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A Symbolic Way to Express Paths and Orbits of Two Iterated Function Systems

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Abstract. Two iterated function systems, resulting in the attractors generally known as the Sierpinski triangle and the von Koch snow flake curve, are studied. The complete attractor is considered to be at the level of precision $\mu = 0$; it is divided into smaller replicas ($\mu = 1$), each of which in turn are considered to be composed of smaller replicas ($\mu = 2$), and so on up to higher levels of precision. An algorithm is designed to represent paths at any (finite) level μ and a very simple procedure is found to simulate random orbits. From these orbits periods may be found. A statistical study of periods shows that, as the level of precision tends to infinity, the mean period also tends to infinity

1. Sierpinski triangle, the addresses of its points and their paths

We study the attractor of the iterated function system

$$\begin{aligned} w_1(z) &= i + Rz \\ w_2(z) &= -0.5(\sqrt{3}+i) + 0.5R(-1+i\sqrt{3})z \\ w_3(z) &= 0.5(\sqrt{3}-i) - 0.5R(1+i\sqrt{3})z \end{aligned} \qquad (1)$$

which is a Sierpinski triangle [1, 2]. In order to have a totally disconnected attractor we shall consider the constant R to be less than 0.5. Each of the equations of the iterated function system has the same opportunity to be chosen during an iteration n.

The main feature of these notes lies in the fact that the Sierpinski triangle is studied at different levels, μ, of precision The coarser level, defined by $\mu = 1$, indicates that the iterated points land inside three triangles, which are replicas of the complete attractor. These three triangles are numbered 0, 1, and 2, counterclockwise, as shown in Figure 1.

At a finer level of precision, $\mu = 2$, each of the previous triangles is divided into three smaller triangles, numbered in the same fashion, and so on, to higher levels of precision. At a specific level μ, a total number of 3^μ triangles may be defined.

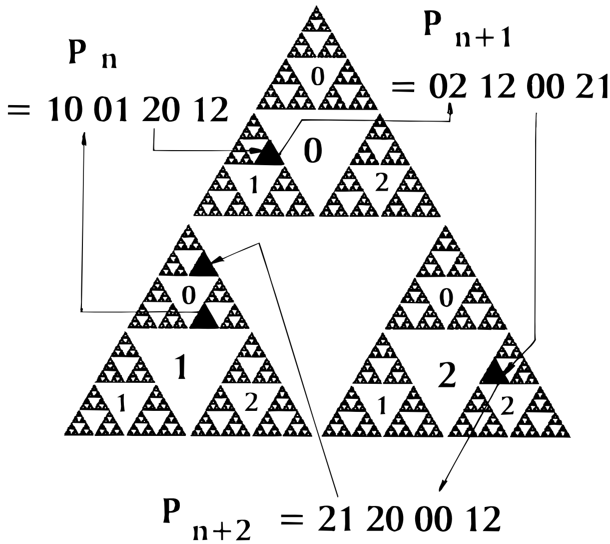

Figure 1: Sierpinski triangle obtained with IFS (1). The whole attractor is assigned a level $\mu = 0$; the three larger triangles, assumed to belong to level $\mu = 1$, are numbered with the larger digits; smaller digits indicate the region where points lie if the level of precision is $\mu = 2$. An orbit of three paths (P_n, P_{n+1}, and P_{n+2}), for $\mu = 4$, is shown. The address where P_n starts is 1021, and it ends at 0102; P_{n+1} ends at address 2201 and the final destination of P_{n+2} is 1002.

When a particular level μ is chosen, the address of a point may be given by a string of μ numbers. For example, the address $a = 1021$ (see Figure 1) indicates that the point belongs to replica 1 of level $\mu = 1$ (first digit of the string); with a further refinement, we may say it is placed at replica 0 of $\mu = 2$ (second number); it is also inside replica 2 of $\mu = 3$ (third number); and finally, the fourth digit of the address identifies replica 1 of level $\mu = 4$.

A path performed by a point during two successive iterations is defined by the address of the point at iteration n and by the address reached by the point at iteration $n + 1$. A path, P_n, may be given by a string of 2μ digits such as

$$P_n = a_1 b_1 \; a_2 b_2 \ldots a_i b_i \ldots a_\mu b_\mu, \tag{2}$$

where a_i is the address at iteration n and b_i is the address for iteration $n+1$. Thus, as an example (see Figure 1), the path $P_1 = 10\ 01\ 20\ 12$ goes from address $a = a_1 a_2 a_3 a_4 = 1021$ at iteration n, to address $b = b_1 b_2 b_3 b_4 = 0102$ at iteration $n + 1$.

