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Computation of Predecessor States for Composite Elementary Cellular Automata

Burton Voorhees
Faculty of Science, Athabasca University,
Box 10,000, Athabasca, AB
CANADA T0G 2R0

Abstract. We consider 3-site binary valued one-dimensional cellular automata rules which are the composition of two 2-site rules. For such rules an algorithmic procedure that is simpler than backward reconstruction is given allowing the computation of all predecessors of any given state. Results obtained are compared to results in [1] on the enumeration of preimages of finite strings. A question relating to Garden-of-Eden states is clarified, and an example is given to illustrate a theorem from [2] to the effect that for nonsurjective rules there exist periodic sequences having an uncountable number of predecessors.

1. Introduction

One of the more interesting questions to ask about cellular automata (CA) rules is: What are the predecessor states of any given configuration? That is, to determine the solution of

$$X(\mu) = \beta \tag{1.1}$$

where X is the global operator defined by the rule, and μ and β are elements of the configuration space.

For linear rules, a method for finding closed form solutions to equation (1.1) has been discovered [3,4,5]. Unfortunately, this method does not extend to the case of nonlinear rules, and to date no general procedure is known for the computation of predecessors under nonlinear rules, other than the lengthy method of backward reconstruction from the rule table. This method, in turn, is applicable only to finite strings, or to periodic configurations.

Progress has been made when the question is changed to that of computing the preimages of finite strings. In [1] it is shown that the number of preimages of any given string can be computed by the solution of a set of recurrence relations, and the elementary (i.e., 3-site) rules are classified into six classes based on the particular nature of the recurrence relations involved.

In [5,6] a method is given for the direct determination of all preimages of any given finite string, and an alternate derivation of the results in [1] is given.

This paper returns to consideration of the predecessor problem. Methods for the computation of predecessors are given for all 2-site rules, and it is shown how these apply to the computation of predecessors for all 3-site nonlinear rules which are composed of two 2-site rules. It turns out that these composite rules fall into three of six categories from [1], and the defining characteristics of these classes appear as a natural consequence of the predecessor formulas.

It is also easy to compute the Garden-of-Eden for these composite rules, and a distinction appears between Garden-of-Eden states which are intrinsic to a rule, and those which appear only as a result of periodic boundary conditions. It is shown that these latter configurations do have predecessors if they are viewed as infinite periodic states rather than as finite "cylindrical" states. An example is also given of a periodic configuration which has a continuum of predecessors.

In what follows two different state spaces are considered. The first corresponds to what is called cylindrical CA in [7]. It consists of all length n binary strings with periodic boundary conditions, and is denoted E_n. The second state space, denoted E^+, consists of all right half-infinite binary strings. The subset of E^+ consisting of all binary strings with period a divisor of n is isomorphic to E_n, but is treated differently in terms of the computation of predecessor states. In particular, certain Garden-of-Eden states in E_n will have predecessors of period $2n$ in E^+. Underlining a string indicates the periodic state defined by that string. In particular, $\underline{0}$ and $\underline{1}$ denote respectively the states consisting of all 0s and all 1s.

2. 2-site rules

There are 16 2-site rules. These will be labeled by a four digit binary number according to their rule table. Thus, the rule with global operator X is defined by its neighborhood components x_i from the table

00	01	10	11
x_0	x_1	x_2	x_3

and this rule is labeled by the binary number $(x_0x_1x_2x_3)$. There are eight linear rules and eight nonlinear rules. These are shown in Table 1.

The linear rules $\mathbf{1}+I$ and $\mathbf{1}+\sigma$ are toggle rules; that is, they satisfy the condition that $(\mathbf{1}+X)^2 = X^2$. This is not true for the rule $\mathbf{1}+D$, which satisfies $(\mathbf{1}+D)^2 = \mathbf{1}+D^2$. The linear rules in the left column are additive and those in the right column are not.

