c}, with $0 \leq c < 1$, may satisfy all exigencies independently from the dimension d. In the present paper, $c = 0$.

If γ is another partition, $E(\alpha \mid \gamma)$ will denote the subcollection of factors in $E(\alpha)$ which are "prime" with γ, that is, such that $\alpha_k \wedge \gamma = \nu$. Clearly,

$$\gamma_1 \leq \gamma_2 \Rightarrow E(\alpha \mid \gamma_1) \supseteq E(\alpha \mid \gamma_2) \tag{1.3}$$

For every α, β, let $\sigma = \alpha \wedge \beta$. Classes $E(\alpha \mid \sigma)$, $E(\beta \mid \sigma)$, the sets of elementary factors which are prime with the common factor σ, are therefore well defined. Two partitions $\alpha' \leq \alpha$, $\beta' \leq \beta$ may be introduced by

$$\begin{cases} \alpha' = \vee_k \alpha_k, & \alpha_k \in E(\alpha \mid \sigma) \\ \beta' = \vee_j \beta_j, & \beta_j \in E(\beta \mid \sigma) \end{cases} \tag{1.4}$$

We define a reduction process π on couples of partitions by

$$(\alpha', \beta') = \pi(\alpha, \beta) \tag{1.5}$$

A certain analogy with the cancellation of common factors in rational numbers motivates the term "rational partitions" for reduced couples. The consistency of the procedure is assured by the following theorem.

Theorem 1. *The reduction π is a projection in an invariant subset $\mathcal{R} \subset \mathcal{Z} \times \mathcal{Z}$; that is, $\pi^2 = \pi$.*

Proof. Denoting $(\alpha', \beta') = \pi(\alpha, \beta)$, $(\alpha'', \beta'') = \pi(\alpha', \beta')$, $\sigma = \alpha \wedge \beta$, $\sigma' = \alpha' \wedge \beta'$, definition (1.4) implies that $\alpha'' \leq \alpha' \leq \alpha$, $\beta'' \leq \beta' \leq \beta$. Since $\sigma' \leq \sigma$, from (1.3) we have

$$E(\alpha' \mid \sigma') \supseteq E(\alpha' \mid \sigma)$$

From the third condition on $\mathcal{P}$, $E(\alpha')$ contains all factors $\alpha_k \in E(\alpha \mid \sigma)$, that is, $E(\alpha') \supseteq E(\alpha \mid \sigma)$; but, since these factors are still prime with σ, it holds that

$$E(\alpha' \mid \sigma) \supseteq E(\alpha \mid \sigma)$$

Therefore

$$E(\alpha \mid \sigma) \subseteq E(\alpha' \mid \sigma) \subseteq E(\alpha' \mid \sigma')$$

which implies that $\alpha'' \geq \alpha'$, $\beta'' \geq \beta'$, that is, $\alpha'' = \alpha'$, $\beta'' = \beta'$. ∎

Note that the definition of π given in the preceding theorem is slightly more general than in [1], where the particular class of simple factors was used from the beginning.

An entropy h_r may be extended from $\mathcal{Z}$ to the space $\mathcal{R}$ of r-partitions, $\mathbf{a} \equiv (\alpha_1', \alpha_2') = \pi(\alpha_1, \alpha_2)$, by the function

$$h_r(\mathbf{a}) = \rho(\alpha_1', \alpha_2') \tag{1.6}$$

where ρ is defined as in (1.1). This entropy represents a measure of the effective difference, or "antisimilarity," between α_1 and α_2. Details of h_r are discussed in [1].

The main quantity we shall use to develop our complexity indicators is a time series, based on distances between couples of partitions (reduced or not) along the same "orbit" in a space $\mathcal{Z}$ of finite partitions.

More precisely, let S be a space of states $q, r, s \ldots$, with a transformation $f : S \to S$, and let Φ be a correspondence $S \to \mathcal{Z}$. Defining $T_f \equiv T$ as the induced transformation in $\mathcal{Z}$, that is,

$$T(\Phi(q)) = \Phi(f(q)) \tag{1.7}$$

if $\alpha^0 = \Phi(q_0)$ and $\alpha^k = \Phi(q_k) \equiv \Phi(f^k(q_0))$, $\mathbf{a}_{n,k}$ will be given for every integer "time delay" n by

$$\mathbf{a}_{n,k} = \pi(\alpha^k, T^n \alpha^k) \equiv \pi(T^k \alpha^0, T^{k+n} \alpha^0)$$

(Of course, the case where $n = 1$ is of special importance.) Regarding the criterion $\mathcal{P}$, the choices of both $\mathcal{Z}$ and Φ are not uniquely defined: they should reflect objective features of the dynamical processes, and the kind of subjective interest the observer has it, respectively.

The entropy which measures the (possibly reduced) distance between α^k and $T^n\alpha^k$, that is, $h_r(\mathbf{a}_{n,k})$, gives the time series we seek:

$$I_n(\alpha_k) \equiv h_r(\mathbf{a}_{n,k}) \tag{1.8}$$

A whole class of numerical indicators may be defined on the time series in (1.8); for example, the time average

$$\langle I_n \rangle = \lim_N \frac{1}{N} \sum_{k=1}^{N} I_n(\alpha_k) \tag{1.9}$$

or the power spectrum, which will be discussed in sections 3 and 4 with reference to numerical experiments. We may also consider a "Lyapunov-like" exponent $L(T)$,

$$L(T,\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log(R_k^\tau / R_k^0) \tag{1.10}$$

where $R_k^0 = h_r(\pi(T^k\alpha, \beta_k)$, $R_k^\tau = h_r(\pi(T^{k+\tau}\alpha, T^\tau\beta_k))$ and, for every k, β_k is a partition very near to $T^k\alpha$.

Omitting the reduction process in (1.8–1.10), analogous quantities are defined in $\mathcal{Z} \times \mathcal{Z}$, instead of in $\mathcal{R}$. For instance, (1.8) is replaced by

$$I'_n(\alpha_k) \equiv H(T^n\alpha_k \mid \alpha_k) + H(\alpha_k \mid T^n\alpha_k) = \rho(\alpha_k, T^n\alpha_k) \tag{1.8'}$$

Comparisons between parameters referring to reduced and nonreduced couples of partitions introduce a further type of parameter; for example, an amplification factor for (1.8),

$$A = \langle I'_n \rangle / \langle I_n \rangle \tag{1.11}$$

where $\langle I'_n \rangle$ uses $I'_n(\alpha_k)$ instead of $I_n(\alpha_k)$ in (1.9). The quantity in (1.11) (and similar quantities) will estimate, depending on the reduction process in time (that is, on the rate of appearance or persistence of equal factors), the relevance of a certain type of similarity and memory during evolution.

The effectiveness of numerical experiments will be improved by considering different quantities simultaneously, since there are situations where a single parameter produces ambiguous results. The fact that several parameters are required to characterize a single process is not incidental. It is important that all of our parameters are implicit, through time series such as (1.8), in one single object, the partition orbit $\{T^k\alpha\}$.

In Section 2 we shall describe the implementation of such an approach for general one-dimensional models. It will be applied to CAs in Section 3 and, in a different way, to maps on the interval in Section 4. Final comments are given in Section 5.