Through the transformation (into a ternary system),

$$\alpha_i = a_i 3^1 + b_i 3^0 \tag{3}$$

equation (2) may be simplified to yield the word for a path,

$$PP_n = \alpha_1\alpha_2\alpha_3 \ldots \alpha_i \ldots \alpha_\mu. \tag{4}$$

Equations (2) and (4) are equivalent words for specifying the same path; the first is a 2μ-digit string, while the latter has only μ digits

In general, the number of paths is given by

$$3^{2\mu}. \tag{5}$$

A very careful numerical study has been performed by the author in order to confirm the existence of all such paths up to a level of $\mu = 3$; some of the paths have been numerically confirmed for higher levels of precision. In the following section we develop an algorithm to express the string of paths for any value of μ.

2. An algorithm to express paths of the Sierpinski triangle

Let us define the following vector:

$$\begin{vmatrix} A(1,1) & A(1,2) & A(1,3) \\ A(2,1) & A(2,2) & A(2,3) \\ A(3,1) & A(3,2) & A(3,3) \end{vmatrix} = \begin{vmatrix} 00 & 01 & 02 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}. \tag{6}$$

The first matrix with numerical values shall be used to express paths in the manner of equation (2) and the second matrix to paths defined by equation (4). Note that each element A of the vector is related to addresses by

$$A(L, M) = a_L b_M = a_L 3^1 + b_M 3^0 \tag{7}$$

where $a_L = L - 1$ and $b_M = M - 1$.

The second (and last) definition is the cyclic vector

$$\begin{vmatrix} B(1,1) & B(1,2) & B(1,3) \\ B(2,1) & B(2,2) & B(2,3) \\ B(3,1) & B(3,2) & B(3,3) \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}. \tag{8}$$

With the aid of equations (6) and (8) a path at any finite level μ may be identified by

$$P_n = 10^{2(\mu-1)} A(J_{1,n}, J_{2,n}) + \sum_{m=1}^{\mu-1} 10^{2(\mu-1-m)} A[J_{m+2,n}, B(J_m, J_{m+1})n] \tag{9}$$

If the indexes $J_{1,n}$, $J_{2,n}$, and $J_{m+2,n}$ are randomly chosen between 1, 2, or 3, then a particular path is selected; if the indexes are given the values 1, 2, and 3, then equation (9) expresses all paths belonging to level μ.

As an example with $\mu = 4$, if we choose

$$J_{1,n} = 2, \quad J_{2,n} = 1, \quad J_{3,n} = 1, \quad J_{4,n} = 3, \quad \text{and} \quad J_{5,n} = 2,$$

the resulting path is, from equations (6) through (9),

$$P_n = 1000000 \times 10 + 10000 \times 01 + 100 \times 20 + 12 = 10\ 01\ 20\ 12$$

or

$$PP_n = 3165.$$

This example is represented in Figure 1; it is the path which goes from address $a = 1021$ to address $b = 0102$.

2.1 Orbits

The successive addresses of a point (as defined in previous paragraphs, for particular levels μ) during iterations n, $n+1$, $n+2$, $n+3\ldots$ form an orbit. The specific iterated function system (IFS) now studied yields only three new possible paths during iteration $n+1$, which may follow a given path performed during iteration n.

Algorithm (9) provides one path, but a similar algorithm may be used to calculate one of the three aforementioned possible paths which may follow. This is achieved by a simple shifting of the elements of P_n one place to the left, in order to start at the address where P_n has arrived at during iteration n. Thus, a new path, P_{n+1}, following P_n, shall be found with,

$$\begin{aligned} P_{n+1} &= 10^{2(\mu-1)} A(J_{1,n+1}, J_{2,n+1}) \\ &\quad + \sum_{m=1}^{\mu-1} 10^{2(\mu-1-m)} A[J_{m+2,n+1}, B(J_m, J_{m+1})n + 1] \end{aligned} \qquad (10)$$

where

$$\begin{aligned} J_{1,n+1} &= J_{2,n}, \\ J_{2,n+1} &= 1,\ 2,\ \text{or } 3 \text{ (chosen at random)}, \end{aligned}$$

and

$$J_{m+2,n+1} = B(J_m, J_{m+1})n.$$

The values of the cyclic vector $B(J_m, J_{m+1})n$ of the last of equations (10), used to calculate the new $J_{m+2,n+1}$, belong to the previous path, P_n.

