Complex Systems **7** (1993) 415–422

Numerical Modeling of Toroidal Dynamical Systems with Invariant Lebesgue Measure

Phil Diamond
Department of Mathematics, University of Queensland,
Brisbane, Queensland 4072, Australia

Peter Kloeden
Department of Computing and Mathematics, Deakin University,
Geelong, Victoria 3217, Australia

Alexei Pokrovskii*
Department of Mathematics, University of Queensland,
Brisbane, Queensland 4072, Australia

Abstract. Computer simulations of dynamical systems can contain both *discretizations*, in which finite machine arithmetic replaces continuum state spaces, and *realizations*, in which a continuous system is replaced by some approximation such as a computational method. In some circumstances, complicated theoretical behavior collapses to trivial and degenerate behavior. In others, the computation may bear little resemblance to the underlying theoretical model. We show here that systems that preserve Lebesgue measure do not suffer such undesirable features. In fact, they can be approximated by such simple maps as permutations of computer state space.

1. Introduction

When investigating complicated dynamical behavior, it is often important to study the *invariant measures* of a system rather than equilibria or cycles. The rich orbital structure of such systems means that more information about the system is eventually found in this way. In particular, absolutely continuous invariant measures are of special interest when they exist.

Frequently, the first recourse when studying complicated dynamics is to begin computer simulation and extensive numerical experimentation. This often involves a *computer realization* of the model, such as a numerical scheme

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*Permanent address: Institute of Information Transmission Problems, Russian Academy of Science, Moscow.

for solving a differential equation. The whole procedure is further affected by *discretization* imposed by the finiteness of computer arithmetic. In such circumstances, the invariant measures of the underlying theoretical model can be very different from those of its computer realization, quite independently of the precision of the machine arithmetic [1]. For example, $g(x) = 2x \bmod 1$ is chaotic, with a unique absolutely continuous invariant measure, and has cycles of all orders. Nevertheless, every N-digit binary realization of g is asymptotically trivial, with $\varphi^k \equiv 0$ if $k \geq N$. This is easily seen if φ is the restriction of g to the set $\mathbf{L}_N = \{i/2^N : i = 0, 1, \ldots, 2^{N-1}\}$.

"Collapsing" phenomena of this sort are not easy to overcome in a computational setting. Effects of introducing small random perturbations, of the order of 2^{-N}, are rather subtle [4]. However, collapsing such as this is always possible in practical situations ([5], Theorem 5 and [6], Theorem 5).

Let $\mathbf{T}^k$ denote the k-dimensional torus, that is, the unit k-dimensional cube $[0,1]\times\cdots\times[0,1] = [0,1]^k$ with the identification that points $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are equivalent if $a_i - b_i \equiv 0 \pmod 1$, $i = 1, \ldots, k$. For a positive integer $\nu > 0$ denote by $\mathbf{L}_\nu$ the set $\{0, 1/\nu, \ldots, (\nu-1)/\nu\} \subseteq [0,1]$ and let $\mathbf{L}_\nu^k = \mathbf{L}_\nu \times \cdots \times \mathbf{L}_\nu \subset \mathbf{T}^k$.

A discrete dynamical system on the torus $\mathbf{T}^k$ is induced by a Borel mapping $f : \mathbf{T}^k \to \mathbf{T}^k$. We shall broadly define a $\mathbf{L}_\nu^k$ realization of the system f as a mapping $\varphi : \mathbf{L}_\nu^k \to \mathbf{L}_\nu^k$, which is close to f in some natural sense. By a $\mathbf{L}_\nu^k$ discretization of f is meant the discretization in the sense of Stetter [12], where we have a projection $P : \mathbf{T}^k \to \mathbf{L}_\nu^k$ and $\varphi = P \circ f : \mathbf{L}_\nu^k \to \mathbf{L}_\nu^k$. Note that a discretization is a realization, but a realization may have other processes involved than mere discretization. For example, if a model consists of a number of differential equations, these might be solved by a numerical method involving approximation by difference equations. Discretization is also involved because all calculations are performed on a computer. However, there is also the realization of the system by a discrete dynamical system that is involved.