2. Algebra of one-dimensional partitions

As a measure space $\mathbf{M}$, consider the finite segment $[a, b)$ and, among all partitions, only *ordinate* partitions generated by a $(n+1)$tuple $x_0, x_1, \ldots, x_n$, where $x_0 \equiv a < x_1 < \ldots < x_{n-1} < x_n \equiv b$ and

$$\alpha = \{A_1, \ldots, A_n\}, \quad A_k = [x_{k-1}, x_k) \tag{2.1}$$

Every atom is a connected set, a segment; and points in the kth atom precede those in the $(k+1)$th. We want to introduce a class of elementary partitions that reflects this structure as much as possible. Simple partitions are not best in this respect, since the complement $\mathbf{M} - A_k$ to A_k is not connected in general. Consider, then, dichotomic partitions defined by

$$\alpha_k = \{A_1^k, A_2^k\}, \quad A_1^k \equiv [a, x_k), \quad A_2^k \equiv [x_k, b) \tag{2.2}$$

This definition trivially satisfies all conditions required for a good criterion $\mathcal{P}$, as defined in Section 1. Elementary factors (2.2) have connected atoms and respect the order of the original partition. Moreover, they are particularly easy to handle: indeed, if β is another partition generated by $\{y_j\}$, the intersection $\sigma = \alpha \wedge \beta$ is generated by $\{z_{kj}\} = \{x_k\} \cap \{y_j\}$, as in Figure 1. (For brevity, we shall omit references to the obvious exceptions in treating the extremes a and b). Order and connection make the reduction process particularly simple: $E(\alpha, \sigma)$ consists of all factors in $E(\alpha)$ except those in $E(\sigma)$, and the same is true for $E(\beta, \sigma)$, thus the reduced partitions α' and β' are generated by erasing all common points from $\{x_k\}$ and $\{y_j\}$. Since $\sigma' \equiv \alpha' \wedge \beta' = \nu$, the fact that $\pi^2 = \pi$ is trivial in this case.

In the following, we shall be concerned with *integer and discrete* partitions; that is, for a lattice with $N + 1$ sites $\{0, 1, 2, \ldots, N\}$, partitions identified by integers $\{k_0 \equiv 0 < k_1 < k_2, \ldots < k_n \equiv N\}$. This subclass fits one-dimensional CAs and, in general, processes with a one-dimensional discretizable state space.

Let $f : S \to S$ be a dynamical process, and $\Phi_1 : S \to \mathcal{Z}$ a first map, as sketched in Section 1. We shall introduce a second map Φ_2 (which associates partitions to binary strings) in order to implement, through the composed map $\Phi = \Phi_2 \circ \Phi_1$, a computable procedure for the complexity parameters (1.8–1.11). The entire scheme is represented in Figure 2, where, as a "state," a string of alphabetical characters has been used.

The map Φ_1, which produces partitions

$$\Phi_1 : \{\text{configurations}\} \to \mathcal{Z}$$

requires that states are explicitly specified, in connection with different dynamical processes; therefore, it will be introduced in subsequent sections. Here we give the (almost trivial) map $\Phi_2 : \mathcal{Z} \to \{\text{binary strings}\}$.

Let $\alpha = \{0, k_1, k_2, \ldots, k_j, \ldots, k_n \equiv N\}$ be an ordered partition of N discrete sites. Because of its order, the left endpoints of atoms are sufficient to identify the partition through a string having 1 at sites $0, k_1, \ldots, k_{n-1}$,

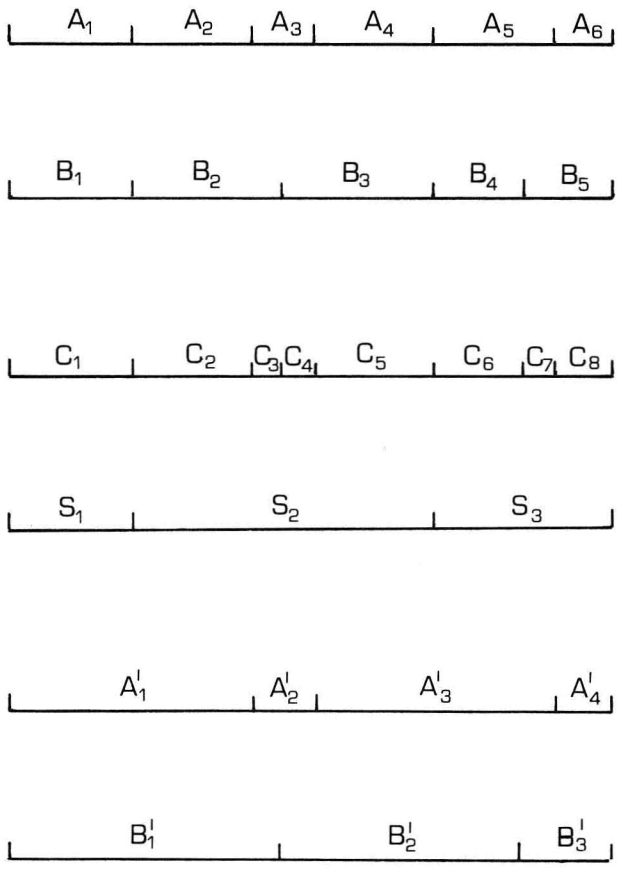

Figure 1: From the top: two partitions of a segment, α and β; their product, $\gamma = \alpha\beta$; their intersection, $\sigma = \alpha \wedge \beta$; the resulting reduced α' and β'

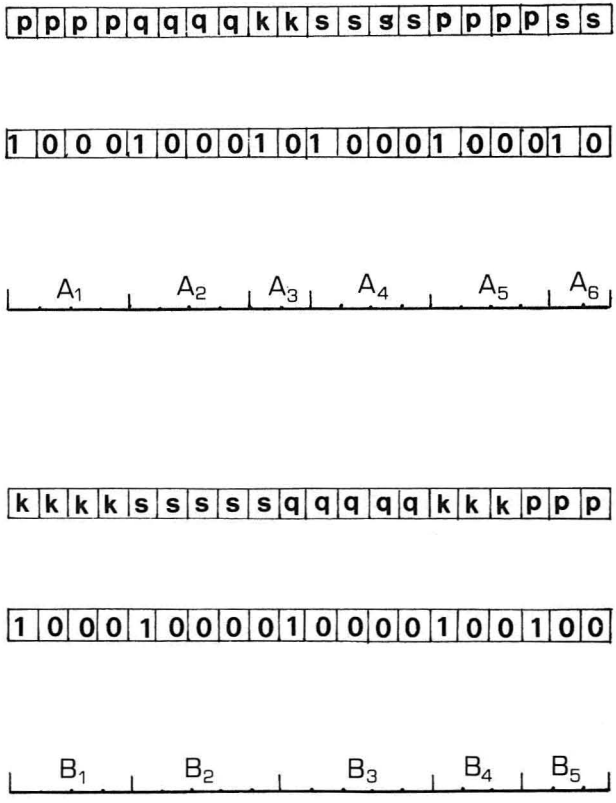

Figure 2: For two strings s_A and s_B of alphabetical characters, the projections $\alpha = \Phi_1(s_A)$ and $\tilde{\alpha} = \Phi_2(\alpha)$, $\beta = \Phi_1(s_B)$ and $\tilde{\beta} = \Phi_2(\beta)$, that is, $\tilde{\alpha} = \Phi(s_A)$ and $\tilde{\beta} = \Phi(s_B)$, where $\Phi = \Phi_2 \circ \Phi_1$. α and β are partitions of a segment equivalent to those of Figure 1.

and 0 elsewhere. The length m_j of A_j is $k_j - k_{j-1}$ for $j < n$, and $k_n - k_{n-1} + 1$ for A_n. In this way, the invertible map Φ_2 is made explicit.

This representation is particularly suitable for the algebra of reduction. For every α and β, the string $\tilde{\sigma} = \Phi_2(\sigma)$ (where $\sigma = \alpha \wedge \beta$) will be given by the logical AND function on $\tilde{\alpha} = \Phi_2(\alpha)$ and $\tilde{\beta} = \Phi_2(\beta)$. (Since Φ_2 is invertible, strings such as $\tilde{\alpha} = \Phi_2(\alpha)$ will be denoted hereafter by the same symbol α, when no confusion is likely.) Therefore,

$$\sigma = \alpha \text{ AND } \beta \tag{2.3}$$

Consistently, we have

$$\alpha' = \alpha \text{ AND NOT } \sigma = \alpha \text{ AND NOT } \beta \tag{2.4}$$

$$\beta' = \beta \text{ AND NOT } \sigma = \beta \text{ AND NOT } \alpha \tag{2.5}$$

$$\alpha\beta = \alpha \text{ OR } \beta \tag{2.6}$$

(Note that the definition of σ is not strictly necessary to define α' and β'). Such logical functions on binary strings make explicit the operative scheme of Figures 1 and 2. The probability measure associated with every atom is assumed by default to be proportional to the number of its sites. Of course, other measures are conceivable in connection with particular exigencies.