As an example for the application of equation (10), we would like to determine one of the paths following $P_n = 10\ 01\ 20\ 12$, given in a previous example. If $J_{2,n+1} = 3$ is (randomly) chosen, then from equation (10),

$$\begin{aligned} J_{1,n+1} &= J_{2,n} = 1, \\ J_{2,n+1} &= 3, \\ J_{3,n+1} &= B(J_1, J_2)n = B(2,1) = 2, \\ J_{4,n+1} &= B(J_2, J_3)n = B(1,1) = 1, \\ J_{5,n+1} &= B(J_3, J_4)n = B(1,3) = 3, \end{aligned}$$

and

$$P_{n+1} = 1000000 \times 02 + 10000 \times 12 + 100 \times 00 + 21 = 02\ 12\ 00\ 21$$

or

$$PP_{n+1} = 2507.$$

(Note that we apply the rule: $100 \times 00 = 00\ 00$.)

The other two possible paths are

$$P_{n+1} = 00\ 10\ 01\ 21, \quad \text{or} \quad PP_{n+1} = 0317 \quad (\text{for } J_{2,n+1} = 1)$$

and

$$P_{n+1} = 01\ 11\ 02\ 21, \quad \text{or} \quad PP_{n+1} = 1427 \quad (\text{for } J_{2,n+1} = 2).$$

It may be seen that the algorithm is austere and specially suited for computation due to the few arithmetic operations needed.

2.2 Recurrent periods

Figure 1 shows an orbit composed of three paths at level $\mu = 4$. It simulates a point lying at the address 1021 at the first iteration n; then it follows a jump to 0102, then a step to 2201, and finally reaches the address 1002. If iterations were to be continued, the point may eventually reach the same address where it started from during a particular orbit. The number of iterations performed in order to repeat an address shall be called recurrent period, and denoted with T. (When μ tends to infinity the recurrent period would tend to a true period.)

As mentioned in previous paragraphs, for a finite level μ there is a finite number of paths. If the orbit is made up with equations (9) and (10), then each of the paths would appear along the orbit with different recurrent periods, in the range

$$T_{\min} \leq T \leq T_{\max}. \tag{11}$$

The value of the minimum recurrent period is precisely unity, thus denoting that during an orbit a point may land in the same address during two successive iterations; this would be the case when one of the equations of IFS (1) is (randomly) chosen twice and the replica of order μ lies in the vicinity of a fixed point. On the order hand, the maximum recurrent period (large but finite) is strongly dependent upon level μ.

Figure 2 shows the results of a typical statistical study of the frequency distribution of recurrent periods for a particular level μ. It is a frequency distribution, $f(T)$, of the Poisson type

$$f(T) = \lambda \exp(-\lambda T) \tag{12}$$

where $\lambda = 1/T_{\text{mean}}$, and T_{mean} is the mean recurrent period.

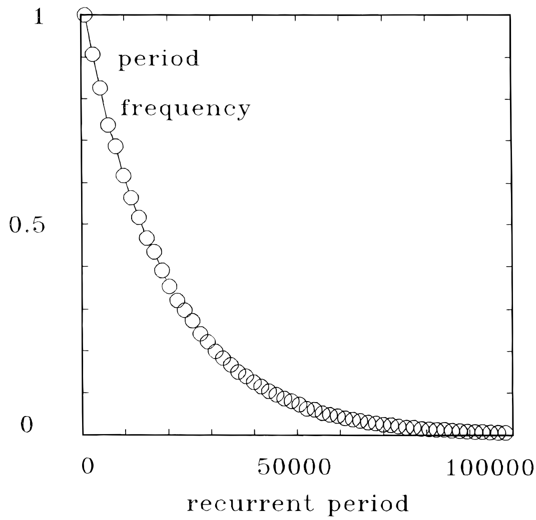

Figure 2: Orbits of two million paths were numerically generated with the algorithm (10) for $\mu > 8$, one million paths for the range $6 \leq \mu \leq 8$, and one-half million for $\mu < 6$. A recurrent period T is defined as the number of iterations necessary to repeat an address. A statistical study for these periods allows the determination of a mean recurrent period T_{mean}, and a maximum mean recurrent period T_{max}. For this level $\mu = 8$ and this particular experiment, $T_{\text{mean}} = 19210$ and $T_{\text{max}} = 226178$. Frequency distribution curves are similar to this one for all levels μ; it may be seen that the most frequent periods are the smaller ones.

