

Additive Cellular Automata with External Inputs

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Abstract. In this paper we consider a form of cellular automata (CA) that allows for external input at each stage of its evolution. The investigation is carried out from an algebraic viewpoint whereby the state structure of the CA, together with its action, can be interpreted ring-theoretically.

Specifically, we consider the structure of attractors for different controls and the associated tree structures of the states that approach the attractor. In particular, we relate the invariant sets that arise for controlled and uncontrolled CA and give properties on the number of states in the invariant set. Algebraic properties of the CA ring are used to qualify properties of cycle lengths and number of cycles.

A qualitative equivalence for CA is introduced and equivalence of attractors is characterized by the cycle set structure for different inputs. Sufficient conditions for qualitative equivalence in the presence of distinct inputs are found, and necessary and sufficient conditions for qualitative equivalence in the presence of distinct inputs are given when an extra condition holds.

Finally, the algebraic properties of CA associated with periodic input are investigated, and some generalizations are discussed.

1. Introduction

A natural question to ask of any autonomous, closed system is: What happens to the behavior of the system in the presence of external influences? Such influences can be viewed as the natural consequence of the system's interaction with the world around it or as a deliberate attempt at influencing or controlling the behavior of the system. Cellular automata (CA) are a class of extended discrete dynamical systems that have attracted considerable attention in recent years. As far as these authors are aware, there has been no serious attempt to study the behavior of deterministic CA in the presence of external inputs with a view to obtaining exact results. (For a

general introduction to CA see [1] and [2] and the references therein.) We consider this question for a class of CA with periodic and eventually periodic external inputs. Of course, the system consisting of CA plus external inputs is not in general a CA itself (for instance, it will not, in general, possess the translational invariance typical of infinite CA or CA with periodic boundary conditions). However, such a system can be considered as a form of hybrid CA.

We restrict our attention to the case of additive CA whose linearity properties make them particularly amenable to algebraic methods but are still capable of producing complex behavior [3]. Several important papers on additive CA have appeared over the last ten years or so, employing a variety of approaches. Detailed results on the reachability of states (configurations), numbers of states on cycles, and some conditions on cycle lengths for additive CA rules is presented in [3]. In the case of the state alphabet as a finite field, [4] provides conditions for reversibility of rules and stronger results on possible cycle lengths, whereas in [5] details of states that can appear on cycles are provided. More recently there has been a resurgence of interest in additive CA in the physics literature, for instance [6, 7, 8, 9, 10].

We shall primarily be concerned with a one-dimensional CA with a state alphabet consisting of a finite commutative ring R , periodic boundary conditions, and an external input at each time step. We shall sometimes refer to these systems as controlled CA and refer to the inputs as controls; however, the emphasis of this paper is not control theory. The generalization of such systems to two or more dimensions is fairly straightforward and is discussed in section 5. We also deal with the case of null (or Dirichlet) boundary conditions, which are treated in parallel with the periodic boundary conditions, though the emphasis is on the latter. The main body of the paper (sections 2 and 3) is concerned with time-independent inputs (constant controls); the extension to inputs with nontrivial periodicity properties is made in section 4. We assume that the reader has some familiarity with basic algebra; for general introductions see [11, 12, 13].

We employ a formalization of the polynomial representation used in [3] that emphasizes algebraic structure to study additive CA in one dimension with periodic boundary conditions (and use a matrix method for null boundary conditions). Using this representation we are able to view the global rule of an additive CA on N cells as a linear map from a ring R_N (related to R) to itself and to view the global map of the corresponding controlled system as an affine map (for null boundary conditions, one has linear and affine maps from the R -module R^N to itself). In section 2 notation is introduced and our method is described in detail. In section 3 we employ this method to study in detail the dynamics of one-dimensional CA with external time-independent inputs. In particular we consider the orbital structure of these systems in terms of the graphical structure obtained with vertices given by the states and the arcs given by the transitions from one state to another. Such a graph gives a pictorial representation of the evolutionary characteristics of the system. The behavior of the system with external inputs is related to that

without inputs; it turns out that for constant inputs the controlled systems exhibit very similar behavior to the uncontrolled system and that much of their behavior can be described in terms of the behavior of the uncontrolled system. Sufficient conditions for different controls to give qualitatively similar behavior are obtained, as well as necessary and sufficient conditions for different controls to give qualitatively similar behavior under certain circumstances. The occurrence of qualitatively similar behavior of different additive CA rules is discussed briefly.

In section 4, periodic and eventually periodic inputs are introduced. We show that much of the behavior of these systems can be understood from the behavior of systems with constant inputs. In section 5, some generalizations are briefly considered. We show that many of our results can be extended to CA in more than one dimension and to hybrids of additive CA rules. We also show that, with appropriate definitions, some of the results still hold for infinite CA. Section 6 is devoted to a discussion of the results.

2. Preliminaries

We shall be considering additive CA with periodic boundary conditions. Many of the results also hold for controls with null or Dirichlet boundary conditions when the state of sites to the left and right of the N cells under consideration are assumed to be held at zero for all time. The possible states that a cell can assume will be the elements of the state alphabet, which we take to be a finite commutative ring R , and we shall sometimes say that the CA is over R . The number of cells will be denoted by $N \in \mathbb{N}_{>0}$, and the cells will be indexed as $0, 1, \dots, N - 1$. A CA rule is additive if for any configurations P and Q of the automaton we have that

$$F(P + Q) = F(P) + F(Q), \quad (2.1)$$

where F is the global rule of the automaton and “addition” is the addition in R taken cell by cell. The global state of such a CA can be taken to be an element of the product space R^N , an R -module; for example, if the state of cell i is $a_i \in R, 0 \leq i < N$, then the element of R^N representing the state of the CA is that with i th component $a_i, 0 \leq i < N$. In this representation the global rule on N cells becomes the map

$$F_N : R^N \longrightarrow R^N. \quad (2.2)$$

The state can also be represented as a polynomial $a(x) \in R[x]$ of degree less than N , with the coefficient of x^i being the state of the i th site.

The configuration consisting entirely of zeros (the quiescent configuration) is left unchanged by any additive CA rule. The global rule \mathbb{T} of the CA can be represented by the multiplicative action of a polynomial $\mathbb{T}(x) \in R[x]$ with periodic boundary conditions implemented by reducing modulo $(x^N - 1)$ after multiplication [3]. If $A^t(x)$ represents the state of the automaton at time t , then

$$A^{t+1}(x) = \mathbb{T}(A^t(x)) \equiv \mathbb{T}(x)A^t(x) \pmod{(x^N - 1)}, \quad (2.3)$$

where $A^{t+1}(x)$ is the state at time $t + 1$. For instance, the local rule

$$a_i^{t+1} = a_{i-1}^t + a_{i+1}^t \tag{2.4}$$

(where the addition is that in R ; when $R = \mathbb{F}_2$, the finite field with two elements, this rule is referred to as rule 90 in [14] and elsewhere) has global rule represented by the polynomial

$$\mathbb{T} = x + x^{N-1}. \tag{2.5}$$

In general, the local rules of the additive CA that we consider are linear, of the form

$$\begin{aligned} a_i^{t+1} &= f(a_{i-r}^t, \dots, a_i^t, \dots, a_{i+r}^t) \\ &= \alpha_{-r} a_{i-r}^t + \dots + \alpha_0 a_i^t + \dots + \alpha_r a_{i+r}^t \end{aligned} \tag{2.6}$$

for a radius r rule where the $\alpha_j, j = -r, \dots, r$ are elements of R . For the case of periodic boundary conditions, the polynomial representation of the global rule is then

$$\mathbb{T}(x) = \alpha_0 + \sum_{i=1}^r (\alpha_i x^{N-i} + \alpha_{-i} x^i) \pmod{(x^N - 1)}. \tag{2.7}$$

The operation $\text{Mod}(x^N - 1)$ is a ring homomorphism from $R[x]$ to the quotient ring $R[x]/((x^N - 1)R[x])$. We shall denote this quotient ring by R_N . R_N is additively indistinguishable from R^N , and each polynomial $a(x) \in R[x]$ of degree less than N can be identified uniquely with the element

$$a = a(x) + (x^N - 1)R[x] \in \frac{R[x]}{(x^N - 1)R[x]}. \tag{2.8}$$

In fact each coset $a \in R_N$ has as canonical representative a polynomial $a(x) \in R[x]$ of degree less than N , and using such representatives we have

$$a = b \quad \Leftrightarrow \quad a(x) = b(x)$$

Thus, global states of the CA can be bijectively identified with the elements of R_N . Further, the polynomial representing the CA rule is uniquely identified with such an element, and we have

$$\begin{aligned} \mathbb{T}(x)a(x) &\equiv b(x) \pmod{(x^N - 1)} && \Leftrightarrow \\ \mathbb{T}(x)a(x) &= b(x) + h(x)(x^N - 1), h(x) \in R[x] && \Leftrightarrow \\ \mathbb{T}(x)a(x) + (x^N - 1)R[x] &= b(x) + (x^N - 1)R[x] && \Leftrightarrow \\ \mathbb{T}a &= b \text{ in } R_N \end{aligned}$$

Thus, we can represent the global CA dynamics by the action of \mathbb{T} on R_N and will use \mathbb{T} to denote both the global rule and the corresponding element of R_N :

$$\mathbb{T} : R_N \longrightarrow R_N \tag{2.9}$$

$$\mathbb{T}(a) = \mathbb{T}a. \tag{2.10}$$

The time evolution is then given by

$$a^{t+1} = \mathbb{T}(a^t).$$

This is a formalization of the method in [3] that emphasizes certain algebraic aspects. Essentially one is viewing the global CA rule as a module endomorphism (i.e., a linear map) of the ring, viewed as a module over itself. (It is a standard result from the theory of rings and modules that the module endomorphisms of the ring R regarded as a module over itself form a ring isomorphic to R . Since R is commutative we may ignore the distinction between left and right R -modules when regarding R as an R -module.)

We shall consider a class of constant controls, where at each time step a constant input is introduced at each cell after the normal single time evolution has been carried out. At the local-rule level, the rule f_i is applied at site i where

$$a_i^{t+1} = f_i(a_{i-r}^t, \dots, a_{i+r}^t) = f(a_{i-r}^t, \dots, a_{i+r}^t) + u_i \tag{2.11}$$

where $u_i \in R$ is the input at site i . At the global level the constant input is represented algebraically by the constant $U = U(x) + (x^N - 1)R[x]$, where the coefficient of the i th power of x in $U(x)$ is u_i for periodic boundary conditions. Thus we have, for each $U \in R_N$, the global mapping \mathbb{T}_U (an affine map)

$$\begin{aligned} \mathbb{T}_U &: R_N \longrightarrow R_N \\ \mathbb{T}_U(a) &= \mathbb{T}a + U \end{aligned} \tag{2.12}$$

and the time evolution is given by

$$a^{t+1} = \mathbb{T}_U(a^t). \tag{2.13}$$

The composition of \mathbb{T}_U with itself k times will be denoted by \mathbb{T}_U^k , and we will write

$$\mathbb{T}_U^k(a^t) = a^{t+k}. \tag{2.14}$$

When the input is uniform, $u_i = u$, $i = 0, \dots, N - 1$, we are in effect applying a different CA rule everywhere; for example, when $R = \mathbb{F}_2$, if $u = 1$ then rule 90 with this control becomes its complement, rule 165 (a nonadditive rule that clearly does not satisfy the quiescent condition), while a nonuniform control gives a hybrid of rules 90 and 165. In general we are applying a hybrid of up to $|R|$ affine CA rules, where by an affine CA rule we mean a rule derived from a linear rule by applying the linear rule and then adding a constant element of the state alphabet.