The reduction procedure we have just sketched is not shift invariant, because of the "reference" of the fixed extremes 0 and N. This seems in contrast to the dynamics of systems using periodic boundary conditions. Once again we stress that the correspondence between states and a certain partition space should reflect the point of view of the observer. While periodic boundary conditions in dynamics constitute a tool to simulate infinite systems, fixed boundaries for configurations correspond to a plausible exigency in looking at them in a non–shift-invariant way. Such exigencies are not incompatible and, moreover, the influence of boundaries becomes very small quantitatively as N grows.

Therefore, even if a corrected shift-invariant reduction on toroidal lattices is conceivable, we think that the present approach is not only the simplest but, in a sense, the most natural one.

3. CAs and complexity parameters

CAs, with their wide, nontrivial variety of dynamical behaviors, provide an ideal scheme for our approach to complexity. They may be defined as a triple $\{K, L, R\}$, where K is a finite alphabet, L is a lattice (states are therefore points in K^L), and $R : K^L \to K^L$ is a "rule" determining at discrete times $t, t+1, t+2, \ldots$, the evolution from one state to another [10–14]. Rules are local in the sense that the element of K in the site p of L at time $t+1$ depends only on the neighbor of $2r+1$ sites $p-r, p-r+1, \ldots, p+r$ at time t. Such a locality is naturally fitted by a parallel computation scheme

but, due to the simplicity of the reduction process in one dimension, the use of a serial computer did not constitute a serious obstacle to the experiments below.

We shall assume, by default, the following map Φ_1 from K^L to a partition space $\mathcal{Z}$: if $\mathbf{M}$ is a finite subset of L (a hypercube, say) the collection of homogeneous connected subsets of $\mathbf{M}$ is a partition of $\mathbf{M}$. In the case of one-dimensional L, $\mathbf{M}$ is the string we discussed in Section 2: when the same symbol of K is located in consecutive cells $k_j, \ldots, k_j + m$, the corresponding places belong to the same atom A_j of length $m + 1$ in the partition α of $\mathbf{M}$ (see Figure 2). This correspondence is quite natural when $K = 2$ (a binary alphabet). More sophisticated correspondences are conceivable; even with larger alphabets, however, Φ_1 is something of a meaningful approximation of degree 0.

In contrast to Φ_2, Φ_1 is not invertible: states obtained by permutations in K, or states which coincide only in $\mathbf{M}$, are projected into the same partition. This dependence of $\mathcal{Z}$ on the choice of $\mathbf{M}$ is quite natural, considering that, in concrete calculations, CAs are finite and the real object under observation is $\mathbf{M}$ with periodic boundary conditions. The choice of a site as the origin specifies Φ_1 completely. As we pointed out in section 2, the influence of this border condition is very small, and it decreases as $\mathbf{M}$ grows.

Using a standard nomenclature [10,11], we shall refer to elementary cellular automata (ECAs) with $K = 2$ and $r = 1$, and to totalistic cellular automata (TCA_{22}s) with $K = 2$ and $r = 2$. The consistency and reliability of the finite approximation to infinite CAs may be checked by varying N. Figures and Tables refer to $N = 100$, but experiments have been performed systematically up to $N = 2 \times 10^3$, and in some cases up to $N = 10^5$. The number of iterations (another crucial parameter when long period cycles are involved), has been chosen to be some ten times N, after the transient. As initial conditions, we considered simple initial conditions (SIC), a string of "white" cells with a single "black" cell; and random initial conditions (RIC), black and white cells randomly distributed in the string.

There are qualitative features directly reflected, through Φ_1, in the behavior of the time series $I_n(\alpha_k)$. For instance, if a certain state is an attractor (that is, if the CA settles down in a steady state after a transient), this implies the annihilation of $I_n(\alpha_k)$. Also, periodicity has an immediate interpretation. Moreover, since similarity is properly measured both in amplitude and in time behavior by the entropy h_τ, the n-periodic onset of similar, not identical, patterns (hereafter called "n-pseudo periodicity" or "PP-n") may also be shown by Fourier analysis. An example of such pseudoperiodic behavior is given by ECA 60 in Figure 3. Note that a direct inspection of patterns becomes extremely difficult when $\mathbf{M}$ is large.

It is quite plausible that pseudoperiodicity is meaningful for complexity; it is related to the "difficulty in describing" the behavior of a system, in the sense of actually performing a full characterization of its main features. Such a difficulty, in general, remains distinct from the "difficulty in forecasting," which consists of the mean amount of information needed to know the future

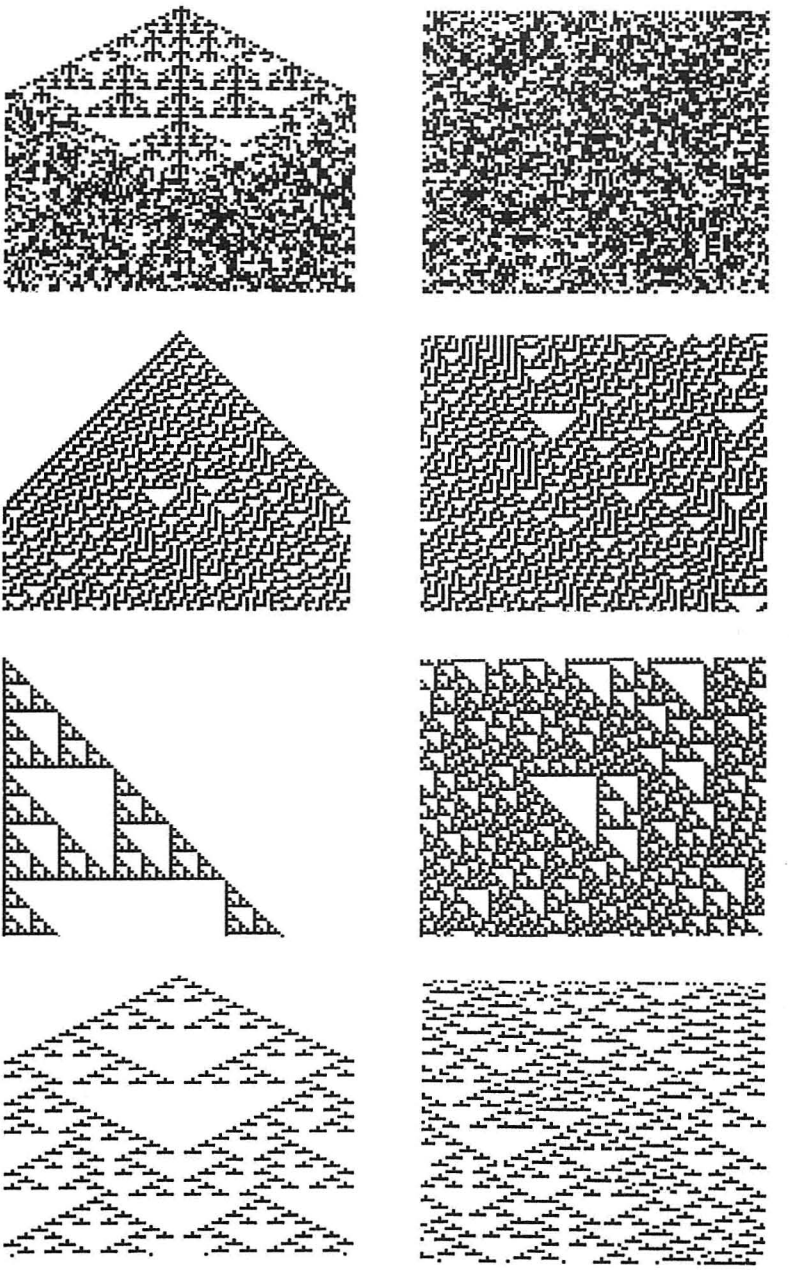

Figure 3: Configurations of typical CAs in Wolfram's class 3. Left column, EIC; right column, RIC. From the top: $\mathrm{TCA}_{22}42$, ECA 30, ECA 60, and $\mathrm{TCA}_{22}2$, with $L = 100$ (horizontal) and $N = 80$ (vertical).

state. In this sense, complexity and randomness are not synonymous, the latter possibly being an aspect of the former.