In this paper we will study some properties of realizations and discretizations under the assumption that Lebesgue measure μ is invariant for the system f. Some interesting systems have this property, and this is a robust situation for computer modeling [2]. In these circumstances it is natural to say that discretizations φ avoid collapsing effects provided that they preserve the natural uniform measure on the finite set $\mathbf{L}_\nu^k$. That is, φ is a bijective mapping of $\mathbf{L}_\nu^k$, or a *permutation* of $\mathbf{L}_\nu^k$.

The results of this paper are as follows.

- Let Lebesgue measure be invariant for f. Then there exists a permutation φ of $\mathbf{L}_\nu^k$ whose graph is within discretization error of the graph of f.

- There exists a fast numerical algorithm that transforms a given computer realization $\psi : \mathbf{L}_\nu^k \to \mathbf{L}_\nu^k$ into a discretization φ having almost the same precision and that is a permutation of $\mathbf{L}_\nu^k$. Thus, computer realizations are closely approximated by permutations of discretized computer state space.

- Suppose that we have a mapping $\varphi : \mathbf{L}_\nu^k \to \mathbf{L}_\nu^k$. Then it is possible to check whether φ is a discretization of a system f with invariant Lebesgue measure, and there exists a fast numerical algorithm to accomplish this.

2. Statement of results

For $x = \left(x^1, \ldots, x^k\right) \in \Re^k$ define $\|x\| = \max_i\{x^i\}$ and denote by

$$\rho(x, y), \qquad x, y \in \mathbf{T}^k \tag{1}$$

the metric induced by this norm on the space $\mathbf{T}^k$. This metric extends naturally to a metric ρ_2 on $\mathbf{T}^k \times \mathbf{T}^k$ according to

$$\rho_2((\xi, x), (\eta, y)) = \max\{\rho(\xi, \eta), \rho(x, y)\}.$$

For any f and φ denote by $d(\varphi, f)$ the *Hausdorff separation* in the metric ρ_2 of the graph $\mathbf{Gr}(\varphi)$ of φ from the graph $\mathbf{Gr}(f)$ of f,

$$\begin{aligned} d(\varphi, f) &= \sup_{\xi \in \mathbf{L}_\nu^k} \inf_{x \in \mathbf{T}^k} \rho_2((\xi, \varphi(\xi)), (x, f(x))) \\ &= \sup_{\xi \in \mathbf{L}_\nu^k} \inf_{x \in \mathbf{T}^k} \max\left\{\rho(\xi, x), \rho(\varphi(\xi), f(x))\right\}. \end{aligned} \tag{2}$$

Note that (2) is the Hausdorff *separation* but not the Hausdorff *metric*, and that $d(\varphi, f) = 0$ if and only if $\mathbf{Gr}(\varphi) \subseteq \mathbf{Gr}(f)$. So, in particular, if $\varphi = f|_{\mathbf{L}_\nu^k}$ is simply the restriction of f to the lattice $\mathbf{L}_\nu^k \subset \mathbf{T}^k$, then $d(\varphi, f) = 0$. The separation of one realization φ from another ψ or of a dynamical system f from its realization φ are defined in a similar manner. The quantity $d(\varphi, f)$ differs from the usual discretization error

$$d_*(\varphi, f) = \sup_{\xi \in \mathbf{L}_\nu^k} \rho(\varphi(\xi), f(\xi))$$

(see [12], [5]).Recall also that a measure μ is said to be an *invariant measure* for f if $\mu(S) = \mu(f^{-1}(S))$ for all Borel sets $S \subseteq \mathbf{T}^k$ [3].

Theorem 1. *(a) Let Lebesgue measure be invariant for the dynamical system $f : \mathbf{T}^k \to \mathbf{T}^k$. Then for any positive integer ν there exists a permutation π of $\mathbf{L}_\nu^k$ satisfying $d(\pi, f) \leq (2\nu)^{-1}$.*

(b) Let the system f be continuous and suppose that for each $\varepsilon > 0$ there exists a positive integer ν and a permutation π of $\mathbf{L}_\nu^k$ satisfying $d(\pi, f) \leq \varepsilon$. Then Lebesgue measure is invariant for f.

For continuity of exposition, the proof is relegated to the next section. For the moment, let us discuss some consequences of the theorem. Consider the situation in which we have a realization $\varphi : \mathbf{L}_\nu^k \to \mathbf{L}_\nu^k$ and an estimate of the error $d(\varphi, f)$, but no other information concerning the underlying theoretical system f. Then instead of Theorem 1 it is perhaps better to use the following corollary.