We now consider the set of elements that appear on orbits under \mathbb{T}_U . Given the finite number of states, all orbits are eventually periodic, so we define $Att(\mathbb{T}_U)$ to be the set

$$Att(\mathbb{T}_U) = \{a : a \in R_N, \mathbb{T}_U^k(a) = a, \text{ some } k > 0 \in \mathbb{N}\}, \tag{2.15}$$

the invariant subset of R_N under \mathbb{T}_U . It is the set of configurations that lie on cycles under \mathbb{T}_U (including fixed points), and will be referred to as the invariant set of \mathbb{T}_U . Let $Fix(\mathbb{T}_U)$ be the subset of fixed points (cycles of length one) under \mathbb{T}_U .

The state transition graph for \mathbb{T}_U is the directed graph with the elements of R_N as vertices and an edge from a to b if $b = \mathbb{T}_U(a)$. The transient elements of R_N under \mathbb{T} are found on the branched structures rooted on the elements of $Att(\mathbb{T}_U)$, these structures are known as trees. If $r \in Att(\mathbb{T}_U)$, denote the tree rooted at r by $T_U(\mathbb{T}, r)$ where we take $r \in T_U(\mathbb{T}, r)$. It is shown in [3] that in the uncontrolled case the trees $T_0(\mathbb{T}, r)$ rooted at each element of $Att(\mathbb{T})$ are identical to each other.

Denote by $O_U(a)$ the set $\{a, \mathbb{T}_U(a), \mathbb{T}_U^2(a), \dots\}$, the orbit of a under \mathbb{T}_U . Let $P_U(a)$ denote the cycle that $a \in R_N$ evolves to under \mathbb{T}_U ; thus, if $a \in Att(\mathbb{T}_U)$ then $P_U(a) = O_U(a)$. We shall define the prime period of $a \in Att(\mathbb{T}_U)$ to be $|P_U(a)|$ and a period of $a \in Att(\mathbb{T}_U)$ to be any multiple of $|P_U(a)|$. Let $Cyc(\mathbb{T}_U)$ be the set of distinct cycles under \mathbb{T}_U . Using terminology from the theory of linear modular systems (see [15]), we define the cycle set $\Sigma(\mathbb{T})$ of \mathbb{T} by

$$\Sigma(\mathbb{T}) = \{N_1[P_1], \dots, N_q[p_q]\} \quad (2.16)$$

where N_i is the number of cycles of length P_i . Each integer pair $N[T]$ is called a cycle term. We sometimes write $\Sigma(\mathbb{T})$ as a formal sum

$$\Sigma(\mathbb{T}) = N_1[P_1] + \dots + N_q[p_q]. \quad (2.17)$$

Let $r \in Att(\mathbb{T}_U)$; then we shall sometimes denote $\mathbb{T}_U^t(r)$ by r^t for $t \in \mathbb{N}$, with $r^0 = r$, and let r^{-t} , $t \in \mathbb{N}$, be the configuration in $P_U(r)$ such that

$$\mathbb{T}_U^t(r^{-t}) = r^0 = r. \quad (2.18)$$

Then, if $0 \leq s < t$,

$$\begin{aligned} r^{-s} &= \mathbb{T}_U^{t-s}(r^{-t}) \\ &= \mathbb{T}^{t-s}r^{-t} + (\mathbb{T}^{t-s+1} + \dots + \mathbb{T} + 1)U. \end{aligned} \quad (2.19)$$

In the case of null or Dirichlet boundary conditions, the values of the sites on either side of the N sites under consideration are regarded as having states fixed at zero for all time. In this case a state of the CA may be represented by an element of R^N , which it is convenient to think of as a vector,

$$(\mathbf{a})_i = a_i, \quad 0 \leq i \leq N - 1.$$

The inputs are represented in the same way and the global CA rule is represented by a matrix \mathbf{T} , which, if the local rule is given by equation (2.6), is

$$(\mathbf{T})_{ij} = \alpha_{j-i} \quad (2.20)$$

where $\alpha_{-r-k} = \alpha_{r+k} = 0$ for each $k \in \mathbb{N}_{>0}$. The dynamics are then given by

$$\begin{aligned} \mathbb{T}_U &: R^N \longrightarrow R^N \\ \mathbb{T}_U(\mathbf{a}) &= \mathbf{T}\mathbf{a} + \mathbf{U} \end{aligned} \tag{2.21}$$

an affine map from R^N to itself, linear (i.e., a module endomorphism) when $\mathbf{U} = \mathbf{0}$.

We note that the case of null boundary conditions and constant input can be used to handle the case of more general fixed boundary conditions. Let the local rule have radius r and suppose that the sites labeled $-1, -2, \dots, -r$ and $N, N + 1, \dots, N + r - 1$ have fixed states $a_{-1}, \dots, a_{-r}, a_N, \dots, a_{N+r-1}$. Then the global dynamics is given by

$$\begin{aligned} \mathcal{T} &: R^N \longrightarrow R^N \\ \mathcal{T}(\mathbf{a}) &= \mathbf{T}\mathbf{a} + \mathbf{U} \end{aligned} \tag{2.22}$$

where \mathbf{T} is the matrix for null boundary conditions and \mathbf{U} is the vector defined by

$$u_i = \sum_{s=1}^{r-i} \alpha_{-i-s} a_{-s} + \sum_{s'=0}^{r+i-N} \alpha_{N+s'-i} a_{N+s'}. \tag{2.23}$$

3. Results for constant inputs

We start by relating the evolution of controlled CA to that of uncontrolled CA in a series of properties (the corresponding results hold for null boundary conditions).

- (a) For any $t \in \mathbb{N}$ and for each $a \in R_N$:

$$\mathbb{T}_U^t(a) = \mathbb{T}^t a + \mathbb{T}_U^t(0). \tag{3.1}$$

- (b) A CA with external inputs has linearity properties in both state and input. Let U, V, a and $b \in R_N$, any $k \in \mathbb{N}$ and any additive CA rule \mathbb{T} :

$$(i) \quad \mathbb{T}_V^k(0) - \mathbb{T}_U^k(0) = \mathbb{T}_{V-U}^k(0); \tag{3.2}$$

$$(ii) \quad \mathbb{T}_U^{k+T}(0) - \mathbb{T}_U^T(0) = \mathbb{T}^T \mathbb{T}_U^k(0); \tag{3.3}$$

$$(iii) \quad \mathbb{T}_V^T(\mathbb{T}_U^k(0)) = \mathbb{T}_U^{k+T}(0) + \mathbb{T}_V^T(0) - \mathbb{T}_U^T(0); \tag{3.4}$$

$$(iv) \quad \mathbb{T}_V^k(a \pm b) = \mathbb{T}_V^k(a) \pm \mathbb{T}_0^k(b); \tag{3.5}$$

$$(v) \quad \mathbb{T}_{U+V}^k(a \pm b) = \mathbb{T}_U^k(a) + \mathbb{T}_V^k(\pm b). \tag{3.6}$$

We can now deduce properties on the existence of periodic orbits from related controlled CA. The proofs are tedious but straightforward.

Lemma 3.1. *For given R, N , and an additive CA rule \mathbb{T} , let U and V be in R_N and let $k \in \mathbb{N}_{>0}$. Then we can have the following.*

1. \mathbb{T}_U has a period- k orbit if and only if \mathbb{T}_{-U} has a period- k orbit.
2. If \mathbb{T}_V and \mathbb{T}_U have period- k orbits then \mathbb{T}_{V+U} and \mathbb{T}_{V-U} have period- k orbits (not necessarily prime period k).
3. If \mathbb{T}_V has a period- k orbit but \mathbb{T}_U does not, then \mathbb{T}_{V+U} and \mathbb{T}_{V-U} do not have period- k orbits.

3.1 Structure of the state transition diagram

In this section we begin to relate the geometrical properties of the system with external inputs to that without inputs. In particular we show that there is always an equivalence between the transient behavior in the two cases, and we relate the invariant set for the controlled case to that for the uncontrolled case.

Theorem 3.1. *For any one-dimensional additive CA \mathbb{T} over R on N cells and for any $U \in R_N$, the trees rooted at each element of $Att(\mathbb{T}_U)$ are identical to one another and to the tree rooted at 0 under \mathbb{T} .*

Proof. We use the notation of equations (2.18) and (2.19). Consider the map $\Psi_{U,r} : T_0(\mathbb{T}, 0) \rightarrow T_U(\mathbb{T}, r), r \in Att(\mathbb{T}_U)$ given by

$$\Psi_{U,r}(a) = a + r^{-t} \tag{3.7}$$

where $\mathbb{T}^t a = 0, \mathbb{T}^s a \neq 0$ for all $s < t$.

We show that (a) $\Psi_{U,r}(0) = r, \mathbb{T}_U^t(\Psi_{U,r}(a)) = r$; (b) $\Psi_{U,r}$ maps configurations at height t in $T_0(\mathbb{T}, 0)$ to configurations at height t in $T_U(\mathbb{T}, r)$; (c) $\Psi_{U,r}$ is injective; (d) $\Psi_{U,r}$ maps configurations with no predecessors under \mathbb{T} to configurations with no predecessors under \mathbb{T}_U ; (e) $\Psi_{U,r}$ is surjective; (f) $\Psi_{U,r}$ preserves the time evolution structure of the tree.

(a) $\Psi_{U,r}(0) = 0 + r^0 = r$. Let $t > 0, \Psi_{U,r}(a) = a + r^{-t}$. Then

$$\begin{aligned} \mathbb{T}_U^t(\Psi_{U,r}(a)) &= \mathbb{T}^t(a + r^{-t}) + \mathbb{T}_U^t(0) \\ &= \mathbb{T}^t a + \mathbb{T}^t r^{-t} + \mathbb{T}_U^t(0) \\ &= 0 + r^0 \\ &= r. \end{aligned}$$

(b) Suppose there is some $m \neq 0, m < t$ such that

$$\begin{aligned} r^{-m} &= \mathbb{T}_U^s(\Psi_{U,r}(a)) \\ &= \mathbb{T}^s a + r^{s-t}, \quad s < t \end{aligned} \tag{3.8}$$

then

$$\begin{aligned} \mathbb{T}_U^{t-s}(r^{-m}) &= \mathbb{T}^{t-s}\mathbb{T}^s a + \mathbb{T}^{t-s}r^{s-t} + \mathbb{T}_U^{t-s}(0) \\ &= \mathbb{T}^t a + r^0 \\ &= r \end{aligned} \tag{3.9}$$

that is, $r^{t-s-m} = r$. Let $k = |P_U(r)|$, then equation (3.9) implies that

$$\begin{aligned} t - s - m &= nk, \quad n \in \mathbb{Z} \\ \Rightarrow \quad -m &= s - t + nk \\ \Rightarrow \quad r^{-m} &= r^{s-t+nk} \\ &= r^{s-t} \end{aligned}$$

Then equation (3.8) implies that $0 = r^{-m} - r^{s-t} = \mathbb{T}^s a$, $s < t$, which contradicts the assumption that t is the smallest positive integer such that $\mathbb{T}^s a = 0$. This, together with (a), proves (b).