Thus, there is a range of features that could contribute to a classification of CAs into classes of complexity. We shall only tentatively indicate such a classification, which overlaps only partially the qualitative scheme proposed by Wolfram [10–12]. Indeed, for our purposes, classes 1, 2, and 4 of Wolfram are trivial (that is, simple, not complex) because, after a finite transient, they all decay into a steady configuration (in some cases the null configuration made up of identical symbols), or into cycles with a short period. But even in trivial cases, reduction may prove useful. Consider, for instance, ECA 13 (Figure 4): in the transient, before the stable final configuration, the reduction not only amplifies the distances (as they are a priori known), it also shows that there are alternatively one or two erased boundaries, a qualitative feature that could escape visual inspection. We are primarily interested in Wolfram's class 3, which presents a wider range of complex situations.

A possible distinction from randomness generally implies the onset of a structure which is increasingly complex, inasmuch as it is increasingly difficult to describe. To check the expected sensitivity of our parameters to these qualitative features, we used as a reference a completely random system (RS), that is, a special case of a probabilistic CA, where every site of L evolves according to a probability distribution on K, typically equiprobability (Figure 5). RS has no memory, since its states are completely independent. Figure 5 shows experimental results for the evolution of RS for RIC (SIC are meaningless for RS), and Figure 6 the evolution of $I_n(\alpha_k)$, both in $\mathcal{R}$ and in $\mathcal{Z} \times \mathcal{Z}$. Figure 7 gives the corresponding power spectra. We assume by default that $n = 1$. The ratio of the time-averaged values gives the mean amplification A due to the reduction process. For RS, $A = 1.6 \pm 0.1$, a value which experiments prove to be largely independent of N in the range $N = 10^2$ to 10^5. The number of iterations varies, in typical experiments, from 10^3 to 10^4, with a convergence of time averages that is increasingly better for large N.

As an account of this type of analysis, the values of typical CAs from Wolfram's class 3 are shown in Table 1, whose entries refer to the asymptotic behaviors of systems in Figures 3 and 5. A qualitative similarity emerges, for instance, between $TCA_{22}42$ and RS. Power spectra (Figure 8) confirm chaotic behavior. The set of these data suggest the classification of $TCA_{22}42$ very near to RS, in a class of highly chaotic CAs. Nevertheless, the Lyapunov-like quantity (1.9) for $TCA_{22}42$, plotted against the time delay τ (Figure 9), approaches a limit (which is precisely the same as that of RS) only for great τ, presumably when orbits are correlated. (We don't report the diagrams for RS, since they obviously do not depend on τ). In other words, the Lyapunov-like parameter gives an excellent insight into the response of the system evolution to small perturbations of initial conditions. This gives evidence of a hidden structure compatible with random configurations of patterns (Figure 3a). There is therefore the possibility of a sharper distinction between RS and systems that behave chaotically with regard to configurations.

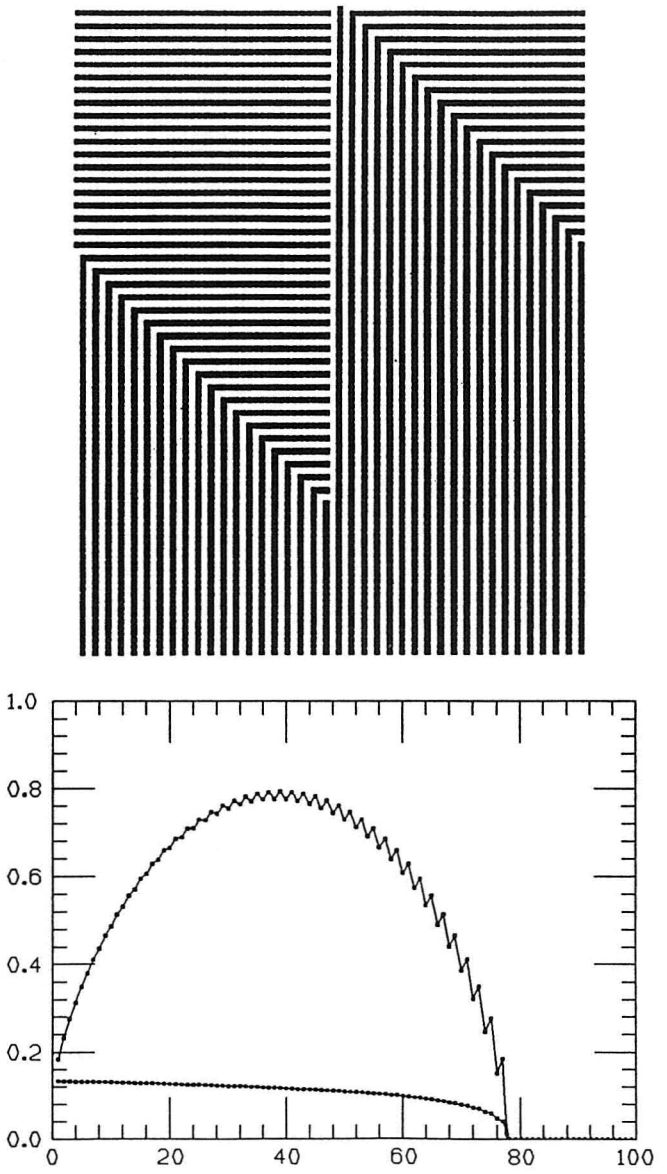

Figure 4: Top, configurations of ECA 13 with EIC; bottom, reduced (upper diagram) and nonreduced (lower diagram) distances.

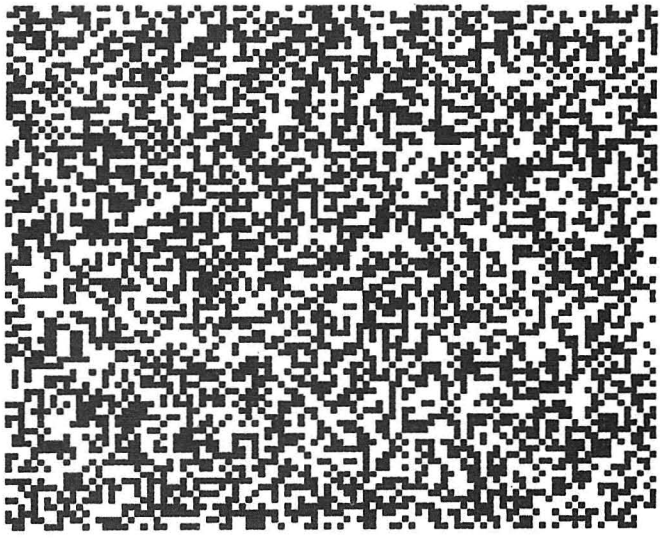

Figure 5: Configurations of RS with $L = 100$ (horizontal) and $N = 80$ (vertical).