Numerical experiments were performed with orbits composed of one-half million paths for $2 \leq \mu \leq 5$, with one million paths for $6 \leq \mu \leq 8$, and with two million paths for $8 < \mu \leq 10$. Higher levels μ, or longer orbits, were not studied due to computational limitations.

Figure 3 shows the results of several numerical experiments trying to describe the behavior of the mean recurrent period as a function of level μ. Its dependence upon μ may be expressed with a function of the type

$$T_{\text{mean}} = 3.53 \exp(1.07\mu) \qquad (r^2 = 0.9998). \tag{13}$$

Equation (13) gives an approximation on how fast the mean recurrent period tends to infinity as μ does so, thus suggesting that the studied IFS has no period at all. This conclusion may not be a simple consequence of the random choice of one of the equations of IFS (1).

With respect to the maximum recurrent period Figure 3 also shows that it varies with a function similar to equation (13), in this case given by

$$T_{\text{max}} = 45.4 \exp(1.07\mu) \qquad (r^2 = 0.9995), \tag{14}$$

denoting that T_{max} is simply a multiple of T_{mean}.

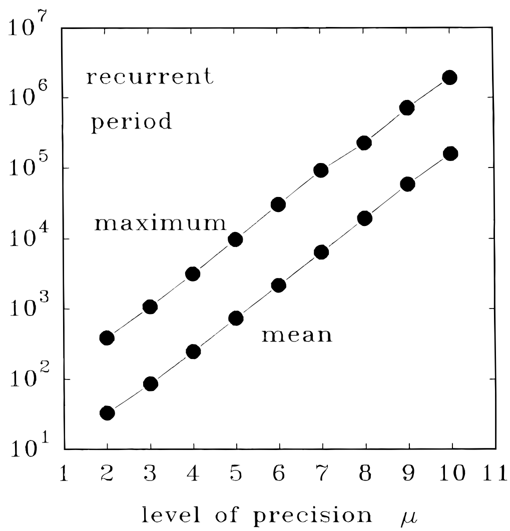

Figure 3: A summary of results for Sierpinski triangle concerning recurrent periods numerically determined with equation (10). As μ tends to infinity it may be assumed that the IFS has no period at all.

Slight variations in the constants of equations (13) and (14) may be expected if longer orbits for $\mu \geq 8$ are studied.

3. von Koch attractor and a simulation of its paths and orbits

This particular attractor may be designed through iterations over the system of equations

$$\begin{aligned} w_1(z) &= -i + R\exp(i\alpha)z, \\ w_2(z) &= -i + R\exp(-i\alpha)z, \end{aligned} \tag{15}$$

where $\alpha = 145°$ and $R = 0.6$ have been chosen for this study (see Figure 4) in order to avoid overlapping. Each of the two members of the IFS have the same opportunity to be chosen for each iteration.

As we have done with the Sierpinski triangle, we consider the whole attractor to belong to level $\mu = 0$. Then we inspect it in more detail by dividing the region into two replicas (identified with the larger digits 1 and 2 in Figure 4); they belong to level $\mu = 1$. A further refinement of the inspection is shown at level $\mu = 2$, also with numbers 1 and 2 (smaller digits in Figure 4). The number of divisions into replicas may reach any level of precision.

The observation of IFS (15), and numerical experiments, show that at level $\mu = 1$ a possible path is $P_n = 12$, meaning that a point travels from address 1 at iteration n to address 2 at iteration $n + 1$. The other three possible paths are $P_n = 21$, 22, and 11; the latter identifies points which remain in the same replica during two consecutive iterations.