(c) Let $a, b \in T_0(\mathbb{T}, 0)$. If $\Psi_{U,r}(a) = \Psi_{U,r}(b)$, then a and b must be at the same height t in $T_0(\mathbb{T}, 0)$ by (b), so

$$\begin{aligned} a + r^{-t} &= b + r^{-t} \\ \Rightarrow a &= b. \end{aligned}$$

Thus $\Psi_{U,r}$ is injective.

(d) Let a be a configuration with no predecessors under \mathbb{T} that goes to 0 in s time steps under \mathbb{T} . Then, for some $b \in T_U(\mathbb{T}, r)$,

$$\Psi_{U,r}(a) = a + r^{-s} = b.$$

Suppose b has a predecessor, b' , under \mathbb{T}_U :

$$b = \mathbb{T}_U(b') = \mathbb{T}b' + U.$$

Then,

$$a + r^{-s} = \mathbb{T}b' + U,$$

so

$$\begin{aligned} a &= \mathbb{T}b' + U - r^{-s} \\ &= \mathbb{T}b' + U - \mathbb{T}r^{-s-1} - U \\ &= \mathbb{T}(b' + r^{-s-1}) \end{aligned}$$

which implies that a has a predecessor—a contradiction—so b has no predecessors.

(e) Let b be at height t in $T_U(\mathbb{T}, r)$, and suppose that $b \neq \Psi_{U,r}(a)$ for any $a \in T_0(\mathbb{T}, 0)$. There must be some minimal s , $0 < s \leq t$ such that

$$\mathbb{T}_U^s(b) = \Psi_{U,r}(a'), \quad \text{for some } a' \in T_0(\mathbb{T}, 0). \tag{3.10}$$

If a has no predecessors, then, by (d), neither does $\mathbb{T}_U^s(b)$. This contradiction implies that a has at least one predecessor, say a'' , and hence

$$\begin{aligned} \Psi_{U,r}(\mathbb{T}a'') &= \mathbb{T}_U^s(b) \\ \Rightarrow \mathbb{T}a'' + r^{-v} &= \mathbb{T}\mathbb{T}_U^{s-1}(b) + U, \quad v = t - s \\ \Rightarrow 0 &= \mathbb{T}a'' + r^{-v} - \mathbb{T}\mathbb{T}_U^{s-1}(b) - U \\ \Rightarrow 0 &= \mathbb{T}a'' + \mathbb{T}r^{-v-1} - \mathbb{T}\mathbb{T}_U^{s-1}(b) \\ \Rightarrow 0 &= \mathbb{T}(a'' + r^{-v-1} - \mathbb{T}_U^{s-1}(b)) \\ \Rightarrow c &= a'' + r^{-v-1} - \mathbb{T}_U^{s-1}(b), \end{aligned}$$

where $\mathbb{T}c = 0$. Thus

$$\mathbb{T}_U^{s-1}(b) = a'' - c + r^{-v-1}.$$

Now, $\mathbb{T}^{t-s+1}(a'' - c) = \mathbb{T}^{t-s+1}a'' + 0$, and a'' is at height $t - s + 1$, so

$$\begin{aligned} \mathbb{T}^{t-s+1}(a'' - c) &= 0 \\ \Rightarrow a'' - c + r^{-v-1} &= a'' - c + r^{s-t+1} \\ &= \Psi_{U,r}(a'' - c), \end{aligned}$$

thus

$$\mathbb{T}_U^{s-1}(b) = \Psi_{U,r}(a'' - c). \tag{3.11}$$

Equation (3.11) contradicts the minimality of s ; thus we have, for all $b \in T_U(\mathbb{T}, r)$, there is some $a \in T_0(\mathbb{T}, 0)$ such that $b = \Psi_{U,r}(a)$; hence Ψ_r is surjective.

(f) Let $a \in T_0(\mathbb{T}, 0)$ be such that $\mathbb{T}^t a = 0$, then

$$\begin{aligned} \mathbb{T}_U(\Psi_{U,r}(a)) &= \mathbb{T}a + \mathbb{T}r^{-t} + U \\ &= a' + r^{-t+1}, \quad \text{where } a' = \mathbb{T}a \\ &= \Psi_{U,r}(a') \end{aligned}$$

so

$$\mathbb{T}_U(\Psi_{U,r}(a)) = \Psi_{U,r}(\mathbb{T}(a)).$$

Thus, $\Psi_{U,r}$ is a bijection that preserves dynamical structure. This proves the theorem. ■

Theorem 3.1 and its corollaries hold for null boundary conditions. The first corollary follows immediately on putting $U = 0$ in Theorem 3.1.

Corollary 3.1. *For any one-dimensional additive CA on N cells over R , a finite commutative ring, the trees rooted at each configuration on each cycle are identical.*

The special case, Corollary 3.1, was proved as a theorem in [3] and, for the cases of additive CA with state alphabet a finite field for several dimensions and higher-order additive CA, in [2].

Corollary 3.2. *For the CA rule represented by $\mathbb{T} \in R_N$ and any $U \in R_N$,*

$$|Att(\mathbb{T}_U)| = |Att(\mathbb{T})|. \tag{3.12}$$

Next we look at two important ideals of R_N associated to each rule \mathbb{T} . The proof of the following result is a simple verification, while its corollary follows from Lagrange’s theorem for finite groups.

Lemma 3.2. *For each $\mathbb{T} \in R_N$, $Att(\mathbb{T})$ and $T_0(\mathbb{T}, 0)$ are ideals of R_N .*

Corollary 3.3. *For each $\mathbb{T} \in R_N$, $|Att(\mathbb{T})|$ divides $|R_N|$ and $|T_0(\mathbb{T}, 0)|$ divides $|R_N|$.*

Note that $Att(\mathbb{T})$ is a subideal of $\mathbb{T}R_N$, the ideal generated by \mathbb{T} . (In fact, $Att(\mathbb{T}) = \mathbb{T}^T R_N$, where T is the maximum tree height occurring under \mathbb{T} .) The rule represented by \mathbb{T} on N cells is reversible if and only if \mathbb{T} is a unit in R_N . Lemma 3.2 and its corollary hold for null boundary conditions with \mathbb{T} replaced by \mathbf{T} , *etcetera*, and “ideal” replaced by “submodule.” The rule on N cells represented by \mathbf{T} for null boundary conditions is reversible if and only if \mathbf{T} is an invertible matrix over R .

It is natural to consider the significance of the quotient rings corresponding to these ideals, $\frac{R_N}{Att(\mathbb{T})}$ and $\frac{R_N}{T_0(\mathbb{T},0)}$. We shall see shortly that $\frac{R_N}{Att(\mathbb{T})}$ is useful in the discussion of $Att(\mathbb{T}_U)$. In order to interpret $\frac{R_N}{T_0(\mathbb{T},0)}$ geometrically, we note that by Theorem 3.1 and its corollaries we must have

$$\left| \frac{R_N}{T_0(\mathbb{T},0)} \right| = |Att(\mathbb{T})|. \tag{3.13}$$

Let $r \in Att(\mathbb{T}) \setminus \{0\}$ and let $a+r \in r+T_0(\mathbb{T},0)$, with $\mathbb{T}^t a = 0$, t minimal. Then $a+r = \Psi_{0,r^t}(a)$, so points at height t in $T_0(\mathbb{T},0)$ correspond to points at height t in $T_0(\mathbb{T},r^t)$. Hence $r + T_0(\mathbb{T},0)$ consists of those elements of R_N at height 1 in $T_0(\mathbb{T},\mathbb{T}r)$, at height 2 in $T_0(\mathbb{T},\mathbb{T}^2r)$, ..., and at height T in $T_0(\mathbb{T},\mathbb{T}^T r)$. It is clear that the distinct cosets are exactly $r + T_0(\mathbb{T},0)$, $r \in Att(\mathbb{T})$.

Theorem 3.2. *The distinct invariant sets that can occur for the CA rule represented by $\mathbb{T} \in R_N$ on N cells with constant input $U \in R_N$ are the elements of $\frac{R_N}{Att(\mathbb{T})}$, with*

$$Att(\mathbb{T}_U) = \alpha + Att(\mathbb{T}) \tag{3.14}$$

where α is any element of $Att(\mathbb{T}_U)$.

Proof. Let $\alpha \in Att(\mathbb{T}_U)$. Then each element of $\alpha + Att(\mathbb{T})$ is of the form $\alpha + b$ where $b \in Att(\mathbb{T})$. Suppose that $\mathbb{T}_U^{k_\alpha}(\alpha) = \alpha$ and $\mathbb{T}^{k_b} b = b$ for strictly positive integers k_α, k_b . Let $k = \text{lcm}(k_\alpha, k_b)$. Then

$$\begin{aligned} \mathbb{T}_U^k(\alpha + b) &= \mathbb{T}_U^k(\alpha) + \mathbb{T}^k b \\ &= \alpha + b. \end{aligned}$$

Thus $\alpha + Att(\mathbb{T}) \subseteq Att(\mathbb{T}_U)$. But $|Att(\mathbb{T})| = |Att(\mathbb{T}_U)|$ by Corollary 3.2 to Theorem 3.1. Hence $Att(\mathbb{T}) = Att(\mathbb{T}_U)$. ■

Thus the elements of $Att(\mathbb{T}_U)$ can be found from those of $Att(\mathbb{T})$ from translation by any particular element of $Att(\mathbb{T}_U)$. In practice, it is often convenient to use $\mathbb{T}_U^T(0)$ where \mathbb{T} is the maximum tree height under \mathbb{T} . Of course, this tells us nothing about the dynamical structure of the attractor.

Applying Theorem 3.2 twice enables us to relate the invariant sets for different inputs.

Corollary 3.4. *For any $U, V \in R_N$ and any $\alpha \in Att(\mathbb{T}_U)$, $\beta \in Att(\mathbb{T}_V)$, one has*

$$Att(\mathbb{T}_U) = \alpha - \beta + Att(\mathbb{T}_V). \tag{3.15}$$

For any $k > 0$, let $Cyc(\mathbb{T}_U, k) = \{a \in Att(\mathbb{T}_U) : \mathbb{T}_U^k(a) = a\}$. In particular, $Cyc(\mathbb{T}, k)$ is an ideal in R_N , a subideal of $Att(\mathbb{T})$. There is a similar relation to that described in Theorem 3.2 between $R_N/Cyc(\mathbb{T}, k)$ and $Cyc(\mathbb{T}_U, k)$ for those \mathbb{T}_U with period- k orbits.

Lemma 3.3. *For given R, N , and \mathbb{T} , if there is a period of k orbit under \mathbb{T}_U , then for any $a \in Cyc(\mathbb{T}_U, k)$ one has*

$$Cyc(\mathbb{T}_U, k) = a + Cyc(\mathbb{T}, k). \tag{3.16}$$

The result analogous to Corollary 3.4 also holds.

The next result gives a set of equivalent conditions for different inputs to yield the same invariant set.

Lemma 3.4. *The following statements are equivalent for any $U, V \in R_N$.*

1. $Att(\mathbb{T}_U) \cap Att(\mathbb{T}_V) \neq \emptyset$.
2. $Att(\mathbb{T}_U) = Att(\mathbb{T}_V)$.
3. $U - V \in Att(\mathbb{T})$.
4. $0 \in P_{U-V}(0)$.
5. $Att(\mathbb{T}) = Att(\mathbb{T}_{U-V})$.