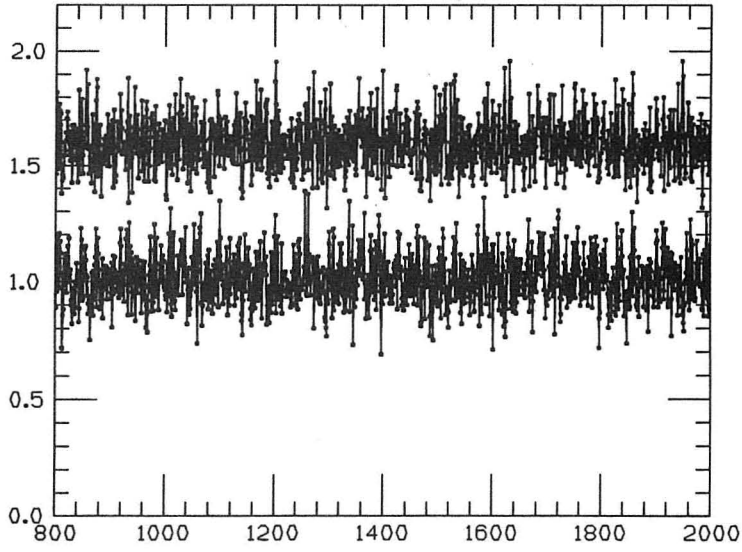

Figure 6: Time behavior of $I_{n,k}$ for RS of Figure 3, in the reduced (upper diagram) and nonreduced (lower diagram) cases.

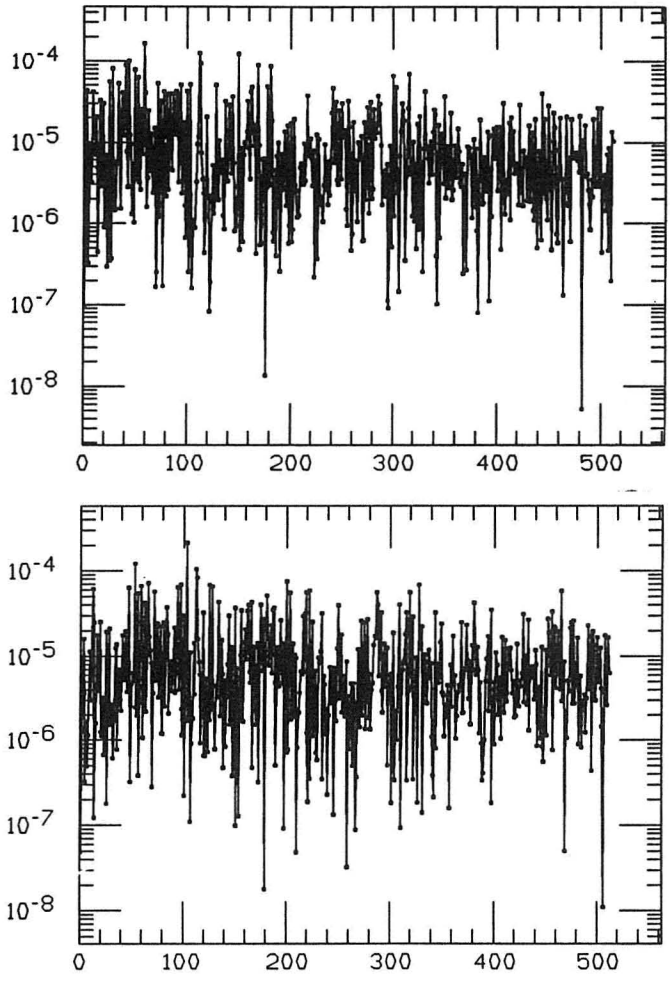

Figure 7: Power spectra of $I_{n,k}$, in the reduced (top) and nonreduced (bottom) cases, for RS of Figures 5–6.

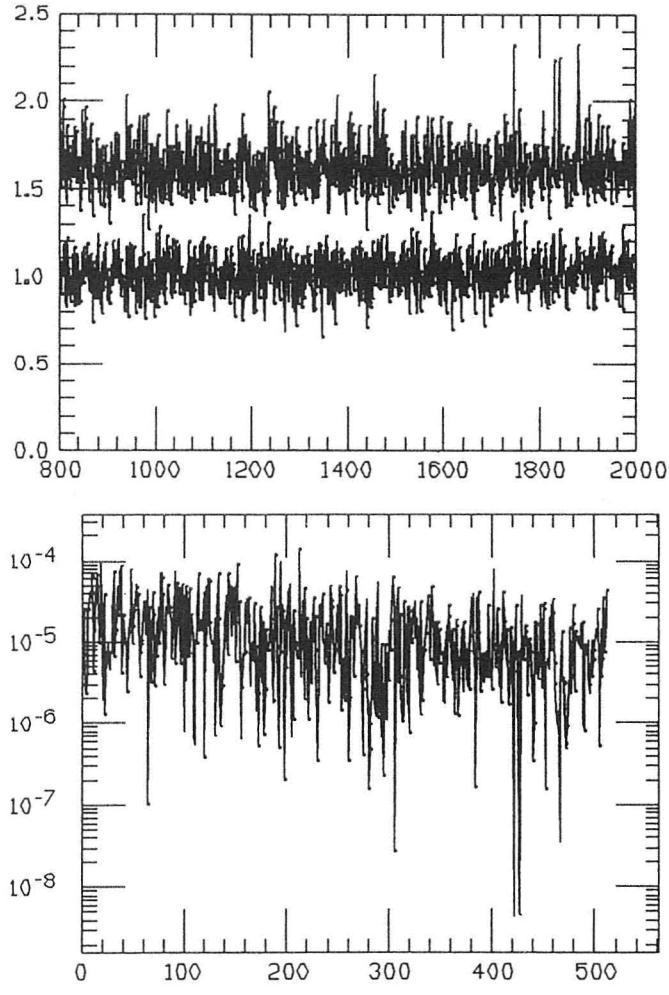

Figure 8: Top, time behavior of $I_{n,k}$ referring to reduced (upper diagram) and nonreduced (lower diagram) partitions for $TCA_{22}42$ with EIC; bottom, power spectrum corresponding to the reduced diagram.

Rule	I.C.	$\langle I_n \rangle$	$\langle I'_n \rangle$	A	Power Spectra
RS	RIC	1.0131	1.6018	1.5811	Random
$TCA_{22}42$	EIC	1.0115	1.6140	1.5956	Random
	RIC	1.0133	1.6152	1.5940	Random
ECA 30	EIC	0.9451	1.3939	1.4749	Random
	RIC	0.9435	1.3931	1.4765	Random
ECA 60	EIC	0.6047	1.8029	2.9815	PP-2
	RIC	0.8931	1.5710	1.7590	Random
$TCA_{22}2$	EIC	1.1346	1.1346	1	PP-2
	RIC	1.1196	1.1196	1	PP-2

Table 1: Values of indicators for *RS* and typical CAs in Wolfram's class 3 (see Figure 3).

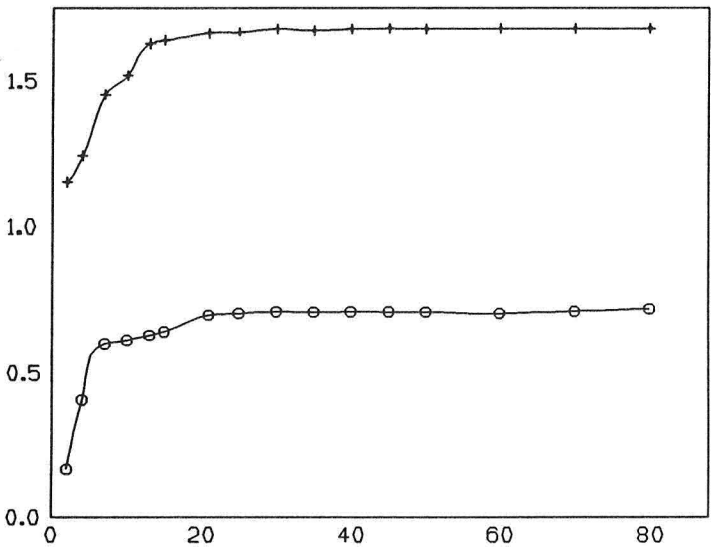

Figure 9: Lyapunov-like exponents versus time delay τ for $TCA_{22}42$, reduced (lower diagram) and nonreduced (upper diagram). Asymptotic values are the same as those of *RS*.