Corollary 1. *Let φ be a given mapping of $\mathbf{L}_\nu^k$ into itself.*

(a) *Suppose that Lebesgue measure is invariant for some system $f : \mathbf{T}^k \to \mathbf{T}^k$ satisfying $d(f, \varphi) \le \varepsilon$. Then there exists a permutation π of $\mathbf{L}_\nu^k$ satisfying $d(\pi, \varphi) \le \varepsilon + (2\nu)^{-1}$.*

(b) *Suppose that there exist a permutation π of $\mathbf{L}_\nu^k$ with $d(\pi, \varphi) \le \varepsilon$. Then there exists a system $f : \mathbf{T}^k \to \mathbf{T}^k$ satisfying $d(f, \varphi) \le \varepsilon + (2\nu)^{-1}$, and such that Lebesgue measure is invariant for f.*

In other words, there exists a natural combinatoric approach that allows a check of the hypothesis that a given computer mapping φ is a realization of some system f that is invariant with respect to Lebesgue measure.

The assertion (a) of Corollary 1 follows from assertion (a) of Theorem 1. For (b), it is sufficient to define f as any Borel mapping that satisfies

$$f(x) = A_{\pi(\xi)-\xi}(x), \qquad x \in \mathbf{T}^k, \quad \xi \in \mathbf{L}_\nu^k, \text{ such that } \rho(x, \xi) < \frac{1}{2\nu}.$$

Here A_η is a shift mapping on $\mathbf{T}^k$ generated by the vector η (see [3], p. 64).

Finally, let us briefly discuss problems associated with practical computation of appropriate permutations. Choose an enumeration $(\eta_1, \dots, \eta_{\nu^k})$ of $\mathbf{L}_\nu^k$. Then to each subset $S \subseteq \mathbf{L}_\nu^k \times \mathbf{L}_\nu^k$ there corresponds the ν^k-square matrix $A(S) = A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (\eta_i, \eta_j) \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For a ν^k-square matrix $A = (a_{ij})$, the set $\left\{a_{i(1)j(1)}, \dots, a_{i(\nu^k)j(\nu^k)}\right\}$ of its entries is called a *diagonal* if all numbers $i(1), \dots, i(\nu^k)$, as well as all numbers $j(1), \dots, j(\nu^k)$, are pairwise distinct.

Consider the set $\Gamma(f, \nu)$ of all $(\xi, \eta) \in \mathbf{L}_\nu^k \times \mathbf{L}_\nu^k$ satisfying the inequality $\rho_2\left((\xi, \eta), \mathbf{Gr}(f)\right) \le (2\nu)^{-1}$. The problem of constructing a permutation φ satisfying $d(\varphi, f) \le (2\nu)^{-1}$ is clearly equivalent to that of choosing a diagonal without zeros in the matrix $A(\Gamma(f, \nu))$. This problem can be resolved by Theorem 1. Therefore, it is equivalent to the optimal assignment problem for the matrix (3) as described, for example, in [8] (Section 8.2.10, p. 232). As such, a solution will follow by the "Hungarian Method" [10], or from the shortest path algorithm [7], and is of order $O((\nu^k)^3)$.

3. Proof of Theorem 1

Let us first prove assertion (a). Let $\mathbf{L}$ be a given finite set and Φ be a mapping from $\mathbf{L}$ into $2^{\mathbf{L}}$. A measure μ on $\mathbf{L}$ is called *semi-invariant* [5] for Φ if for any subset $\mathbf{L}_* \subseteq \mathbf{L}$

$$\mu(\mathbf{L}_*) \leq \mu(\Phi^{-1}(\mathbf{L}_*)), \tag{4}$$

where $\Phi^{-1}(\mathbf{L}_*) = \{\xi \in \mathbf{L} : \Phi(\xi) \cap \mathbf{L}_* \neq \emptyset\}$.

Denote by μ_* the uniform measure on $\mathbf{L}$ defined by $\mu_*(\xi) = 1/\#(\mathbf{L})$ for all $\xi \in \mathbf{L}$, where $\#(\mathbf{L})$ denotes the cardinality of the set $\mathbf{L}$. We will say that a multi-valued mapping Φ *contains* a single-valued mapping $\varphi : \mathbf{L} \to \mathbf{L}$ if $\varphi(\xi) \in \Phi(\xi)$ for all $\xi \in \mathbf{L}$.