In particular

$$Att(\mathbb{T}) = Att(\mathbb{T}_U) \Leftrightarrow 0 \in P_U(0) \Leftrightarrow U \in Att(\mathbb{T}). \tag{3.17}$$

Corollary 3.5. *For each $U \in R_N$ there are $|Att(\mathbb{T})| - 1$ elements U' such that $U' \neq U$ but $Att(\mathbb{T}_{U'}) = Att(\mathbb{T}_U)$.*

Theorem 3.3. *Let \mathbb{T} be a nonunit in R_N and, for each $r \in Att(\mathbb{T})$, let $\Psi_{0,r}$ be the structure preserving bijection defined in the proof of Theorem 3.1. Then with $U, V, a \in T_0(\mathbb{T}, 0)$ we have the following.*

1. $Att(\mathbb{T}_U) = Att(\mathbb{T}_{\Psi_{0,r}(U)})$, each $r \in Att(\mathbb{T})$.
2. $Att(\mathbb{T}_U) \neq Att(\mathbb{T}_V)$ if $U \neq V$.
3. $a + Att(\mathbb{T}) = \Psi_{0,r}(a) + Att(\mathbb{T})$, each $r \in Att(\mathbb{T})$.
4. $b + Att(\mathbb{T}) \neq a + Att(\mathbb{T})$ for $b \in T_0(\mathbb{T}, 0) \setminus \{a\}$.

Proof.

1. Let U be at height t in $T_0(\mathbb{T}, 0)$, let $r \in Att(\mathbb{T})$, then

$$\mathbb{T}_{U-\Psi_{0,r}(U)} = \mathbb{T}_{-r^{-t}}.$$

$Att(\mathbb{T})$ is an ideal, so $-r^{-t} \in Att(\mathbb{T})$ and so by statements 2 and 3 of Lemma 3.4 we have $Att(\mathbb{T}_U) = Att(\mathbb{T}_{\Psi_{0,r}(U)})$.

2. This result follows from statement 1 and Corollary 3.5.

3. Let a be at height t in $T_0(\mathbb{T}, 0)$, then

$$\begin{aligned} \Psi_{0,r}(a) + \text{Att}(\mathbb{T}) &= a + r^{-t} + \text{Att}(\mathbb{T}) \\ &= a + \text{Att}(\mathbb{T}) \end{aligned}$$

as $r^{-t} \in \text{Att}(\mathbb{T})$ as $\text{Att}(\mathbb{T})$ is an ideal.

4. This follows from statement 3 on noting that for any $a \in R_N$ there are exactly $|\text{Att}(\mathbb{T})|$ elements $c \in R_N$ (including a) such that $c + \text{Att}(\mathbb{T}) = a + \text{Att}(\mathbb{T})$. ■

The above holds for null boundary conditions, with nonunit \mathbb{T} replaced by noninvertible matrix \mathbf{T} .

By Theorem 3.3, $T_0(\mathbb{T}, 0)$ can be thought of as an “index” for the possible invariant sets, that is, choosing distinct elements of $T_0(\mathbb{T}, 0)$ as inputs gives distinct invariant sets. Immediate consequences of Theorem 3.3 are that $\text{Att}(\mathbb{T}_U) = \text{Att}(\mathbb{T}_V)$ for $U \in T_0(\mathbb{T}, 0)$ if and only if $V = \Psi_{0,r}(U)$ for some $r \in \text{Att}(\mathbb{T})$, and that $b + \text{Att}(\mathbb{T}) = a + \text{Att}(\mathbb{T})$ for some $a \in T_0(\mathbb{T}, 0)$ if and only if $b = \Psi_{0,r}$ for some $r \in \text{Att}(\mathbb{T})$. In general, choosing distinct $a, b \in T_0(\mathbb{T}, 0)$ gives

$$a + \text{Att}(\mathbb{T}) \neq b + \text{Att}(\mathbb{T}). \tag{3.18}$$

This result follows from statement 3 of Theorem 3.3. However, we note that, in general, for a at height greater than 1 in $T_0(\mathbb{T}, 0)$,

$$a + \text{Att}(\mathbb{T}) \neq \text{Att}(\mathbb{T}_a).$$

In contrast, if a is at height 0 or 1 in $T_0(\mathbb{T}, 0)$, then we have $a + \text{Att}(\mathbb{T}) = \text{Att}(\mathbb{T}_a)$.

Example 1. Consider the case $R = \mathbb{Z}/_3$, $N = 4$, and the rule given by $\mathbb{T} = x + x^3 + (x^4 - 1)\mathbb{Z}/_3[x]$. We omit the “ $+(x^4 - 1)\mathbb{Z}/_3[x]$ ” from now on for brevity. In this case, each $a \in \text{Att}(\mathbb{T})$ has exactly nine predecessors, and one can show that $|\text{Att}(\mathbb{T})| = 9$; thus the maximum tree height is one. Consider the configuration $2x + 2x^3 \in \text{Att}(\mathbb{T})$ (easily verified). Suppose we have

$$\mathbb{T}b = 2x + 2x^3. \tag{3.19}$$

Then

$$(x + x^3)(b_0 + b_1x + b_2x^2 + b_3x^3) = 2x + 2x^3 \tag{3.20}$$

where the $b_i \in \mathbb{Z}/_3, i = 0, \dots, 3$.

$$\begin{aligned} \Rightarrow b_1 + b_3 &= 0, \text{ and} \\ b_0 + b_2 &= 2 \\ \Rightarrow (b_1, b_3) &\in \{(0, 0), (1, 2), (2, 1)\}, \text{ and} \\ (b_0, b_2) &\in \{(0, 2), (1, 1), (2, 0)\} \end{aligned}$$

Thus there are nine distinct choices of the $\{b_0, b_1, b_2, b_3\}$ leading to $2x + 2x^3$. Since the trees rooted at all elements of $\text{Att}(\mathbb{T})$ are identical, all such elements

have exactly nine predecessors. Since $|\frac{\mathbb{Z}_3[x]}{(x^4-1)\mathbb{Z}_3[x]}| = 81$, all configurations are accounted for, and the maximum tree height can only be one. Thus, here

$$Att(\mathbb{T}_U) = U + (Att(\mathbb{T})) \text{ for all } U \in \frac{\mathbb{Z}_3[x]}{(x^4 - 1)\mathbb{Z}_3[x]}.$$

Theorem 3.1 shows that all the results in [3] about numbers of configurations without predecessors, tree structure, maximum tree height, *etcetera* hold for the constant control case. Moreover, the number of configurations on cycles remains the same. However, the number of cycles and size of cycles may change, that is, the cycle set may change. In [3] it is shown that the length of any cycle under \mathbb{T} must divide $\Pi_N(\mathbb{T})$, where

$$\Pi_N(\mathbb{T}) = |P_0(x^i)|. \tag{3.21}$$

For null boundary conditions, define $\Pi_N(\mathbb{T})$ to be the period (or eventual period) of the matrix (\mathbb{T}) ; that is, $\Pi_N(\mathbb{T})$ is the least integer such that, for some minimal $T \in \mathbb{N}$, $\mathbb{T}^{T+\Pi_N} = \mathbb{T}^T$. The value $\Pi_N(\mathbb{T})$ for periodic boundary conditions can be defined in the same way as $\Pi_N(\mathbb{T}) = |P_0(1)| = |P_0(\mathbb{T})|$. The next two lemmas are the analogs of that result for the constant-control case. Denote $|P_0(a)|$ by $(\phi_a)_0$, $|P_U(a)|$ by $(\phi_a)_U$ for any $a \in R_N$.

Lemma 3.5. *Let $a \in R_N$. Then*

$$(\phi_a)_U | d \text{ and } (\phi_a)_U | D \tag{3.22}$$

where

$$d = \text{LCM}((\phi_a)_0, (\phi_0)_U) \text{ and } D = \text{LCM}(\Pi_N(\mathbb{T}), (\phi_0)_U) \tag{3.23}$$

with $d \leq D$. Further, for each $U \in R_N$,

$$(\phi_0)_U | (\phi_0)_1. \tag{3.24}$$

It is worth noting that, for all $a \in R_N$,

- (a) $(\phi_a)_U | \Pi_N \Leftrightarrow (\phi_0)_U | \Pi_N(\mathbb{T})$;
- (b) $\Pi_N(\mathbb{T}) | (\phi_0)_U \Rightarrow (\phi_a)_U | (\phi_0)_U$.

Lemma 3.6. *Let m be the characteristic of R , and let $U \in R_N$. Then for given \mathbb{T} and any $a \in R_N$,*

$$(\phi_a)_U | m \Pi_N(\mathbb{T}). \tag{3.25}$$

We shall denote the set of units in R_N by $U(R_N)$. The next lemma concerns the case $U = 0$. There is no analog of this result for null boundary conditions.

Lemma 3.7. *If $\mathbb{T} \in R_N$ is not a unit, then any $a \in U(R_N)$ has no predecessors under \mathbb{T} . If \mathbb{T} is a unit (and hence the corresponding rule is reversible), then the units lie on cycles of length $\Pi_N(\mathbb{T})$ under \mathbb{T} , and if one element on a cycle is a unit, then every element on the cycle is a unit. If \mathbb{T} is any element of R_N and R_N contains nonzero nilpotent elements, then if a cycle contains such an element, all its elements are nilpotent, and the trees rooted at the elements of this cycle contain no units. The set of nilpotent elements on cycles (including 0) is an ideal.*

Using Lemma 3.7, we can obtain a result about the nonoccurrence of cycles of length $|R_N| - 1$ for linear CA.

Theorem 3.4. *An additive CA over a finite commutative ring R with periodic boundary conditions on $N > 1$ cells cannot have a cycle of length $|R_N| - 1$.*

Proof. Let $\mathbb{T} \in R_N$, $N > 1$, represent an additive CA. If \mathbb{T} is not a unit, then the number of elements on cycles is less than $|R_N| - 1$, so it suffices to consider \mathbb{T} a unit. By Lemma 3.7, any unit is on a cycle under \mathbb{T} consisting entirely of units; thus, if there is a cycle of length $|R_N| - 1$, then all nonzero elements of R_N are units, so R_N is a field, but for $N > 1$ the elements $x - 1 + (x^N - 1)R[x]$ and $x^{N-1} + x^{N-2} + \dots + x + 1 + (x^N - 1)R[x]$ are nontrivial zero-divisors, so R_N cannot be a field. Hence there can be no cycle of length $|R_N| - 1$. ■

One of the motivations for recent studies of linear CA is interest in the use of CA and hybrid CA in very-large-scale integrated circuits, in particular as pseudorandom number generators; see [16, 17, 18]. This interest is focused mainly on the occurrence of cycles of maximal possible length, that is, of length $|R_N| - 1$. Theorem 3.4 shows that linear CA with periodic boundary conditions cannot generate such cycles.

Note that when $N = 1$, a cycle of length $|R| - 1$ will occur if and only if R is a finite field and \mathbb{T} is a primitive element of R , that is, a generator of the (cyclic) group of units in R .

We conclude this section by characterizing the configurations with no predecessors under \mathbb{T}_U in terms of those configurations with no predecessors under \mathbb{T} . This result holds for null boundary conditions.

Lemma 3.8. *For given R , N , and \mathbb{T} and any $U \in R_N$, a configuration $a \in R_N$ has no predecessors under \mathbb{T} if and only if $a + U$ has no predecessors under \mathbb{T}_U .*

3.2 Conditions for qualitatively similar dynamics

In this section we consider, first, the question of determining how different inputs give qualitatively similar behavior when applied with the same CA rule and, secondly, the problem of when distinct additive CA rules give qualitatively similar behavior for a given number of cells.