In contrast to $TCA_{22}42$, diagrams of ECA 30 exhibit the emergence of a structure that is detectable also by visual inspection (Figure 3b). Table 1 begins to give different values, while configurations (which we do not reproduce) analogous to those in Figure 8 would not exhibit relevant differences; on the contrary, diagrams such as those of Figure 9 would indicate a longer delay τ to correlate. ECA 30 remains a highly random system, although a structured one.

A further step toward a structured behavior is given by ECA 60 (Figure 10), which is sensitive to initial conditions: SIC give behaviors that are quasiperiodic, as stressed both by power spectra and amplification factors; on the contrary, RIC still keep a certain degree of chaos. Similar features are exhibited by ECA 22, which Grassberger [2] notes for its nonstandard geometry.

We omit a full report of numerical experiments with increasing N (from 100 to 2000), stressing only that the fractal structure of the CA configurations with SIC is faithfully detected by our parameters (see Figure 11, and note that the power spectra show a typical cascade of peaks).

Finally, in Table I we report the case of $TCA_{22}2$, a CA extremely stable both for SIC and RIC. Note the amplification factor, whose value depends on the fact that successive partitions never overlap. The regularity of patterns is matched by the pseudoperiodic character of the spectra.

Regarding experiments with various types of probabilistic noise, we mention only the fact that there exist correlations between the influence of noise on evolution and the response of complexity parameters. In other words, in addition to the qualitative indications obtained in the analysis of deterministic systems, we can get indications on their robustness under the influence of a random perturbation. For instance, $TCA_{22}2$ proves to be highly robust.

To achieve confidence about a process, independently from visualization, the simultaneous consideration of a whole set of parameters is required. Patterns of values such as those in Table 1 confirm, on the one hand, that a single parameter may be misleading or meaningless but, on the other, that the time series developed with the partition generation described in Section 1 implicitly contains an extremely rich set of qualitative and quantitative information, largely sufficient to characterize the process.

4. One-dimensional maps

The approach described in the previous section may be applied to iterated maps in the interval, that is, systems of the form $x_{n+1} = f(x_n)$. For definiteness, we shall refer to the well known case of the quadratic map

$$x_{n+1} = 4\mu x_n(1 - x_n) \tag{4.1}$$

with

$$x_n \in [0, 1], \quad 0 < \mu < 1$$

For $\mu < 0.25$, 0 is a fixed-point attractor; for $0.25 \leq \mu < 1$ there is an increasingly rich variety of dynamical behaviors. In particular, a "period

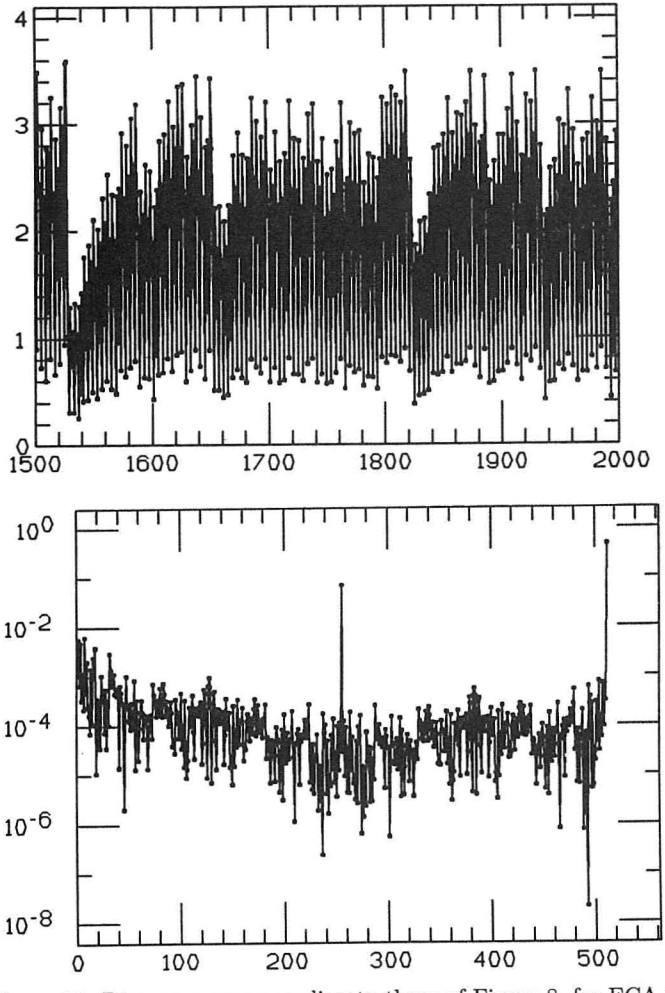

Figure 10: Diagrams corresponding to those of Figure 8, for ECA 60 with EIC. Only the reduced distances are plotted.

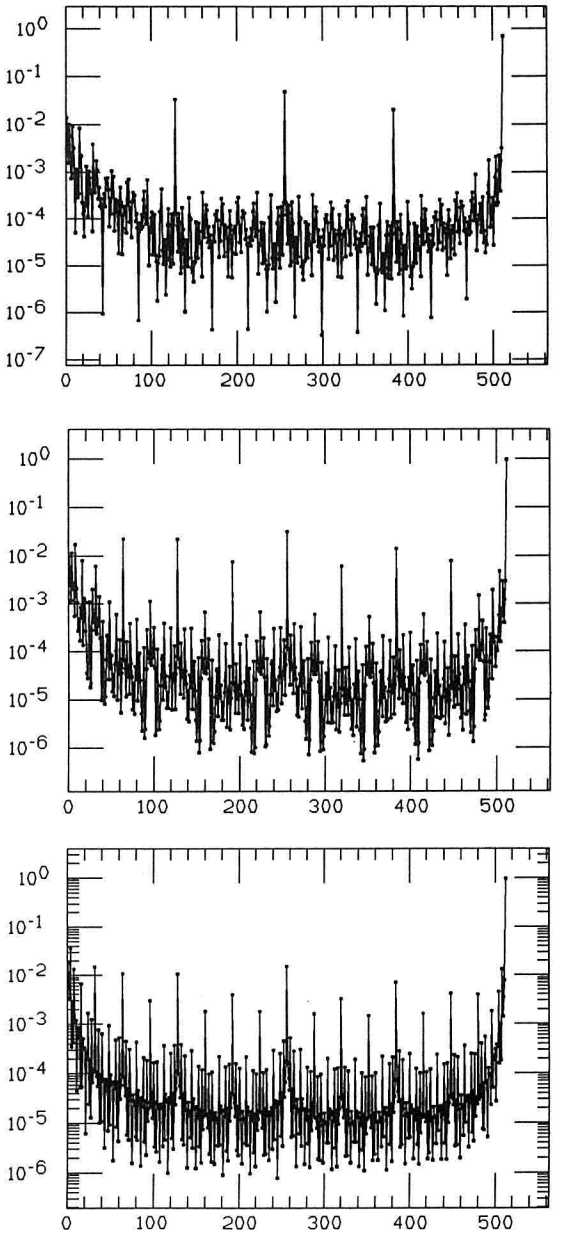

Figure 11: Power spectra for ECA 60 with $N = 200$, 400, and 800 (from the top down).

doubling" phenomenon occurs at certain μ_k culminating in a limit value $\mu_\infty = 0.892486418\ldots$. When μ overpasses μ_∞, orbits move chaotically on a fractal attractor. The value μ_∞ is related to universal features of one-dimensional maps, in the sense that the reduction factor associated with consecutive bifurcations,

$$\delta_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

for $n \to \infty$ tends to a constant δ independent of the details of the function f. For reviews of these facts (for example, regarding the successive dependence of the regime of motion on μ), there is an extensive bibliography, but [15–20] give a sufficient overview. We perform a comparison between such features and the results of an analysis performed with the complexity parameters of the previous section. In particular, we look at possible correlations between the appearance of a strange attractor and the complexity of the trajectory.