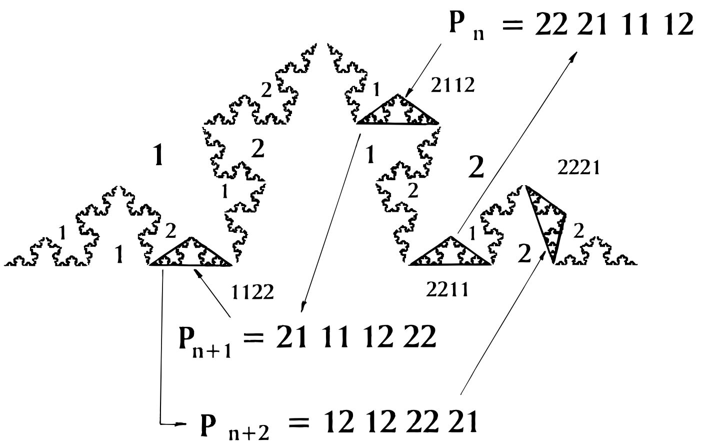

Figure 4: von Koch attractor resulting from IFS (15) The whole attractor is assumed to belong to a level of precision $\mu = 0$; if it is inspected with more detail, $\mu = 1$, two replicas, numbered 1 and 2 with the larger digits, may be identified; smaller digits (also 1 and 2) correspond to level $\mu = 2$. The digits for each level are placed approximately at the center of gravity of the replica they belong to. Three paths, obtained with equations (19) are identified with lines and arrows, together with the addresses where each path starts and ends its traveling. If an address is repeated after N iterations, then a recurrent period $T = N$ is said to exist.

A path at level μ may be expressed with the string,

$$P_n = a_1b_1\ a_2b_2\ a_3b_3 \ldots a_ib_i \ldots a_\mu b_\mu,$$

where a_i is the original position of the point and b_i is its final destination, both at level $\mu = i$ and during two successive iterations.

If we define the matrices

$$\begin{vmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{vmatrix} = \begin{vmatrix} 11 & 12 \\ 21 & 22 \end{vmatrix} \tag{16}$$

and

$$\begin{vmatrix} B(1) \\ B(2) \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \end{vmatrix}, \tag{17}$$

the word for a path at level μ is given by

$$P_n = 10^{2(\mu-1)} A(J_{1,n}, J_{2,n}) + \sum_{m=1}^{\mu-1} 10^{2(\mu-1-m)} A[J_{2m+1,n}, B(J_{2m-1})n] \tag{18}$$

where the indexes $J_{1,n}$, $J_{2,n}$, and $J_{2m+1,n}$ are chosen at random between 1 and 2, and the factors $10^{2(\mu-1)}$ and $10^{2(\mu-1-m)}$ (as it was done with the Sierpinski triangle) are designed to place a_ib_i in the correct place inside the string.

There are only two possible paths which may follow P_n; the paths are obtained by a random choice and a simple shifting of the addresses, since P_{n+1} at iteration $n+1$ should start at the address where P_n has arrived at during iteration n. The two possible paths following equation (18) are given by the set of equations

$$\begin{aligned} P_{n+1} &= 10^{2(\mu-1)} A(J_{1,n+1}, J_{2,n+1}) \\ &\quad + \sum_{m=1}^{\mu-1} 10^{2(\mu-1-m)} A[J_{2m+1,n+1}, B(J_{2m-1})n+1] \end{aligned} \tag{19}$$

where

$$\begin{aligned} J_{1,n+1} &= J_{2,n}, \\ J_{2,n+1} &= 1 \text{ or } 2 \text{ (chosen at random)}, \end{aligned}$$

and

$$J_{2m+1,n+1} = B(J_{2m-1})n.$$

The orbits in equation (19) for von Koch attractor have the same structure of the orbits in equation (10) for Sierpinski triangles.

If a computational algorithm is used to draw orbits of the von Koch attractor it is more convenient to use a binary system since it will use half the amount of digits, if compared with an orbit given by equation (19). For this purpose, the path may be given by

$$PP_n = \alpha_1\ \alpha_2\ \alpha_3 \ldots \alpha_i \ldots \alpha_\mu \tag{20}$$

where

$$\alpha_i = (\alpha_i - 1) \times 2^1 + (b_i - 1) \times 2^0. \tag{21}$$

The matrix given by equation (16) should be replaced by

$$\begin{vmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}, \tag{16.1}$$

and for the pointers, $10^{2(\mu-1)}$ and $10^{2(\mu-1-m)}$, a new pair should be used: $10^{\mu-1}$ and $10^{\mu-1-m}$.