Definition 3.1. We shall say that $Att(\mathbb{T}_U)$ and $Att(\mathbb{T}_V)$ are qualitatively dynamically similar (QDS) if, for each cycle length l occurring under \mathbb{T}_U , there is a bijection between cycles of length l in $Cyc(\mathbb{T}_U)$ and cycles of length l in $Cyc(\mathbb{T}_V)$. We shall say that \mathbb{T}_U and \mathbb{T}_V are QDS if $Att(\mathbb{T}_U)$ and $Att(\mathbb{T}_V)$ are QDS.

We can use the cycle set notation introduced in section 2 to state this definition more succinctly as

$$Att(\mathbb{T}_U) \text{ and } Att(\mathbb{T}_V) \text{ are QDS} \Leftrightarrow \Sigma(\mathbb{T}_U) = \Sigma(\mathbb{T}_V). \quad (3.26)$$

We make the same definition for the case of null boundary conditions, with the obvious substitutions. Thus the dynamical structure of $Att(\mathbb{T}_U)$ is qualitatively different from that of $Att(\mathbb{T}_V)$ if there is a cycle in $Cyc(\mathbb{T}_U)$ of a length that does not occur in $Cyc(\mathbb{T}_V)$ or if there are more cycles of a given length in one than in the other. The next theorem tells us exactly when the dynamical structure of $Att(\mathbb{T}_U)$ is qualitatively the same as that of $Att(\mathbb{T})$. Its proof relies on the following lemma.

Lemma 3.9. Let $\phi : Att(\mathbb{T}_U) \rightarrow Att(\mathbb{T}_V)$ be a bijection that satisfies

$$\mathbb{T}_V(\phi(b)) = \phi(\mathbb{T}_U(b)) \quad (3.27)$$

for all $b \in Att(\mathbb{T}_U)$ (i.e., ϕ preserves the time evolution structure of $Att(\mathbb{T}_U)$). Then ϕ induces a bijection $\Phi : Cyc(\mathbb{T}_U) \rightarrow Cyc(\mathbb{T}_V)$ that preserves cycle lengths.

In the language of dynamical systems, if there is a map ϕ satisfying the conditions of Lemma 3.9, then \mathbb{T}_U is conjugate to \mathbb{T}_V .

Corollary 3.6. If $\phi : Att(\mathbb{T}_U) \rightarrow Att(\mathbb{T}_V)$ is a bijection satisfying

$$\mathbb{T}_V \circ \phi = \phi \circ \mathbb{T}_U$$

then \mathbb{T}_U and \mathbb{T}_V are QDS.

Proof. This is immediate from Lemma 3.9 and Definition 3.1. ■

Corollary 3.7. If $\phi : R_N \rightarrow R_N$ is a bijection satisfying $\mathbb{T}_V \circ \phi = \phi \circ \mathbb{T}_U$, then \mathbb{T}_U and \mathbb{T}_V are QDS. If ϕ is any mapping from R_N to itself satisfying $\mathbb{T}_V \circ \phi = \phi \circ \mathbb{T}_U$ that when restricted to $Att(\mathbb{T}_U)$, is bijective with its image, then \mathbb{T}_U and \mathbb{T}_V are QDS.

Theorem 3.5. The dynamical structure of $Att(\mathbb{T}_U)$ is qualitatively the same as that of $Att(\mathbb{T})$ if and only if there is a cycle of length one under \mathbb{T}_U .

It is easily shown that for given R , N , and \mathbb{T} , each configuration $a \in R_N$ is a fixed point for the controlled rule \mathbb{T}_U for one and only one $U \in R_N$. The condition for the existence of inputs U such that $Att(\mathbb{T})$ and $Att(\mathbb{T}_U)$ are not QDS is given in the following.

Corollary 3.8. *The dynamical structure of $\text{Att}(\mathbb{T}_U)$ is qualitatively the same as that of $\text{Att}(\mathbb{T})$ for all $U \in R_N$ if and only if \mathbb{T} has only the zero fixed point. If $|\text{Fix}(\mathbb{T})| > 1$ then there are*

$$\frac{|R_N|}{|\text{Fix}(\mathbb{T})|} (|\text{Fix}(\mathbb{T})| - 1) \tag{3.28}$$

elements U of R_N such that $\text{Att}(\mathbb{T}_U)$ is not QDS to $\text{Att}(\mathbb{T})$ and

$$\frac{|R_N|}{|\text{Fix}(\mathbb{T})|} \tag{3.29}$$

elements U of R_N such that $\text{Att}(\mathbb{T}_U)$ and $\text{Att}(\mathbb{T})$ are QDS.

For given rule R, \mathbb{T}, N and any $U, V \in R_N$, consider the relation $U \sim_{\wedge} V$ if $\mathbb{T}_{V-\nu U}$ has a fixed point where $\nu \in U(R_N)$.

Lemma 3.10. \sim_{\wedge} is an equivalence relation on R_N .

There are two important restricted cases of the \sim_{\wedge} equivalence relation. First, for a given rule \mathbb{T} and any $U, V \in R_N$, consider the relation $U \sim_1 V$ if \mathbb{T}_{V-U} has a fixed point. Secondly, consider the relation \sim_0 defined as $U \sim_0 V$ if $V = \nu U$ for some $\nu \in U(R_N)$. Clearly $U \sim_0 V \Rightarrow U \sim_{\wedge} V$.

Lemma 3.11. \sim_1 and \sim_0 are equivalence relations on R_N .

It is easily shown that all the U such that \mathbb{T}_U has a fixed point are in the same \sim_1 equivalence class. By Corollary 3.8, there are $\frac{|R_N|}{|\text{Fix}(\mathbb{T})|}$ members in the \sim_1 class of 0; hence for a given V , as U runs through R_N , $V - U$ runs through R_N , so there will be exactly $\frac{|R_N|}{|\text{Fix}(\mathbb{T})|}$ configurations U such that \mathbb{T}_{V-U} has a fixed point. Therefore, each \sim_1 equivalence class has $\frac{|R_N|}{|\text{Fix}(\mathbb{T})|}$ members, and we have the following corollary.

Corollary 3.9. *For given R, N , and \mathbb{T} there are $|\text{Fix}(\mathbb{T})| \sim_1$ equivalence classes, and each such class has $\frac{|R_N|}{|\text{Fix}(\mathbb{T})|}$ members.*

Lemma 3.12. *All controls U that are on or evolve to the same cycle under \mathbb{T} are in the same \sim_1 equivalence class.*

Proof. This follows from the fact that for any $U \in R_N$

$$\mathbb{T}_{U-\mathbb{T}U}(U) = \mathbb{T}U + U - \mathbb{T}U = U. \blacksquare$$

Thus the size of a \sim_1 equivalence class is an upper bound on cycle size for the uncontrolled rule.

Note that the \sim_0 equivalence classes are independent of the rule \mathbb{T} being considered. By the next lemma, the \sim_{\wedge} equivalence classes can be generated from the \sim_0 and \sim_1 equivalence classes. This is useful if more than one rule is being considered, since the \sim_0 classes are independent of the rule and the \sim_1 classes are simple to compute.

Lemma 3.13. *For all $W, U \in R_N$, $W \sim_\wedge U$ if and only if there exists $V \in R_N$ such that $U \sim_1 V$ and $V \sim_0 W$.*

We show in the next theorem that being a member of the same \sim_\wedge equivalence class is a sufficient condition for qualitative dynamical similarity.

Theorem 3.6. *For given R, N, \mathbb{T} , if U and V are in the same \sim_\wedge equivalence class, then $Att(\mathbb{T}_U)$ and $Att(\mathbb{T}_V)$ are QDS.*

Proof. Let U and V be in the same \sim_\wedge equivalence class. Suppose that \mathbb{T}_{V-DU} has a fixed point a where $D \in U(R_N)$. Consider the map $\phi_{D,a} : Att(\mathbb{T}_U) \rightarrow Att(\mathbb{T}_V)$ given by

$$\phi_{D,a}(b) = Db + a. \tag{3.30}$$

$\phi_{D,a}$ is clearly a bijection, since D is invertible in R_N . For any $b \in R_N$ we have

$$\begin{aligned} \mathbb{T}_V(\phi_{D,a}(b)) &= \mathbb{T}Db + \mathbb{T}a + V \\ &= \mathbb{T}Db + DU + a \\ &= D(\mathbb{T}b + U) + a \\ &= \phi_{D,a}(\mathbb{T}_U(b)) \end{aligned}$$

since

$$\mathbb{T}a + V = DU + a.$$

Hence $\phi_{D,a}$ satisfies the conditions of Corollary 3.6, and the proof is complete. ■

In particular if $U \sim_0 V$ or $U \sim_1 V$, then \mathbb{T}_U and \mathbb{T}_V are QDS. The next result follows from Theorem 3.6 and Lemma 3.12.

Corollary 3.10. *Let $U, V \in R_N$ be on the same cycle or evolve to that cycle under \mathbb{T} , then \mathbb{T}_U and \mathbb{T}_V are QDS.*

Note that controls in separate \sim_\wedge equivalence classes can give qualitatively similar dynamical structures, but being in the same class guarantees qualitative similarity, and being in a distinct class from 0 guarantees qualitative dissimilarity from $Att(\mathbb{T})$. The \sim_\wedge classes are, in general, larger than the \sim_1 equivalence classes.

Example 2. Let $R = \mathbb{Z}/2$ and $N = 3$. Let $\mathbb{T} = x + x^2 + (x^3 - 1)\mathbb{Z}/2[x]$ (rule 90). We omit the “ $+(x^3 - 1)\mathbb{Z}/2[x]$ ” from now on. One finds that $Fix(\mathbb{T}) = \{0, 1 + x, 1 + x^2, x + x^2\}$, so by Corollary 3.8 there are only two inputs U such that \mathbb{T}_U has a fixed point. Hence, there are six inputs U such that \mathbb{T} and \mathbb{T}_U are not QDS. One finds that the \sim_0 equivalence classes are $\{0\}$, $\{1 + x + x^2\}$, $\{1 + x, 1 + x^2, x + x^2\}$, and $\{1, x, x^2\}$. The \sim_1 equivalence

classes are $\{0, 1 + x + x^2\}$, $\{x^2, 1 + x\}$, $\{x, 1 + x^2\}$, and $\{1, x + x^2\}$. It then follows that there are two \sim_\wedge equivalence classes:

$$\{0, 1 + x + x^2\} \text{ and } \{1, x, x^2, 1 + x, 1 + x^2, x + x^2\}.$$

The second set is the set of inputs U such that \mathbb{T} and \mathbb{T}_U are not QDS. These inputs are all in the same class, so we see that only two qualitatively distinct types of behavior occur for this \mathbb{T} in the presence of time-independent inputs.

In the case of null boundary conditions these results still hold if $U(R_N)$ is replaced by the set of $N \times N$ invertible matrices over R that commute with the matrix \mathbb{T} representing the rule.

So far we have found only sufficient conditions for QDS. In the next theorem we find both necessary and sufficient conditions under certain circumstances, but to do so we need the following lemma.