Predecessors for the additive linear rules are easily obtained. For the rule $\mathbf{0}$ every state is a predecessor of $\underline{0}$, and no other states have predecessors. Every state is its own predecessor under the identity rule I, while the predecessors of a state β under the left shift σ depend on whether the state space is E_n or E^+. On E_n the predecessor of β is just $\sigma^{-1}(\beta)$ where σ^{-1} is the right

Table 1: 2-site CA rules.

Linear 2-Site Rules	
$\mathbf{0} = (0000)$	$\mathbf{1} = (1111)$
$I = (0011)$ identity	$\mathbf{1} + I = (1100)$
$\sigma = (0101)$ left shift	$\mathbf{1} + \sigma = (1010)$
$D = (0110)$ binary difference	$\mathbf{1} + D = (1001)$
Nonlinear 2-Site Rules	
$Z_1 = (0100)$	$Y_1 = \mathbf{1} + Z_1 = (1011)$
$Z_2 = (0010)$	$Y_2 = \mathbf{1} + Z_2 = (1101)$
$Z_3 = (0001)$	$Y_3 = \mathbf{1} + Z_3 = (1110)$
$Y_0 = (0111)$	$Z_0 = \mathbf{1} + Y_0 = (1000)$

Table 2: $GE^*(X)$ for nonlinear 2-site rules.

X	Z_1	Z_2	Z_3	Y_0	Y_1	Y_2	Y_3	Z_0
$GE^*(X)$	$\{11\}$	$\{11\}$	$\{101\}$	$\{010\}$	$\{00\}$	$\{00\}$	$\{010\}$	$\{101\}$

shift. On E^+ the predecessor of β is given by $c1\underline{0} + \sigma^{-1}(\beta)$ where c takes on both 0 and 1 values, and $\sigma^{-1}(\beta)$ is defined on E^+ by

$$[\sigma^{-1}(\beta)]_i = \begin{cases} 0 & i = 1 \\ \beta_{i-1} & \text{otherwise.} \end{cases} \tag{2.1}$$

Predecessors for a state β under the binary difference rule D are given by [3]

$$\mu = c\underline{1} + B\sigma^{-1}(\beta) \tag{2.2}$$

where again c takes on both 0 and 1 values, and B is an "integration" operator defined on E^+ by

$$[B(\beta)]_i = \sum_{j=1}^{i} \bmod(2). \tag{2.3}$$

Predecessors under D on E_n are computed by embedding E_n into E^+. Finally, predecessors for the linear rules in the right column are computed from the identity $(\mathbf{1} + X)(\mu) = \beta \Leftrightarrow X(\mu) = \underline{1} + \beta$ where addition is sitewise mod(2).

All of the linear rules have an empty Garden-of-Eden. The Garden-of-Eden for the nonlinear rules is defined in terms of a minimal seed set $GE^*(X)$ of finite strings such that if $s_1 \dots s_n \in GE^*(X)$ then any configuration which contains $s_1 \dots s_n$ as a substring is a Garden-of-Eden configuration for the rule X. It is easy to verify that the sets $GE^*(X)$ for the nonlinear rules of Table 1 are given by the sets listed in Table 2.

If a state β contains a string from $GE^*(X)$ then it has no predecessors under rule X. If this is not the case, then all predecessors can be explicitly constructed. Using notation from [1], let a_i be the number of 0s in the ith

0-block of β and let b_i be the number of 1s in the ith 1-block of β, counting from left to right (in [1] the counts are from right to left). Let $s_i(0)$ and $s_i(1)$ label the respective site values of the left-most endpoint of the ith 0-block and the ith 1-block. The construction of predecessors for the nonlinear 2-site rules is carried out as follows.