A study of the frequency distribution of recurrent periods was performed with the algorithm (19) for increasing values of μ ($2 \leq \mu \leq 6$) and for orbits composed of 600000 paths (see Figure 5). Though the number of paths for each orbit and the range of levels μ do not seem to be large enough, the results for mean recurrent periods follow a curve given by (see Figure 6)

$$T_{\text{mean}} = 1.01 \exp(0.73\mu). \tag{22}$$

Maximum recurrent periods can be represented by the function

$$T_{\max} = 37.7 \exp(0.67\mu). \tag{23}$$

The exponential increase of periods with an increase in the level μ suggests that periods would tend to infinity when μ does so, thus denoting that IFS (15), just like IFS (1), have orbits without a period.

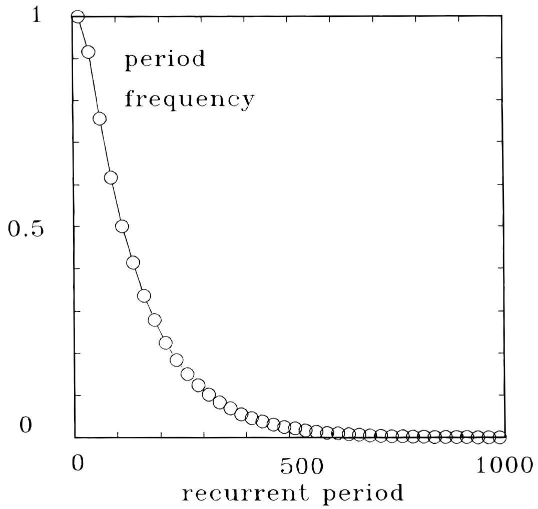

Figure 5: Frequency distribution for recurrent periods for the von Koch attractor at $\mu = 6$ with orbits of 600000 paths. For this particular level of precision the mean recurrent period is found to be $T_{\text{mean}} = 75.15$, and a maximum was found with $T_{\max} = 2013$. As with Sierpinski triangles, the most frequent periods are the smaller ones.

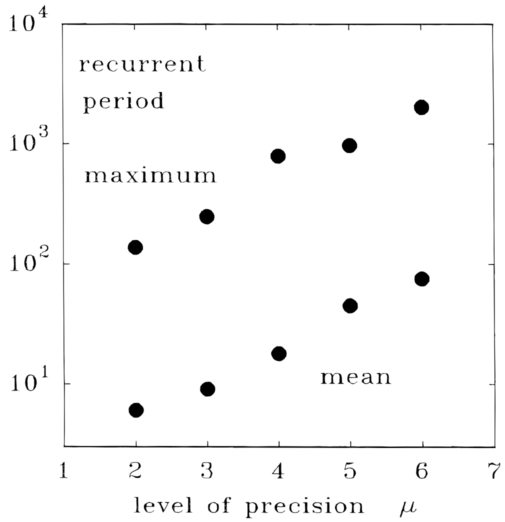

Figure 6: Summary of results for mean and maximum recurrent periods for increasing levels of precision μ, for the von Koch attractor.

4. Conclusions

An iterated function system of three equations yielding a Sierpinski triangle, and another IFS of two equations drawing a von Koch attractor (both without overlapping), are divided into replicas of decreasing size, each replica corresponding to an assigned level of precision (or depth) μ.

Since there is a finite number of paths for each level μ, two very simple algorithms were devised for the studied IFSs to express all possible existing paths for each level μ. Similar algorithms are devised to calculate orbits. The condition imposed upon these algorithms is that a new path starts at the address where the previous path has arrived at, thus simulating an iterative procedure.

A particular orbit for a specific level μ, randomly made up with the algorithms, may repeat addresses; the number of paths generated in order to repeat an address is herein called recurrent period.

An orbit thus generated shall have a number of different recurrent periods. Statistical studies show that the frequency distributions of recurrent periods are of the Poisson type, defined by a mean recurrent period, T_{mean}.

Numerical results show that the mean recurrent period increases exponentially with μ. This result should be an indirect (but strong) suggestion that the IFSs under consideration (and perhaps all IFSs, with or without overlapping) have no period at all when μ tends to infinity.

References

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[2] Mandelbrot, Benoit B., *The Fractal Geometry of Nature* (W. H. Freeman and Co., New York, 1983).