Lemma 3.14. *Given \mathbb{T} and U in R_N such that the minimal orbit length occurring under \mathbb{T}_U is m . Then there is an element $U^* \in R_N$ such that $U \sim_1 U^*$ and $\mathbb{T}_{U^*}^m(0) = 0$.*

Proof. If $m = 1$, then $U \sim_1 0$, so take $U^* = 0$. If $m > 1$, then by Corollary 3.12, there is some $\bar{U} \in \text{Att}(\mathbb{T})$ with $U \sim_1 \bar{U}$. Let $a \in R_N$ be such that $\mathbb{T}_{\bar{U}}^m(a) = a$, and let $U^* = \bar{U} - a(\mathbb{T} - 1)$. Then $\bar{U} - U^* = a(\mathbb{T} - 1)$ and $\mathbb{T}_{a(\mathbb{T}-1)}(-a) = -a$, so $\bar{U} \sim_1 U^*$. Applying the orbit structure-preserving map $\phi_{1,-a}$ defined in the proof of Theorem 3.6, we see that $\phi_{1,-a}(a) = 0$, and hence $\mathbb{T}_{U^*}^m(0) = 0$. The result then follows because \sim_1 is an equivalence relation. ■

Theorem 3.7. *Suppose that for a given U the minimum orbit length occurring under \mathbb{T}_U is m and that for any $V \in R_N$ the minimum orbit length under \mathbb{T}_V divides all other orbit lengths occurring under \mathbb{T}_V . Then, for any $V \in R_N$, $\Sigma(\mathbb{T}_U) = \Sigma(\mathbb{T}_V)$ if and only if m is the minimum orbit length occurring under \mathbb{T}_V .*

Proof. By Lemma 3.14 we know that there is some $U^* \in R_N$ such that $U \sim_1 U^*$ where $\mathbb{T}_{U^*}^m(0) = 0$, and hence \mathbb{T}_U and \mathbb{T}_{U^*} are QDS by Theorem 3.6. If V is such that the minimum orbit length under \mathbb{T}_V is m , we define a map

$$\begin{aligned} \chi_a &: \text{Att}(\mathbb{T}_{U^*}) \longrightarrow R_N \\ &b \mapsto a + b, \end{aligned}$$

where $\mathbb{T}_V^m(a) = a$. Now, for all $b \in \text{Att}(\mathbb{T}_{U^*})$,

$$\chi_a(\mathbb{T}_{U^*}^m(b)) = \mathbb{T}^m b + \mathbb{T}_{U^*}^m(0) + a = \mathbb{T}^m b + \mathbb{T}_V^m(a) = \mathbb{T}_V^m(\chi_a(b))$$

and hence for any integer $n > 0$, one has

$$\chi_a(\mathbb{T}_{U^*}^{nm}(b)) = \mathbb{T}_V^{nm}(\chi_a(b)).$$

Note that in general we do not have $\chi_a \circ \mathbb{T}_{U^*} = \mathbb{T}_V \circ \chi_a$. It follows, because χ_a is clearly a bijection, that $\text{Im } \chi_a = \text{Att}(\mathbb{T}_V)$ and that χ_a maps points on prime period nm orbits under \mathbb{T}_{U^*} onto points on period nm orbits under \mathbb{T}_V . Suppose that, under \mathbb{T}_{U^*} , b is on an orbit of length nm . We show that $\chi_a(b)$ is on an orbit of length nm , for under the conditions of the theorem and by the preceding remarks, $\chi_a(b)$ must be on an orbit of length $n'm$ for some integer $n' \leq n$, but if $n' < n$, then

$$\mathbb{T}_V^{n'm}(\chi_a(b)) = \chi_a(b) \Rightarrow \chi_a(\mathbb{T}_{U^*}^{n'm}(b)) = \chi_a(b) \Rightarrow \mathbb{T}_{U^*}^{n'm}(b) = b$$

contradicting the minimality of nm as the orbit length of b under \mathbb{T}_{U^*} . Because χ_a is a bijection and maps points on orbits of length nm under \mathbb{T}_{U^*} to points on orbits of length nm under \mathbb{T}_V , it is clear that $\Sigma(\mathbb{T}_{U^*}) = \Sigma(\mathbb{T}_V)$ and hence that $\Sigma(\mathbb{T}_U) = \Sigma(\mathbb{T}_V)$.

If the minimum orbit length occurring under \mathbb{T}_V is not m , then clearly \mathbb{T}_U and \mathbb{T}_V are not QDS. ■

The condition that for any $V \in R_N$ the minimum orbit length under \mathbb{T}_V divides all other orbit lengths occurring under \mathbb{T}_V seems somewhat restrictive at first sight. However, it turns out that it often holds. For instance, we show in [19] and [20] that it is always the case when R is a finite field or $R = \mathbb{Z}/m\mathbb{Z}$ for any integer $m > 1$. Note that Lemma 3.14 and Theorem 3.7 hold for null boundary conditions.

The notion of qualitative dynamical similarity can be extended with the following definition.

Definition 3.2. We shall say that the rules \mathbb{T} and \mathbb{S} are QDS on N cells if there is a bijection $\omega : R_N \rightarrow R_N$ such that for each $U \in R_N$, \mathbb{T}_U and $\mathbb{S}_{\omega(U)}$ have identical cycle sets.

We make the same definition (with the obvious alterations) for null boundary conditions. The remaining results in this section concern qualitative dynamical similarity of rules.

Lemma 3.15. *If $\mathbb{T} \in U(R_N)$, then \mathbb{T} and \mathbb{T}^{-1} are QDS, with*

$$\omega(U) = -\mathbb{T}^{-1}U. \tag{3.31}$$

Lemma 3.16. *If $\mathbb{T}^{\Pi_N(\mathbb{T})+1} = \mathbb{T}$, then, defining $\mathbb{T}^{\Pi_N(\mathbb{T})-1}$ to be the rule whose action is represented by $(\mathbb{T})^{\Pi_N(\mathbb{T})-1}$, \mathbb{T} and $\mathbb{T}^{\Pi_N(\mathbb{T})-1}$ are QDS, with*

$$\omega(U) = \mathbb{T}_U^{\Pi_N(\mathbb{T})-1}(0). \tag{3.32}$$

Note that if \mathbb{T} is invertible, then $\mathbb{T}^{\Pi_N-1} = \mathbb{T}^{-1}$. Lemmas 3.15 and 3.16 hold for null boundary conditions also.

The previous two cases depend on N in the sense that for local rules f and g , with f giving global rules \mathbb{T}_N and \mathbb{T}_{N+1} on N and $N + 1$ cells and g giving global rules \mathbb{S}_N and \mathbb{S}_{N+1} , the proposition that \mathbb{T}_N and \mathbb{S}_N are QDS

does not imply that \mathbb{T}_{N+1} and \mathbb{S}_{N+1} are QDS. The next case is independent of N .

Suppose an additive CA rule has local rule

$$f(a_{i-r}, \dots, a_i, \dots, a_{i+r}) = \alpha_{-r}a_{i-r} + \dots + \alpha_0a_i + \dots + \alpha_ra_{i+r}. \tag{3.33}$$

If $\alpha_{-j} = \alpha_j$, $j = 1, \dots, r$, we say that f is reflection-symmetric (RS). One can consider the reflected rule

$$\hat{f}(a_{i-r}, \dots, a_i, \dots, a_{i+r}) = \alpha_ra_{i-r} + \dots + \alpha_0a_i + \dots + \alpha_{-r}a_{i+r}. \tag{3.34}$$

If f is RS, then clearly $f = \hat{f}$, and at the global level $\mathbb{T} = \hat{\mathbb{T}}$. If f is not RS, then, with \mathbb{T} written as

$$\mathbb{T} = \alpha_0 + \sum_{i=1}^{N-1} \beta_i x^i + (x^N - 1)R[x], \tag{3.35}$$

the action of $\hat{\mathbb{T}}$ is given by

$$\hat{\mathbb{T}} = \alpha_0 + \sum_{i=1}^{N-1} \beta_i x^{N-i} + (x^N - 1)R[x]. \tag{3.36}$$

For a control $U = u_0 + u_1x + \dots + u_{N-1}x^{N-1} + (x^N - 1)R[x]$, define \tilde{U} to be

$$\tilde{U} = u_0x^{N-1} + \dots + u_{N-2}x + u_{N-1} + (x^N - 1)R[x]. \tag{3.37}$$

Lemma 3.17. \mathbb{T} and $\hat{\mathbb{T}}$ are qualitatively dynamically similar with

$$\omega(U) = \tilde{U}. \tag{3.38}$$

For null boundary conditions the above holds if $\tilde{\mathbb{U}}$ is obtained from \mathbb{U} by reversing the order of entries and if

$$\mathbb{T} = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0N-1} \\ a_{10} & a_{11} & \dots & a_{1N-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N-10} & a_{N-11} & \dots & a_{N-1N-1} \end{pmatrix} \tag{3.39}$$

then

$$\hat{\mathbb{T}} = \begin{pmatrix} a_{N-1N-1} & \dots & a_{N-11} & a_{N-10} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1N-1} & \dots & a_{11} & a_{10} \\ a_{0N-1} & \dots & a_{01} & a_{00} \end{pmatrix}. \tag{3.40}$$

4. Periodic Inputs

In this section we consider nonconstant periodic or eventually periodic inputs. These inputs will be represented as

$$U(t) = U(t, x) + (x^N - 1)R[x]. \tag{4.1}$$

The obvious analogous definitions are made for null boundary conditions, and results in that case hold with the same qualifications (if any) as stated in section 3.

We show that results in this case can be obtained from those for time-independent inputs. An interesting case of periodic inputs is that of “semi-coupled” CA, where the input to an additive CA is supplied by one or more CA running independently.

We make the following definition.

Definition 4.1. The sequence of inputs $\{U(1), U(2), \dots\}$ is eventually periodic with period $P_{U(t)} > 0$ if there is an integer $T_{U(t)} \geq 0$ (the transient time), chosen to be minimal, such that

$$U(T_{U(t)} + k + P_{U(t)}) = U(T_{U(t)} + k), \text{ all } 0 < k \leq P_{U(t)}. \tag{4.2}$$

The sequence is periodic with period $P_{U(t)}$ if $T_{U(t)} = 0$.

We shall denote the global mapping corresponding to a rule \mathbb{T} acting with control sequence $U(t)$ as input by $\mathbb{T}_{U(t)}$. The k th iterate will be written, for any $a \in R_N$, as

$$\begin{aligned} \mathbb{T}_{U(t)}^k(a) &= \mathbb{T}_{U(t)}(\mathbb{T}_{U(t-1)}(\dots(\mathbb{T}_{U(1)}(a))\dots)) \\ &= \mathbb{T}^k a + \sum_{i=1}^k \mathbb{T}^{k-i} U(i). \end{aligned}$$

We can now examine the reoccurrence of states of the system. For time-independent inputs it is no longer sufficient to say that $a \in R_N$ is periodic with period k if $\mathbb{T}_{U(t)}^k(a) = a$.

Definition 4.2. Say $a \in R_N$ is eventually periodic under $\mathbb{T}_{U(t)}$ with period $P(U(t), a)$, if there is some integer $n > 0$ such that for each $k, 1 \leq k \leq P(U(t), a)$, we have

$$\mathbb{T}_{U(t)}^{n+k+P(U(t), a)}(a) = \mathbb{T}_{U(t)}^{n+k}(a). \tag{4.3}$$

When $n = 0$, a is said to be periodic.

Lemma 4.1. For any global rule \mathbb{T} and periodic input $U(t)$

$$P_{U(t)} \mid P(U(t), a) \tag{4.4}$$

for each $a \in R_N$.