Z_1: If β contains any 11 pair it will have no predecessors. Assuming that β contains only isolated 1s, replace the ith 0-block of $\sigma^{-1}(\beta)$ by a block consisting of r_i consecutive 1s followed by $a_i - r_i$ consecutive 0s, $0 \leq r_i \leq a_i - 1$. Thus there are a_i possible replacements for the ith 0-block, if the state space is E_n. If it is E^+, however, and β starts with a 0-block, there will be $a_1 + 1$ possible substitutions for this initial 0-block. Thus the number of predecessors of a configuration β is given by

$$N(Z_1;\beta \in E_n) = \begin{cases} 0 & \beta \text{ contains a 11 block} \\ \prod_{i=1}^{n(0)} a_i & \beta \text{ contains only isolated 1s.} \end{cases} \tag{2.4a}$$

$$N(Z_1;\beta \in E^+) = \begin{cases} 0 & \beta \text{ contains a 11 block} \\ \prod_{i=1}^{n(0)} a_i & \beta \text{ starts with 1} \\ (a_1+1)\prod_{i=2}^{n(0)} a_i & \beta \text{ starts with 0} \end{cases} \tag{2.4b}$$

where $n(0)$ is the number of 0-blocks contained in β. If the state space is E^+ and β has only isolated 1s then either β contains an infinite number of 1s, or there is an $M < \infty$ such that $\beta_i = 0$ for all $i \geq M$. In the latter case there will always be an infinite number of predecessors. In the former, the product in equation (2.4b) will be infinite unless β contains only a finite number of 0-blocks for which $1 < a_i < \infty$.

Z_2: The construction of predecessors for this rule is the same as for Z_1 except that replacements are made for 0-blocks in β rather than $\sigma^{-1}(\beta)$, and the substitution is a block of r_i consecutive 0s followed by $a_i - r_i$ consecutive 1s with $1 \leq r_i \leq a_i$. The same distinction between E_n and E^+ also holds. Thus, the number of predecessors is again given by equation (2.4).

Z_3: For this case, and for the rule Y_0, it is necessary to distinguish between the state spaces E_n and E^+.

On E_n a configuration β will have predecessors if and only if it contains no isolated 0s. If this is so, replace the ith 1-block in β by a 1-block of length b_i+1 which starts at the same initial site $s_i(1)$. Replace the ith 0-block by a block of the form $0r_1 \ldots r_{a_i-3}0$, starting at site $s_i(0)+1$, with the block $r_1 \ldots r_{a_i-3}$ containing only isolated 1s. The number of blocks $0r_1 \ldots r_{a_i-3}0$ which satisfy this condition is given by the a_i-1 Fibonacci number $F(a_i-1)$, defined by the recurrence relation $F(0) = 0$, $F(1) = F(2) = 1$, $F(n) = F(n-1) + F(n-2)$. Thus the number of predecessors of a configuration $\beta \in E_n$ is given by

$$N(Z_3;\beta \in E_n) = \prod_{i=1}^{n(0)} F(a_i - 1). \tag{2.5}$$

Note that if β contains an isolated 0 then the right side of equation (2.5) has $F(0)$ as a factor, hence β has no predecessors.

On E^+ a configuration can start with an isolated 0, but can have no others. Two cases must be distinguished. If β starts with a 1-block the construction of predecessors is exactly the same as for configurations in E_n, and the number of predecessors is again given by equation (2.5). If β starts with a 0-block the procedure is the same except that the initial 0-block is replaced by $r_1 \ldots r_{a_1-1}0$ where again $r_1 \ldots r_{a_1-1}$ contains only isolated 1s. Thus, in this case, the number of predecessors is given by

$$N(Z_3; \beta \in E^+,\ \beta \text{ starts with a 0-block}) = F(a_1+1) \prod_{i=2}^{n(0)} F(a_i - 1). \tag{2.6}$$

In both cases, the products of equation (2.5) or (2.6) will be infinite unless $\beta i = 1$ for all $i \geq M$, for some $M < \infty$; or, there are only a finite number of 0-blocks with lengths such that $4 \leq a_i < \infty$.