To regain the formalism of one-dimensional partitions, the range of the quadratic map (that is, the unit interval $I = [0, 1]$) will be divided into M equal subintervals, or cells; then, considering an N-points segment of trajectory $\{x_0, \ldots, x_{N-1}\}$, a cell will be labeled as black if at least one x_j there falls in it, otherwise it will remain white. We obtain a configuration of black or white cells that, as a CA configuration at a fixed time, may be projected by Φ_1 into a partition space. The successive segment of trajectory $\{x_N, \ldots, x_{2N-1}\}$ defines a second configuration, and so on. There is therefore a sequence of configurations (and partitions) that depend on the trajectory and the parameters M and N. The analysis may proceed as for CAs, keeping in mind that the true time step has become the length N of every segment of the trajectory. As noted in [1], this approach applies to every iterative process whose range is bounded and may be discretized.

Clearly, if μ is such that the orbit lies on a finite attractor of period $P \leq N$, then there is a steady set of black subintervals, and the configurations (or the associated partitions) remain identical during evolution; therefore complexity parameters are 0. If $P > N$, successive configurations in general differ, but they repeat periodically with a period that cannot be larger than the minimal common multiple between P and N (it depends also on the discretization M). In this case, complexity remains at a low degree, and numerical experiments shall only confirm a behavior that is completely known. Only in the case when $P \gg N$, and the observation time is not adequately long, should one notice a progressive onset of apparently aperiodic (but not really chaotic) behavior.

If the trajectory becomes aperiodic (as happens when μ overpasses μ_∞), the chaotic behavior of the trajectory of the strange attractor will correspond to an irregular sequence of configurations (however, in general, not a random sequence with probability 0.5 for each cell). For these reasons, an analysis of the complexity parameters should reveal the onset of a strange attractor for $\mu \to \mu_\infty$ from below, and it should give a qualitative estimate of the behavior of the attractor for $\mu > \mu_\infty$. Numerical experiments have been performed with this conceptual background.

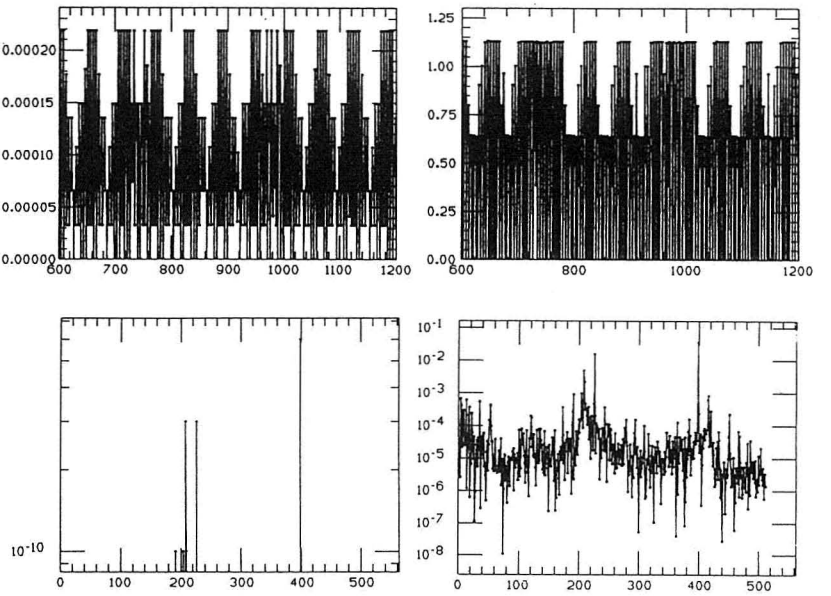

Figure 12: Top: time behavior of $I_{n,k}$ for nonreduced (left) and reduced (right) cases, with $\mu = .892486418$. Bottom: the corresponding power spectra.

Figures 12–14 show the reduced distances and the corresponding power spectra for three values of μ. In Figure 12, $\mu = .892486418$ (an approximation of the ideal μ_∞ obtained by dropping the tenth decimal digit). Evidently, there is still a quasiperiodic behavior due to the fact that (since μ_∞ is approximated from below) the trajectory lies on a P-periodic attractor, even if P is very large. The same happens in Figure 13 with $\mu = 0.892486418118$, while in Figure 14 ($\mu = 0.892486418119$) a clear passage to chaotic behavior appears. The sensitivity of our parameters allows us, therefore, to distinguish between two values of μ that differ at the twelfth decimal digit. The reported figures refer to $M = 10^5$, $N = 10^4$, and iterations up to 1200 configurations (that is, 1.2×10^7 points in the orbit). A typical run for these values of the parameters takes about one hour on a VAX 3900. (Of course, these estimates easily could be improved by increasing the magnitude of M and N, compatibly with computer features.) Even if the sequence of partitions obtained for these mappings is formally the same as the one produced by a CA, the dynamical process is intrinsically different. One consequence of this difference, for instance, is that in the present case it is not possible to replicate the computation of the Lyapunov-like exponents. In fact, the clarity of the results is such that a further analysis would be useless.

Awaiting an implementation of the present approach on a parallel machine, we did not perform numerical experiments that, in the same spirit, would have given a further interesting characterization of the geometrical

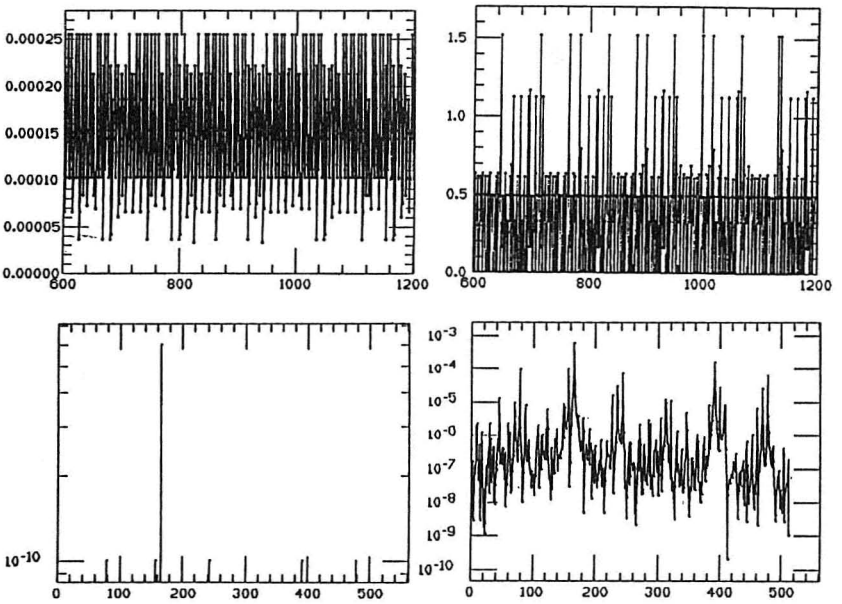

Figure 13: Same diagrams as in Figure 12, with $\mu = .892486418118$.

complexity of the maps. For instance, an attractor could, in principle, be explored by the comparison between the partition of the whole range of the map, and the partition of a subrange $[a, 1-a]$ enlarged up to $[0, 1]$. The scale factor in the enlargement would play the role of the time step in previous computations. Such a procedure, which should reflect the self-similarity properties of the attractor, does not in principle present any difficulty; but it has strong bounds in the high resolution required in the computation, and in the lengthening segments of trajectories to consider in order to have a comparable occupation of the diminishing cells.

5. Conclusions

We have tested, in a number of nontrivial situations, the efficiency of complexity parameters based on the r-partitions formalism, with the following results.

- There is a good correspondence between the indications extracted from our indicators and the intuitive sense of complexity when present (e.g. at visual inspection).
- In the absence of an intuitive tendency, complexity parameters are useful in indicating hidden features, such as quasiperiodicity, false relationship to chaotic behavior, and the existence of internal memory between states.