Corollary 4.1. *For any global rule \mathbb{T} and periodic input $U(t)$ there are no fixed points unless $P_{U(t)} = 1$.*

Consider first the case in which $U(t)$ is periodic; that is, $T_{U(t)} = 0$. Consider a cycle C of length $nP_{U(t)}$ for some integer $n > 0$:

$$C = \{a, \mathbb{T}_{U(t)}(a), \dots, \mathbb{T}_{U(t)}^{nP_{U(t)}-1}(a)\}.$$

Points on such a cycle fall into two classes: those occurring at times $t \equiv 0$ modulo $P_{U(t)}$ and those occurring at times $t \not\equiv 0$ modulo $P_{U(t)}$. Those in the first class behave as one would normally expect a periodic point to do, that is, if $b = \mathbb{T}_{U(t)}^{P_{U(t)}}(a)$, then if b is taken as initial condition, it evolves to the same cycle C . However, those in the second class are different; their presence on C tells us nothing about their behavior when taken as initial conditions. We shall refer to points of the first type as primary periodic points and those of the second type as secondary periodic points.

Example 3. Let $R = \mathbb{F}_2$, $N = 3$, and let $\mathbb{T} = 1 + x^2 + (x^3 - 1)\mathbb{F}_2[x]$. Let $U(t) = (t + 1 \bmod 2)x + (x^3 - 1)\mathbb{F}_2[x]$ (so $U(2n + 1) = 0$, $U(2n) = 1$ for all positive integers n). Then $P_{U(t)} = 2$. We shall omit the “ $+(x^3 - 1)\mathbb{F}_2[x]$ ” for the sake of brevity. With initial condition x^2 one obtains the cycle

$$x^2 \rightarrow 1 + x^2 \rightarrow x^2.$$

However, with $1 + x^2$ as initial condition, the system evolves to the cycle

$$1 \rightarrow 1 + x \rightarrow 1 + x + x^2 \rightarrow 0 \rightarrow x \rightarrow x + x^2 \rightarrow 1$$

where the primary periodic points are $1, 1 + x + x^2, x$.

We note that in this case, as $P_{U(t)}|P(U(t), a)$, the qualitative behavior under $\mathbb{T}_{U(t)}$ is completely determined by the behavior of the primary periodic points.

When $T_{U(t)} > 0$, we define the primary periodic points as those elements of R_N that occur on cycles under $\mathbb{T}_{U(t)}$ at times $t = T_{U(t)} + LP_{U(t)}$, $L \geq 0$. It is no longer true that if b is a primary periodic point, then b or other primary periodic points on the same cycle will evolve to that cycle if taken as initial conditions, as the following example shows.

Example 4. Let $R = \mathbb{F}_2$, $N = 3$, $\mathbb{T} = 1 + x + (x^3 - 1)\mathbb{F}_2[x]$. Let $U(t)$ be defined (omitting the “ $+(x^3 - 1)\mathbb{F}_2[x]$ ” for brevity) by $U(1) = 0$, $U(2) = 1$, $U(3) = x$, $U(4 + 2i) = 1 + x$, and $U(5 + 2i) = x^2$ for each $i \geq 0$. Then $T_{U(t)} = 3$ and $P_{U(t)} = 2$. With initial condition 0 one obtains, after three time steps, the cycle

$$1 \rightarrow 0 \rightarrow x^2 \rightarrow x + x^2 \rightarrow 1 + x + x^2 \rightarrow 1 + x \rightarrow 1$$

with primary periodic points $1, x^2$ and $1 + x + x^2$. If we use x^2 as an initial condition, we do not regain the above cycle; instead the system evolves, again after three time steps, to the cycle

$$x \rightarrow 1 + x^2 \rightarrow x.$$

Despite the preceding example, the primary periodic points are still important when $T_{U(t)} > 0$.

The next lemma relates the dynamics under $\mathbb{T}_{U(t)}$ to that of the CA rule on N cells represented by $\mathbb{T}^{P_{U(t)}}$ with a constant control. Its proof is a simple verification.

Lemma 4.2. *For the additive CA rule over R represented by \mathbb{T} on N cells, let $U(t)$ be a periodic input sequence, and let*

$$W(U) = \mathbb{T}^{P_{U(t)}-1}U(1) + \dots + \mathbb{T}U(P_{U(t)} - 1) + U(P_{U(t)}). \tag{4.5}$$

Then, for each $a \in R_N$ and each integer $n > 0$,

$$\mathbb{T}_{U(t)}^{nP_{U(t)}}(a) = (\mathbb{T}^{P_{U(t)}})_{W(U)}^n(a). \tag{4.6}$$

More generally, let $U(t)$ be eventually periodic with transient time $T_{U(t)}$, and let

$$W(U) = \mathbb{T}^{P_{U(t)}-1}U(T_{U(t)} + 1) + \mathbb{T}^{P_{U(t)}-2}U(T_{U(t)} + 2) + \dots + \mathbb{T}U(T_{U(t)} + P_{U(t)} - 1) + U(T_{U(t)} + P_{U(t)}) \tag{4.7}$$

and let

$$S(U) = \mathbb{T}^{T_{U(t)}-1}U(1) + \dots + \mathbb{T}U(T_{U(t)} - 1) + U(T_{U(t)}), \tag{4.8}$$

then

$$\mathbb{T}_{U(t)}^{T_{U(t)}+nP_{U(t)}}(a) = (\mathbb{T}^{P_{U(t)}})_{W(U)}^n(\mathbb{T}^{T_{U(t)}}a + S(U)), \tag{4.9}$$

for each $a \in R_N$ and all integers $n > 0$.

Corollary 4.2. *When $T_{U(t)} = 0$, $a \in R_N$ is a primary periodic point of prime period $nP_{U(t)}$ under $\mathbb{T}_{U(t)}$ if and only if a is on a cycle of length n under $(\mathbb{T}^{P_{U(t)}})_{W(U)}$.*

Thus the set of primary periodic points under $\mathbb{T}_{U(t)}$ when $T_{U(t)} = 0$ is equal to $Att((\mathbb{T}^{P_{U(t)}})_{W(U)})$. It is evident from the proof of the next theorem that the same is true when $T_{U(t)} > 0$. The following theorem relates the cycle set of $\mathbb{T}_{U(t)}$ to that of $(\mathbb{T}^{P_{U(t)}})_{W(U)}$.

Theorem 4.1. *If*

$$\Sigma((\mathbb{T}^{P_{U(t)}})_{W(U)}) = \sum_{i=1}^k n_i [m_i], \tag{4.10}$$

then

$$\Sigma(\mathbb{T}_{U(t)}) = \sum_{i=1}^k n_i [m_i P_{U(t)}]. \tag{4.11}$$

Proof. When $T_{U(t)} = 0$, the proof is immediate from Corollary 4.2.

For $T_{U(t)} > 0$ it is sufficient to prove that there is some positive integer K such that

$$\mathbb{T}_{U(t)}^{T_{U(t)}+KP_{U(t)}}(R_N) = \text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)}),$$

because then all primary periodic points will be in $\text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)})$.

We note first that $\text{Att}(\mathbb{T}^L) = \text{Att}(\mathbb{T})$ for any $L > 0$ and that

$$\text{Att}((\mathbb{T}^{T_{U(t)}})_{S(U)}) \subseteq (\mathbb{T}^{T_{U(t)}})_{S(U)}(R_N) = \mathbb{T}_{U(t)}^{T_{U(t)}}(R_N). \tag{4.12}$$

Set $K = T(\mathbb{T})$, the maximum tree height that occurs under \mathbb{T} . Let $a \in R_N$; then

$$\mathbb{T}_{U(t)}^{T_{U(t)}}(a) = b \in \text{Att}((\mathbb{T}^{T_{U(t)}})_{S(U)}),$$

and by equation (4.12) every element of $\text{Att}((\mathbb{T}^{T_{U(t)}})_{S(U)})$ is $\mathbb{T}_{U(t)}^{T_{U(t)}}(a)$ for some $a \in R_N$. Then, by Theorem 3.2, $b = b' + c$, where $c \in \text{Att}(\mathbb{T}^{T_{U(t)}}) = \text{Att}(\mathbb{T})$ and $b' \in \text{Att}((\mathbb{T}^{T_{U(t)}})_{S(U)})$ (and, as c ranges over $\text{Att}(\mathbb{T})$, b ranges over $\text{Att}((\mathbb{T}^{T_{U(t)}})_{S(U)})$). Then

$$\begin{aligned} \mathbb{T}_{U(t)}^{T_{U(t)}+KP_{U(t)}}(a) &= (\mathbb{T}^{P_{U(t)}})_{W(U)}^K(b) \\ &= (\mathbb{T}^{P_{U(t)}})^K b' + (\mathbb{T}^{P_{U(t)}})^K c + (\mathbb{T}^{P_{U(t)}})_{W(U)}^K(0). \end{aligned}$$

Now, $(\mathbb{T}^{P_{U(t)}})^K c \in \text{Att}(\mathbb{T}) = \text{Att}(\mathbb{T}^{P_{U(t)}})$, and by choice of K ,

$$(\mathbb{T}^{P_{U(t)}})^K b' + (\mathbb{T}^{P_{U(t)}})_{W(U)}^K(0) \in \text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)}).$$

Hence, by Theorem 3.2,

$$\mathbb{T}_{U(t)}^{T_{U(t)}+KP_{U(t)}}(a) \in \text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)}).$$

Further, as c ranges over $\text{Att}(\mathbb{T})$, $(\mathbb{T}^{P_{U(t)}})^K c$ ranges over $\text{Att}(\mathbb{T})$ ($c \mapsto (\mathbb{T}^{P_{U(t)}})^K c$ is a bijection on $\text{Att}(\mathbb{T})$). Thus, every element of $\text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)})$ is obtained in this manner. Hence,

$$\text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)}) \subseteq \mathbb{T}_{U(t)}^{T_{U(t)}+KP_{U(t)}}(R_N).$$

It remains to consider those elements $a \in R_N$ such that

$$\mathbb{T}_{U(t)}^{T_{U(t)}}(a) = \mathbb{T}^{T_{U(t)}} a + S(U) = b \in (\mathbb{T}^{T_{U(t)}})_{S(U)}(R_N) \setminus \text{Att}((\mathbb{T}^{T_{U(t)}})_{S(U)}).$$

Then, because $KP_{U(t)} \geq T(\mathbb{T})$,

$$\mathbb{T}_{U(t)}^{T_{U(t)}+KP_{U(t)}}(a) = (\mathbb{T}^{P_{U(t)}})_{W(U)}^K(b) \in \text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)}).$$

Hence

$$\text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)}) = \mathbb{T}_{U(t)}^{T_{U(t)}+KP_{U(t)}}(R_N).$$

The result now follows as $P_{U(t)}|P(U(t), a)$. ■

Thus, the cycle set for $\mathbb{T}_{U(t)}$ can be obtained from that of $\mathbb{T}^v = (\mathbb{T}^{P_{U(t)}})_{W(U)}$, whatever the value of $T_{U(t)}$.

The full set of points lying on cycles under $\mathbb{T}_{U(t)}$, for any value of $T_{U(t)}$ is given in terms of $\text{Att}((\mathbb{T}^{P_{U(t)}})_{W(U)})$ as follows.