Y_0: On E_n, β will have predecessors if and only if it contains no isolated 1s. If this is so, predecessors are constructed by replacing the ith 0-block with a 0-block of length $a_i + 1$ starting at the same site $s_i(0)$; and by replacing the ith 1-block by a block $1r_1 \ldots r_{b_i-3}1$ of length $b_i - 1$ starting at site $s_i(1) + 1$, with $r_1 \ldots r_{b_i-3}$ containing only isolated 0s. If $n(1)$ is the number of 1-blocks in β, the number of predecessors is given by

$$N(Y_0; \beta \in E_n) = \prod_{i=1}^{n(1)} F(b_i - 1). \tag{2.7}$$

On E^+, β can start with an isolated 1, but can have no others. If β starts with a 0-block the procedure is exactly the same as on E_n and the number of predecessors is given by equation (2.7). If β starts with a 1-block the procedure is the same, except that the initial 1-block is replaced by a block $r_1 \ldots r_{b_1-1}1$ with $r_1 \ldots r_{b_1-1}$ containing only isolated 0s. In this case the number of predecessors is given by

$$N(Y_0; \beta \in E^+,\ \beta \text{ starts with a 1-block}) = F(b_1+1) \prod_{i=2}^{n(1)} F(b_i - 1). \tag{2.8}$$

As with Z_3, the products of equations (2.7) and (2.8) will diverge for states in E^+ unless either $\beta_i = 0$ for all $i \geq M$; or, there are only finitely many 1-blocks of length $4 \leq b_i < \infty$.

3. Composite 3-site rules

Excluding **0** and **1**, there are 60 3-site rules that are composed as the product of two 2-site rules, 12 of these are linear. Also, if a rule X is composite, then $\mathbf{1} + X$ is composite as well. The composite 3-site rules together with their composite form are listed in Table 3. The 3-site rules are listed in terms of their standard numeration, followed by their composite form.

The forms given in Table 3, together with the methods for computing predecessors for 2-site rules, allows immediate computation of predecessors

Table 3: Composite forms of 3-site rules.

A. Linear rules			
$15 = (\mathbf{1}+I)I$	$51 = (\mathbf{1}+I)\sigma$	$60 = ID$	$85 = (\mathbf{1}+\sigma)\sigma$
$90 = D^2$	$102 = D\sigma$	$153 = (\mathbf{1}+D)\sigma$	$165 = (\mathbf{1}+D)D$
$170 = \sigma^2$	$195 = (\mathbf{1}+I)D$	$204 = I\sigma$	$240 = I^2$
B. Nonlinear rules: I			
$12 = IZ_1$	$243 = (\mathbf{1}+I)Z_1$	$24 = Z_2D$	$231 = (\mathbf{1}+Z_2)D$
$34 = \sigma Z_1$	$221 = (\mathbf{1}+\sigma)Z_1$	$46 = DZ_1$	$209 = (\mathbf{1}+D)Z_1$
$48 = IZ_2$	$207 = (\mathbf{1}+I)Z_2$	$66 = Z_1D$	$189 = (\mathbf{1}+Z_1)D$
$68 = \sigma Z_2$	$187 = (1+\sigma)Z_2$	$116 = DZ_2$	$139 = (\mathbf{1}+D)Z_2$
C. Nonlinear rules: II			
$1 = (\mathbf{1}+Y_0)Y_0$	$254 = Y_0Y_0$	$127 = (\mathbf{1}+Z_3)Z_3$	$128 = Z_3Z_3$
$2 = Z_1Y_0$	$253 = (\mathbf{1}+Z_1)Y_0$	$8 = Z_1Z_3$	$247 = (\mathbf{1}+Z_1)Z_3$
$16 = Z_2Y_0$	$239 = (\mathbf{1}+Z_2)Y_0$	$64 = Z_2Z_3$	$191 = (\mathbf{1}+Z_2)Z_3$
$18 = DY_0$	$237 = (\mathbf{1}+D)Y_0$	$72 = DZ_3$	$183 = (\mathbf{1}+D)Z_3$
$36 = Z_3D$	$219 = (\mathbf{1}+Z_3)D$	$126 = Y_0D$	$129 = (\mathbf{1}+Y_0)D$
$136 = \sigma Z_3$	$119 = (\mathbf{1}+\sigma)Z_3$	$192 = IZ_3$	$63 = (\mathbf{1}+I)Z_3$
$200 = Y_0Z_3$	$55 = (\mathbf{1}+Y_0)Z_3$	$236 = Z_3Y_0$	$19 = (\mathbf{1}+Z_3)Y_0$
$238 = \sigma Y_0$	$17 = (\mathbf{1}+\sigma)Y_0$	$252 = IY_0$	$3 = (\mathbf{1}+I)Y_0$

for all the composite 3-site rules. The kind of analysis involved will be illustrated with the examples of rules 18 and 126.