Lemma 1. *The measure μ_* is semi-invariant for Φ if and only if Φ contains some permutation φ.*

Proof. Choose an enumeration $(\eta_1, \ldots, \eta_{\nu^k})$ of the set $\mathbf{L}$. Then to each multivalued mapping $\Phi : \mathbf{L} \to 2^{\mathbf{L}}$ there corresponds a ν^k-square matrix $A_\Phi = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \eta_j \in \Phi(\eta_i), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the matrix A_Φ coincides with the matrix (3) constructed for the graph $\mathbf{Gr}(\Phi)$. From the definitions, the measure μ_* is semi-invariant for Φ if and only if A_Φ contains no $s \times t$ zero submatrix such that $s+t = \nu^k+1$. Therefore, Lemma 1 follows from the following classical result attributed to Frobenius and König ([11], Theorem 2.1).

> *Let A be an n-square matrix. A necessary and sufficient condition for every diagonal of A to contain a zero entry is that A contain an $s \times t$ zero submatrix such that $s + t = n + 1$.* ∎

By virtue of Lemma 1, to complete the proof of (a) it is sufficient to establish the following assertion.

Lemma 2. *Let Lebesgue measure μ be invariant for the system f on $\mathbf{T}^k$. Then there exists a multi-valued discretization Φ satisfying*

$$d(\mathbf{Gr}(\Phi), \mathbf{Gr}(f)) \leq (2\nu)^{-1} \tag{5}$$

and having the uniform measure μ_* on $\mathbf{L}_\nu^k$ as a semi-invariant measure.

Proof. For any subset $X \subseteq \mathbf{T}^k$ denote by $\overline{\mathcal{O}}_\varepsilon(X)$ the closed ε-neighborhood of X in the metric ρ defined by (1). Without confusion, the same symbol $\overline{\mathcal{O}}_\varepsilon$ will be used for the corresponding ε-neighborhoods of sets $P \subseteq \mathbf{T}^k \times \mathbf{T}^k$ in the extended metric ρ_2. Write

$$\Gamma_\nu(f) = \overline{\mathcal{O}}_{(2\nu)^{-1}}(\mathbf{Gr}(f)) \bigcap \left(\mathbf{L}_\nu^k \times \mathbf{L}_\nu^k\right).$$

Define the multi-valued discretization $\Phi_\nu : \mathbf{L} \to 2^{\mathbf{L}}$ by $\mathbf{Gr}(\Phi_\nu) = \Gamma_\nu$. Clearly, this discretization satisfies (5), and it remains to prove that the uniform measure μ_* is semi-invariant for the mapping Φ_ν, that is,

$$\mu_*(\Phi_\nu^{-1}(\Lambda)) \geq \mu_*(\Lambda) \qquad \text{for all } \Lambda \subseteq \mathbf{L}_\nu^\kappa. \tag{6}$$

Suppose that Λ is a given subset of $\mathbf{L}_\nu^k$. Because μ_* is the uniform measure on $\mathbf{L}_\nu^k$, (6) will follow from the inequality

$$\#(\Phi_\nu^{-1}(\Lambda)) \geq \#(\Lambda) \tag{7}$$

where $\#(S)$ denotes cardinality of the set S. By the definition of Lebesgue measure μ,

$$\mu(\overline{\mathcal{O}}_{(2\nu)^{-1}}(\Lambda)) = \frac{\#(\Lambda)}{\nu^k}.$$

Because the measure μ is invariant for f, this last inequality implies that

$$\mu\left(f^{-1}\left(\overline{\mathcal{O}}_{(2\nu)^{-1}}(\Lambda)\right)\right) = \frac{\#(\Lambda)}{\nu^k}.$$

In particular, this means that the set $f^{-1}(\overline{\mathcal{O}}_{(2\nu)^{-1}}(\Lambda))$ intersects at least $\#(\Lambda)$ different balls of the form $\overline{\mathcal{O}}_{(2\nu)^{-1}}(\xi)$, $\xi \in \mathbf{L}_\nu^k$, or, what comes to the same thing,

$$\#\left(\overline{\mathcal{O}}_{(2\nu)^{-1}}\left(f^{-1}(\overline{\mathcal{O}}_{(2\nu)^{-1}}(\Lambda))\right) \bigcap \mathbf{L}_\nu^k\right) \geq \#(\Lambda).$$

The inequality (7) immediately follows and the proof of Lemma 2, and thus of assertion (a), is complete.