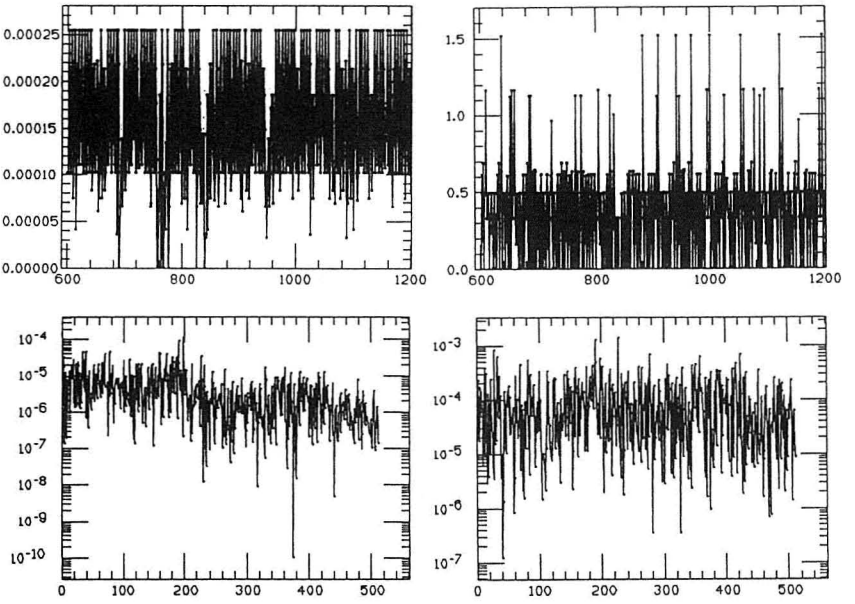

Figure 14: Same diagrams as in Figure 12, with $\mu = .892486418119$.

- The possible distinction between randomness as difficulty in forecasting (related to the mean uncertainty of the process variables, for which dynamical entropy is an excellent indicator), and complexity as difficulty in describing (related to the full characterization of a process, even if it is deterministic), is faithfully pointed out.

Such results emerge from the whole set of parameters, confirming that complexity cannot be reduced to a single index. As frequently stressed (see, for example, [4]), the necessity of a multiparameter analysis should be seen not as a weakness of the theory, but as a consequence of the richness of perspectives contained in the term "complexity." In this sense, the fitness of the partition-space formalism to such multiparameter study is in itself a good result. (It is noteworthy that all the parameters that we have considered substantially depend on only *one class of objects*, time series in the spaces of (possibly reduced) partitions.) This fitness may be understood on the basis of the following considerations.

In a dynamical process, there are two possible sources for complexity:

- a "timelike" source, that is, algorithmic features of sequences. It is this aspect (historically the most relevant for the genesis of ideas about complexity—see [5]) that appears when one explores the compressibility of a sequence of characters, or other features related to the degree of dynamical regularity; and

- a "spacelike" source, that is, complexity of configurations (or, more generally, states) in a given space. In this sense, for instance, three-dimensional sets are intrinsically more complex objects than one-dimensional sets. In the simplest case of a sequence of characters, such a spacelike source reduces to possible outcomes from a finite probability space (the alphabet). This is very different from the case when the evolving sequence is constituted by (possibly structured) patterns, as for CAs.

Moreover, complexity may depend on interactions between these two facets of the problem.

Of course, our spacelike source may be analyzed (in many cases) by methods invented to study our timelike source, for example, through a digitalization (reduction to a bit map) of configurations. But it is not clear whether such methods may be satisfactory in every case—for instance, when the "shapes" of many dimensional sets are involved, with both their metric and topological features.

Since these two aspects arise, in some sense, in all characterizations of complexity, a computable sensitivity to both of them would provide a weak but meaningful approach to quantitative estimation. ("Weak" in the sense that such a sensitivity does not directly face the problem of determining the intrinsic difficulty of certain tasks, as in more formalized (but, generally, not computable) approaches.) An appreciable feature of the methods we have presented is that the object of measurements (that is, the time series on which various parameters are built), is synthetically linked on the one hand to the Shannon entropy in $\mathcal{Z}$ and, on the other, to the dynamical features of the process (in other words, to the complexity of the state space and to the time evolution law).

This implies, conversely, that the analysis cannot be completely automated. The necessity of preadapting the formalism to each definite problem appears from the very beginning, when one chooses the correspondence $\Phi = \Phi_2 \circ \Phi_1$, defining the partition space. In such a preliminary procedure, the concrete features of the model (one-dimensionality, for example) play an important role. In this instance, exploiting one-dimensionality permitted an easy implementation of the reduction algorithm. The extension of the formalism to two-dimensional models, which is in progress [21, 22], will give new evidence for this aspect.

Acknowledgments

We would like to thank Paolo Celli for useful discussions during the preparation of this paper.

References

[1] M.Casartelli, "Partitions, Rational Partitions, and Characterization of Complexity," *Complex Systems*, **4** (1990) 491–507.

[2] P. Grassberger, "Randomness, Information and Complexity," Technical Report WU B 89-26, Department of Physics, University of Wuppertal (1989).

[3] P. Grassberger, "How to Measure Self-Generated Complexity," *Physica A*, **140** (1986) 319–325.

[4] L. Peliti and A. Vulpiani, *Measure of Complexity* (Berlin, Springer, 1988).

[5] Ming Li and P. M. B. Vitanyi, "Kolmogorov Complexity and Its Applications," Centre for Mathematics and Computer Science, Report CS-R8901, P.O. Box 4079, 1009 AB Amsterdam. To appear in *Handbook of Theoretical Computer Science*, edited by J. van Leeuwen (Amsterdam, North-Holland, in press).

[6] V.I. Arnold and A. Avez, *Problemes Ergodiques de la Mecanique Classique* (Paris, Gauthier-Villars Editeur, 1967).

[7] P. Billingsley, *Ergodic Theory and Information* (New York, Wiley, 1965).

[8] I.P Cornfeld, S.V. Fomin, and Y. G. Sinai, *Ergodic Theory* (New York, Springer-Verlag, 1982).

[9] P. Grassberger, "Chaos and Diffusion in Deterministic Cellular Automata," *Physica D*, **10** (1984) 52–64.

[10] S. Wolfram, "Statistical Mechanics of Cellular Automata," *Reviews of Modern Physics*, **55**(3) (1983) 601–644.

[11] S. Wolfram, "Universality and complexity in Cellular Automata," *Physica D*, **10** (1984) 1–35.

[12] S. Wolfram, "Computation Theory of Cellular Automata," *Communications in Mathematical Physics*, **96** (1984) 15–57.

[13] T. Toffoli, "Cellular Automata Mechanics," Technical Report No. 208, University of Michigan, (1977).

[14] T. Toffoli and N. Margolous, *Cellular Automata Machines* (Cambridge, MIT Press, 1987).

[15] M. J. Feigenbaum, "Quantitative universality for a class of non-linear trasformations," *Journal of Statistical Physics*, **19** (1978) 25–52.

[16] M. J. Feigenbaum, "Universal Behavior in Nonlinear Systems," *Los Alamos Science*, **1** (1980) 4–27.

[17] P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Boston, Birkhauser, 1980).

[18] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems* (Menlo Park, CA, Benjamin/Cummings, 1986).

[19] P. Cvitanović (editor), *Universality in Chaos* (Bristol, Adam Hilger Ltd., 1984).

[20] P. Bergé, Y. Pomeau, and C. Vidal, *Order within Chaos* (New York, Wiley, and Paris, Hermann, 1984).

[21] P. Celli, "Partizioni Misurabili e Complessità di Sistemi Dinamici su Reticoli Bidimensionali," unpublished dissertation, University of Parma, Parma, Italy, 1991.

[22] M. Casartelli and P. Celli, "Partitions in Two-Dimensional Lattices" (in preparation).