Rule 18: For this rule equation (1.1) becomes $DY_0(\mu) = \beta$. Making use of equation (2.2) this reduces to

$$Y_0(\mu) = c\underline{1} + B\sigma^{-1}(\beta). \tag{3.1}$$

If the term on the right in this equation contains an isolated 1 (other than a possible initial isolated 1 if the state space is E^+) then β is a Garden-of-Eden state. By inspection, this can occur if and only if the term $B\sigma^{-1}(\beta)$ contains both an isolated 1 (for the case $c = 0$) and an isolated 0 (for the case $c = 1$). This immediatly eliminates states β containing blocks 111 since for $\beta_{i-1} = 0$, $\beta_i = \beta_{i+1} = \beta_{i+2} = 1$, $B(\beta)$ will have

$$B(\ldots 0111\ldots) = \begin{cases} \ldots 0101\ldots & [B(\beta)]_{i-1} = 0 \\ \ldots 1010\ldots & [B(\beta)]_{i-1} = 1. \end{cases}$$

(Even if β starts with $111\ldots$, $\sigma^{-1}(\beta)$ will start with $0111\ldots$.) Hence, if β is not to be a Garden-of-Eden state it can contain only single and double 1s.

Assume this to be so and consider segments of β having the form $0110r_1\ldots r_n110$ where $r_n = 0$ and $r_1\ldots r_{n-1}$ contains only isolated 1s. Let $k(r)$ be the number of these isolated 1s and suppose that the block $0110r_1\ldots r_n110$ starts at site $i-1$. Taking $m_j = [B(\beta)]_{j+i+2}$, and $m'_j =$

$1 + m_j \bmod(2)$:

$$B(\ldots 0110 r_1 \ldots r_n 110 \ldots) = \begin{cases} \ldots 0100 m_1 \ldots m_{n-2} 00100 \ldots & m_{i-1} - 0,\ k(r) \text{ even} \\ \ldots 0100 m_1 \ldots m_{n-2} 11011 \ldots & m_{i-1} = 0,\ k(r) \text{ odd} \\ \ldots 1011 m'_1 \ldots m'_{n-2} 11011 \ldots & m_{i-1} = 1,\ k(r) \text{ even} \\ \ldots 1011 m'_1 \ldots m'_{n-2} 00100 \ldots & m_{i-1} - 1,\ k(r) \text{ odd} \end{cases}$$

with $m_1 \ldots m_{n-1}$ and $m'_1 \ldots m'_{n-2}$ containing no isolated 0s or 1s. Inspection of these forms shows that $B\sigma^{-1}(\beta)$ will have both an isolated 0 and an isolated 1 if and only if $k(r)$ is odd.

There is one final consideration. If the state space is E_n and β contains an odd number of 1s then $B\sigma^{-1}(\beta)$ will have period $2n$ rather than period n [8], and so will not be contained in E_n. Thus, on E^+, $GE^*(18)$ is given by the set containing 111, and all segments of the form $0110 r_1 \ldots r_{n-1} 0110$ where $r_1 \ldots r_{n-1}$ contains an odd number of isolated 1s. If the state space is E_n, $GE^*(18)$ also contains all segments having an odd number of 1s, whether or not they satisfy the other conditions. It is the E_n case that is presented in [9], where the membership conditions for $GE^*(18)$ are derived only after much (unpublished) computation.

Suppose that β is not a Garden-of-Eden configuration. There are three cases to be considered.

1. $B\sigma^{-1}(\beta)$ contains no isolated 0s or 1s.