Lemma 4.3. *The set $Att(\mathbb{T}_{U(t)})$ of all elements of R_N lying on cycles under $\mathbb{T}_{U(t)}$ is given by*

$$Att(\mathbb{T}_{U(t)}) = Att((\mathbb{T}^{P_{U(t)}})_{W(U)}) \cup_{i=1}^{P_{U(t)}-1} \mathbb{T}_{U(t)}^i (Att((\mathbb{T}^{P_{U(t)}})_{W(U)})) \quad (4.13)$$

when $T_{U(t)} = 0$ and

$$Att(\mathbb{T}_{U(t)}) = Att((\mathbb{T}^{P_{U(t)}})_{W(U)}) \cup_{i=1}^{P_{U(t)}-1} (\mathbb{T}^i)_{V(T_{U(t)}+i)} (Att((\mathbb{T}^{P_{U(t)}})_{W(U)})) \quad (4.14)$$

otherwise, where

$$V(T_{U(t)} + i) = \sum_{j=1}^i \mathbb{T}^{i-j} U(T_{U(t)} + j). \quad (4.15)$$

As in the case of constant inputs, we can define QDS for periodic and eventually periodic inputs, provided that the periods of the inputs are the same.

Definition 4.3. Let $U(t)$ and $V(t)$ both have period $P > 1$. Then $\mathbb{T}_{U(t)}$ and $\mathbb{T}_{V(t)}$ are QDS if

$$\Sigma(\mathbb{T}_{U(t)}) = \Sigma(\mathbb{T}_{V(t)}). \quad (4.16)$$

From Theorem 4.1 we have that

$$\Sigma(\mathbb{T}_{U(t)}) = \Sigma(\mathbb{T}_{V(t)}) \Leftrightarrow \Sigma((\mathbb{T}^{P_{U(t)}})_{W(U)}) = \Sigma((\mathbb{T}^{P_{V(t)}})_{W(V)}). \quad (4.17)$$

The next result follows immediately.

Theorem 4.2. $\mathbb{T}_{U(t)}$ and $\mathbb{T}_{V(t)}$ are QDS if and only if $(\mathbb{T}^{P_{U(t)}})_{W(U)}$ and $(\mathbb{T}^{P_{V(t)}})_{W(V)}$ are QDS.

Thus, we can use the results from section 3.2 to determine whether $\mathbb{T}_{U(t)}$ and $\mathbb{T}_{V(t)}$ are QDS for time-dependent inputs $U(t)$ and $V(t)$.

Example 5. Let $N = 4$, $R = \mathbb{F}_2$, and $\mathbb{T} = 1 + x + x^2 + (x^4 - 1)\mathbb{F}_2[x]$. We omit the “ $+(x^4 - 1)\mathbb{F}_2[x]$ ” for the rest of this example. Let $U(t)$ be defined by $U(1) = 1$, $U(2i) = x$, $U(2i + 1) = x^2$ for $i \geq 1$, so $T_{U(t)} = 1$ and $P_{U(t)} = 2$. Then

$$W(U) = \mathbb{T}U(2) + U(3) = x + x^3.$$

Let $V(t)$ be defined by $V(1) = 1$, $V(2) = 1 + x$, $V(2i - 1) = x^3$, and $V(2i) = x^2$ for all $i \geq 2$. Thus $T_{V(t)} = 2$ and $P_{V(t)} = 2$. Then

$$W(V) = \mathbb{T}V(3) + V(4) = 1 + x + x^2 + x^3.$$

Now,

$$\mathbb{T}^{P_{U(t)}} = \mathbb{T}^{P_{V(t)}} = \mathbb{T}^2 = x^2$$

and one finds that

$$(\mathbb{T}^2)_{W(U)-W(V)} = (\mathbb{T}^2)_{1+x^2},$$

which has a fixed point; hence, $\mathbb{T}_{U(t)}$ and $\mathbb{T}_{V(t)}$ are QDS by Theorem 3.6 and Theorem 4.2. Moreover, one finds that $(\mathbb{T}^2)_{W(U)}$ is QDS to \mathbb{T}^2 and

$$\Sigma(\mathbb{T}^2) = 4[1] + 6[2].$$

Hence

$$\Sigma((\mathbb{T}^2)_{W(U)}) = \Sigma((\mathbb{T}^2)_{W(V)}) = 4[2] + 6[4]$$

by Theorem 4.1.

5. Generalizations

In this section we briefly consider some extensions of the results we have derived. The key to this is to note that for many of the results in sections 3 and 4, either the proofs do not rely on any specific property of the rings R and R_N (or the R -module R^N for null boundary conditions), or the only specific property used is the commutativity of R (and hence R_N). The results could be viewed as results about the iteration of linear and affine mappings of R_N to itself (or R^N to itself for null boundary conditions), and in the light of these remarks, can immediately be generalized to results on iteration of linear and affine mappings of arbitrary finite commutative rings.

The most immediate consequence of the foregoing discussion is in the case of additive CA with periodic boundary conditions in more than one dimension. It is shown in [3] that the states of an additive CA in D dimensions with cells arranged in a rectangular lattice of N_i cells in the i th direction, $1 \leq i \leq D$, with periodic boundary conditions, can be represented as polynomials in D commuting variables x_1, \dots, x_D of degree at most $N_i - 1$ in the i th variable. The global action of the rule can be represented by multiplication by a polynomial in these variables derived from the local rule and taking the result modulo $x_i^{N_i} - 1$ for each i . By the same arguments as in section 2, this is equivalent to representing the states of the CA as elements of

$$R(N_1, \dots, N_D) = \frac{R[x_1, \dots, x_D]}{(x_1^{N_1} - 1, \dots, x_D^{N_D} - 1)R[x_1, \dots, x_D]} \tag{5.1}$$

and representing the action of the global rule by multiplication by an element $\mathbb{T} \in R(N_1, \dots, N_D)$,

$$\mathbb{T} = \mathbb{T}(x_1, \dots, x_D) + (x_1^{N_1} - 1, \dots, x_D^{N_D} - 1)R[x_1, \dots, x_D]. \tag{5.2}$$

External inputs are represented in the obvious way. By the foregoing remarks results in sections 3 and 4 hold by just replacing R_N by $R(N_1, \dots, N_D)$, *etcetera*, in the proofs. The only exception is that the specific results on QDS of rules at the end of section 3.2 require modifications in higher dimensions.

Another case is that of additive CA on an infinite rectangular array of cells in one or more dimensions. We shall discuss the one-dimensional case, the generalization to higher dimensions should be obvious. The cells are indexed by \mathbb{Z} . We note that studying a CA on N cells with periodic boundary conditions is equivalent to the same CA in the infinite case but with attention restricted to those configurations consisting entirely of infinite repetitions of blocks of N cells (in [21] it is shown that a one-dimensional CA is reversible if and only if it is reversible on all such periodic configurations). Formal power series have been used for representing the states of such CA (e.g., [22]). We represent the states of the CA over R by the ring of formal Laurent series over R , $R[[x, x^{-1}]]$, using the identification

$$(\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots) \mapsto \sum_{i=-\infty}^{i=\infty} a_i x^i. \tag{5.3}$$

The global action of the CA rule is given by multiplication by a Laurent polynomial derived from the local rule (referred to as a dipolynomial in [3]). In general, if the local rule is

$$f(a_{i-r}, \dots, a_{i+r}) = \sum_{j=-r}^r \alpha_j a_{i+j}$$

then the Laurent polynomial is

$$\mathbb{T}(x, x^{-1}) = \sum_{j=-r}^r \alpha_j x^{-j}, \tag{5.4}$$

and the global action is represented by the iteration of the mapping

$$\begin{aligned} \mathbb{T} : R[[x, x^{-1}]] &\longrightarrow R[[x, x^{-1}]], \\ \mathbb{T}(a(x, x^{-1})) &= \mathbb{T}(x, x^{-1})a(x, x^{-1}) \end{aligned} \tag{5.5}$$

for each $a(x, x^{-1}) \in R[[x, x^{-1}]]$.

Define $Att(\mathbb{T})$ to be the set of cycles of finite length under such a CA, and define $T_0(\mathbb{T}, r)$, $r \in Att(\mathbb{T})$ to be the set of elements of $R[[x, x^{-1}]]$ evolving to r in finite time. With these definitions, for instance, Theorem 3.1 still holds, and $Att(\mathbb{T})$ and $T_0(\mathbb{T}, 0)$ are ideals of $R[[x, x^{-1}]]$. Indeed, one can show here that $Att(\mathbb{T}_U) = a(x, x^{-1}) + Att(\mathbb{T})$ where $a(x, x^{-1})$ is any element of $Att(\mathbb{T}_U)$, though the proof is different from that in the finite case.

One may also consider hybrids of additive CA for either null or periodic boundary conditions. In either case we shall represent such hybrids by matrices over R acting on the R -module R^N . Let \mathbf{T} be the matrix representing the hybrid CA with local rule f_i acting at site i ,

$$f_i(a_{i-r}, \dots, a_{i+r}) = \sum_{j=-r}^r \alpha_{i,j} a_{i+j} \tag{5.6}$$

then \mathbf{T} is defined by

$$\alpha_{i,j} = \{\mathbf{T}\}_{i(i+j)} \text{ Mod}(N) \tag{5.7}$$

and all other entries zero for periodic boundary conditions, and by

$$\alpha_{i,j} = \{\mathbf{T}\}_{i(i+j)} \quad (5.8)$$

whenever $0 \leq i+j \leq N-1$ and zero otherwise for null boundary conditions.

Introducing a constant input into this system can be viewed as increasing the number of possible rules involved in the hybrid. Clearly all results that hold for null boundary conditions in sections 3 and 4 hold for these systems, as no special properties of the matrix representing the rule for null boundary conditions are used in proving their results.

6. Discussion

We have presented an analysis of a class of finite CA in the presence of external inputs. We have shown that knowledge of the system without inputs gives insight into the behavior of the system with constant inputs and, in turn, that knowledge of these systems in the presence of constant inputs gives insight into their behavior under periodic inputs. We have introduced a notion of qualitative dynamical similarity (QDS) for these systems, and sufficient conditions for this to occur are found in terms of an equivalence class structure on the set of possible inputs for a given state alphabet and number of cells. A necessary and sufficient condition for QDS is given, provided another condition is satisfied. The dynamical behavior of the input when considered as a state acted on by the uncontrolled rule is shown to be important in the study of the controlled system. The results are presented for a general class of state alphabets, and, although this paper concentrates on one-dimensional finite CA, we show how the results may be extended in several ways.

In the case of periodic boundary conditions the results were obtained using an algebraic representation that is a formalization of that introduced in [3]. This formalization, in addition to being notationally more elegant (in the opinion of these authors) than the original, has the advantage of emphasizing algebraic structure and the relation between this structure and the dynamics. Another advantage is that all operations take place within the ring R_N . This advantage is further exploited for some specific classes of state alphabet in [19] and [20] to determine the behavior of these systems completely.

Another property of this representation is that it brings out an analogy between the iteration of one-dimensional real, linear dynamical systems and linear CA. Indeed, if in equations (2.10) and (2.12), one changes R_N to \mathbb{R} , we obtain such a system. However, despite this similarity behavior, the CA case is quite different.

These authors believe that the study of CA, as the simplest examples of extended systems, is interesting in its own right. Also, real-world systems do not exist in isolation, so if one is interested, however distantly, in applications of extended systems, it is important to study the behavior of such systems in the presence of external input. We also believe that algebraic techniques,

though primarily developed for linear systems (but see the recent works [23] and [24] relating the properties of certain nonlinear CA to certain linear CA, thus providing a nonlinear application of the algebraic methods used to study linear CA), will be important in the study of CA in general.

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