It remains to prove assertion (b), namely that the Lebesgue measure μ on $\mathbf{T}^k$ is invariant for f. It is well known that it is sufficient to prove the equality

$$\int \alpha(f(x))\, d\mu = \int \alpha(x)\, d\mu \tag{8}$$

for all continuous functions $\alpha : \mathbf{T}^k \to \Re$ ([3], p. 37, equation (1)). Under the conditions of (b) there exist a sequence of positive integers $\nu(n)$, $n = 1, 2, \ldots, \nu(n) \to \infty$, and permutations π_n of the sets $\mathbf{L}_{\nu(n)}^k$, $n = 1, 2, \ldots$ satisfying

$$d(\pi_n, f) \leq 1/n \qquad n = 1, 2, \ldots. \tag{9}$$

Using (9) and the continuity of $\alpha(x)$ and $f(x)$ it follows that

$$\int \alpha(f(x))\, d\mu = \lim_{n\to\infty} \sum_{\xi \in \mathbf{L}_{\nu(n)}^k} \alpha(\pi_n(\xi)). \tag{10}$$

Because π_n is a permutation we can rewrite (10) as

$$\int \alpha(f(x))\, d\mu = \lim_{n\to\infty} \sum_{\xi \in \mathbf{L}_{\nu(n)}^k} \alpha(\xi). \tag{11}$$

On the other hand, from the continuity of $\alpha(x)$ it follows that

$$\int \alpha(x)\, d\mu = \lim_{n \to \infty} \sum_{\xi \in \mathbf{L}^k_{\nu(n)}} \alpha(\xi). \tag{12}$$

The equalities (10) and (12) imply (8), and the proof of the assertion (b) is complete. ∎

Note that the continuity of the mapping f in Theorem 1(b) is essential. For instance, for the mapping $h : \mathbf{T}^1 \mapsto \mathbf{T}^1$ defined by $h(x) = x$ for rational x and $h(x) = 0$ otherwise, the restriction on any $\mathbf{L}^1_\nu$ is the identity permutation. However, Lebesgue measure is not invariant for h.

Acknowledgments

The authors would like to thank Tony Watts for pointing out the equivalence of the Assignment Problem and that of finding diagonals, and for suggesting the reference [7].This research has been supported by the Australian Research Council Grant A8913 2609.

References

[1] M. Blank, "Small Perturbations of Chaotic Dynamical Systems," *Russian Mathematical Society Surveys*, **44**(6) (1989) 1–33.

[2] A. Boyarsky and P. Gora, "Why Computers Like Lebesgue Measure," *Computers & Mathematics with Applications*, **16** (1988) 321–329.

[3] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (New York: Springer-Verlag, 1982).

[4] P. Diamond, P. Kloeden, and A. Pokrovskii, "An Invariant Measure Arising in Computer Simulation of a Chaotic Dynamical System," *Journal of Nonlinear Sciences*, **4** (1994) 59–68.

[5] P. Diamond, P. Kloeden, and A. Pokrovskii, "Weakly Chain Recurrent Points and Spatial Discretizations of Dynamical Systems," *Random & Computational Dynamics*, **2**(1) (1994) 97–110.

[6] P. Diamond, P. Kloeden, and A. Pokrovskii, "Cycles of Spatial Discretizations of Shadowing Dynamical Systems," *Mathematische Nachrichten* (in press).

[7] M. Engquist, "A Successive Shortest Path Algorithm for the Assignment Problem, *Information*, **20** (1982) 370–384.

[8] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization* (Berlin: Springer-Verlag, 1988).

[9] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (New York: Springer-Verlag, 1983).

[10] H. W. Kuhn, "The Hungarian Method for the Assignment Problem," *Naval Research Logistic Quarterly*, **2** (1955) 83–97.

[11] H. Minc, *Encyclopedia of Mathematics and Applications. Volume 6: Permaments* (London: Addison-Wesley, 1978).

[12] H. J. Stetter, *Analysis of Discretization Methods for Ordinary Differential Equations* (Berlin: Springer-Verlag, 1976).