2. $B\sigma^{-1}(\beta)$ contains isolated 0s but no isolated 1s.

3. $B\sigma^{-1}(\beta)$ contains isolated 1s but no isolated 0s.

Note that cases 2 and 3 will occur if and only if β contains a 11 block.

In case 1, $B\sigma^{-1}(\beta)$ contains no isolated 0s or 1s, hence $\underline{1} + B\sigma^{-1}(\beta)$ will also be free of isolated 0s and 1s. Predecessors of β are obtained by application of the construction procedure given for predecessors of Y_0 to the right-hand side of equation (3.1), both for the $c = 0$ and the $c = 1$ cases. The total number of predecessors is therefore given by making use of equations (2.7) and (2.8), noting that for $c = 0$ on E^+, the term $B\sigma^{-1}(\beta)$ will always start with a 0-block while for $c = 1$, $\underline{1} + B\sigma^{-1}(\beta)$ will always start with a 1-block. Thus, the number of predecessors is given by

$$\prod_{i=1}^{n(0)} F(a_i - 1) + \prod_{i=1}^{n(1)} F(b_i - 1) \quad \beta \in E_n$$

$$F(a_1 + 1) \prod_{i=2}^{n(0)} F(a_i - 1) + \prod_{i=1}^{n(1)} F(b_i - 1) \quad \beta \in E^+. \tag{3.2a}$$

In case 2, $c = 1$ yields a configuration with isolated 1s, for which equation (3.1) has no solution. Likewise, in case 3, $c = 0$ yields a configuration with isolated 1s. Thus the total number of predecessors in these cases are given respectively by

$$\prod_{i=1}^{n(1)} F(b_i - 1) \quad \beta \in E_n \text{ or } E^+ \tag{3.2b}$$

and

$$\prod_{i=1}^{n(0)} F(a_i - 1) \quad \beta \in E_n$$

$$F(a_1 + 1) \prod_{i=2}^{n(0)} F(a_i - 1) \quad \beta \in E^+. \tag{3.2c}$$

Rule 126: The equation to be solved for this rule is $Y_0 D(\mu) = \beta$, which reduces to $D(\mu) = \gamma$ where γ is any solution of $Y_0(\gamma) = \beta$. Thus, by equation (2.2), the general solution is given by $\mu = c\underline{1} + B\sigma^{-1}(\gamma)$ and the number of solutions is twice the number of solutions of $Y_0(\gamma) = \beta$, as given in equations (2.7) and (2.8).

4. Comparison with Jen's enumeration of preimages

In [1] Jen has derived a set of recurrence relations to count preimages of finite strings, and finds that the elementary CA can be grouped into six classes on the basis of the particular natures of these recurrence relations. The six classes and the composite rules belonging to each class are listed in Table 4.

The labels in Table 3 were chosen to match Jen's classes, indicating that all composite elementary rules belong to her A, B, and C classes. Further, class A consists of surjective rules, or in the case of composite rules, linear rules. Comparison of Tables 3 and 4 show several correspondences.

1. All linear rules are found in class A.
2. All composite rules in class B are a product of a linear and a nonlinear factor, with the nonlinear factor being either Z_1 or Z_2, or $\mathbf{1}$ plus one of these rules.
3. All composite rules in class C have a factor of either Z_3 or Y_0, or $\mathbf{1}$ plus one of these rules. The formulas computed for numbers of predecessors for the 2-site rules can be seen to bear out the defining characteristics of Jen's classes. There are some differences, however, in the specific number of predecessors computed versus the number of preimages. This is a result of the fact that in counting preimages only finite sequences without periodic boundary conditions are considered. If this difference is taken into consideration then analysis of the 2-site rules Z_1, Z_2, Z_3,

Table 4: Composite rules in recurrence classes from [1].

Class	Defining Characteristic	Composite Rules in Class
A	Constant number of preimages.	15, 51, 60, 85, 90, 102, 153, 165, 170, 195, 204, 240
B	Number of preimages is a product of integers representing lengths of blocks of consecutive units (0s and 1s, or combinations thereof).	12, 24, 34, 46, 48, 66, 68, 116, 139, 187, 189, 207, 209, 221, 231, 243
C	Number of preimages is a product of integers in Fibonacci-like sequences.	1, 2, 3, 8, 16, 17, 18, 19, 36, 55, 63, 64, 72, 119, 126, 127, 128, 129, 136, 183, 191, 192, 200, 219, 236, 237, 238, 239, 247, 252, 253, 254
D	Number of preimages satisfies telescoping recurrence relation.	no composite rules
E	Number of preimages is given by terms in sequences whose values vary periodically.	no composite rules
F	Number of preimages is given by other recurrence relations which do not easily reduce to any of the above.	no composite rules

and Y_0 yields the following formulas for the number of preimages of a string $s = s_1 \ldots s_n$:

$$N(Z_1; s) = N(Z_2; s) = \begin{cases} 0 & s \text{ contains a 11 block} \\ (a_1+1)(a_n+1)\prod_{i=2}^{n(0)-1} a_i & s \text{ starts with 0} \\ 2(a_1+1)(a_n+1)\prod_{i=2}^{n(0)-1} a_i & s \text{ starts with 1} \end{cases} \tag{4.1}$$

$$N(Z_3; s) = F(a_1+1)(a_n+1) \prod_{i=2}^{n(0)-1} F(a_i - 1) \tag{4.2}$$

$$N(Y_0; s) = F(b_1+1)(b_n+1) \prod_{i=2}^{n(1)-1} F(b_i - 1) \tag{4.3}$$

where a_1 and a_n (or b_1 and b_n) are the respective lengths of the initial and final blocks of s. For example, if s is the string 0011100110111 then $a_1 = 2$, $a_n = 0$, $b_1 = 0$, and $b_n = 3$.

5. Discussion

In the computation of $GE^*(18)$ it was noted that states in E_n which contained an odd number of 1s had no predecessors in E_n. If these states are considered as period n configurations in E^+, on the other hand, and they do not contain any other substring contained in $GE^*(18)$, they will have period

$2n$ predecessors in E^+. This is a result of the period doubling property of the operator B, discussed in [8]. In fact, these period $2n$ predecessors will be composed of a length n string γ concatenated with its binary compliment string $\underline{1} + \gamma$. This highlights a distinction between configurations which are always Garden-of-Eden for a given rule, and those which are only conditionally so as a result of periodic boundary conditions. Only the former can be seen as intrinsically characteristic of a rule.

For 2-site rules it is only the binary difference operator D for which computation of predecessors may result in period doubling, hence only those rules in Table 3 which involve D as a factor will have these conditional Garden-of-Eden states.

Another point of interest relates to a theorem of Hedlund [2], which states that if a CA rule is not surjective then there will be periodic configurations having an uncountable number of predecessors under this rule. Consideration of the procedures given for computing predecessors of the 2-site rules indicates how this comes about, and allows the construction of a simple example.

Consider the configuration $\underline{100} \in E^+$, and the 2-site rule Z_2. According to the procedure for the construction of predecessors, each 00 block is to be replaced by either 00, or 01. Let $\alpha \in E^+$ be defined by taking

$$\alpha_i \begin{cases} 0 & \text{the } i\text{th block is replaced by } 00 \\ 1 & \text{the } i\text{th block is replaced by } 01. \end{cases}$$

Then, since for each i the choice is independent of choices made for all other values of i, any element of E^+ can be so constructed; or equivalently, each element of E^+ indexes an unique predecessor of $\underline{100}$.

In [2] it is also proved that surjective rules map eventually periodic configurations to eventually periodic sequences, and nonperiodic configurations to nonperiodic configurations. Since E^+ maps naturally to [0,1], CA rules defined on E^+ also define maps of the interval, and considered as such, surjective rules will map rationals to rationals and irrationals to irrationals. Nonsurjective rules, on the other hand, will have uncountable sets of irrationals with the same rational image. This explains the fact that nonsurjective rules acting on E^+ reduce entropy, while surjective rules do not.

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