

Language Complexity of Unimodal Systems

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Abstract. The complexity of formal languages which are generated by S -unimodal systems on interval covers is studied. It is shown that there exist S -unimodal systems with nonrecursive languages, those with recursive but not context-sensitive languages, and those with context-sensitive but not regular languages. It is also shown that S -unimodal systems with regular languages include all systems with finite, periodic, and preperiodic kneading sequences, but also all infinitely renormalizable systems. Finally, it is shown that systems with zero topological entropy might be characterized by periodic languages (a subclass of regular languages), and systems with unique fixed points can be characterized by bounded periodic languages.

1. Introduction

The rich variation of dynamics encountered in the simple quadratic family $f_r(x) = rx(1 - x)$ has been a source of fascination resulting in deep mathematical theorems (e.g., [6, 13–15]). Better understanding of the quadratic family and more general unimodal families calls for classification schemes which could evaluate their complexity. Their statistical complexity is well captured by topological entropy, which is nondecreasing with the parameter r (see [14]).

Another approach to the complexity of unimodal systems was pioneered in [7] (see also [8, 18]) that is based on the theory of formal languages and computational complexity. The decomposition of the real interval into two intervals separated by the critical point yields a language of possible itineraries. The obtained language is regular if the kneading sequence is finite or periodic (when the system has a stable periodic orbit), or when its kneading sequence is preperiodic (when the system is at a band merging bifurcation). Thus the language complexity is not monotonous with r : the really complex behaviors can be found only in the interior of the parameter range.

This approach could be generalized. Every closed cover of the state space of a dynamical system yields a language of its finite itineraries and the corresponding subshift of its infinite itineraries. For classification purposes, suitable test covers have to be chosen. For zero-dimensional systems, appropriate test covers seem to be clopen partitions (see [11]). In the one-dimensional case we use interval covers that consist of intervals which intersect at most in their end points. For a given class \mathfrak{L} of languages we say that a dynamical system on a real interval is of class \mathfrak{L} , if there exists a separating sequence of interval covers, each of which yields a language belonging to \mathfrak{L} (here separating means that the diameter of the covers tends to 0, and every cover of the sequence refines the preceding one). Using this concept, the language complexity of circle rotations has been investigated in [3].

In this paper we investigate unimodal systems in a similar manner. We show that there exist S -unimodal systems with nonrecursive languages, those with recursive but not context-sensitive languages, and also those with context-sensitive but nonregular languages (no results have been obtained for context-free languages). This has been demonstrated on S -unimodal systems with nonrecurrent critical points, where the kneading sequence can be reconstructed from the language of a sufficiently fine interval cover. The systems with regular languages include again systems with finite, periodic, or preperiodic kneading sequences. However, in contrast to the results in [7], infinitely renormalizable systems yield regular languages also. This is possible because the canonical cover plays no distinguishable role in our approach.

We also consider two subclasses of regular languages. We say that a regular language is periodic if it can be generated by a finite graph whose every communicating class is a cycle. This happens exactly when the corresponding subshift is countable. We say that a periodic language is bounded periodic if, in addition, every communicating set is final, that is, no edge goes out of it. This happens exactly when the corresponding subshift is finite. We show that a unimodal system has periodic languages if and only if it has zero topological entropy, that is, if it has finitely many periodic points (whose periods must then be powers of 2) or if it is the Feigenbaum system. Finally, the system has bounded periodic languages if and only if it has a unique fixed point.

2. Subshifts and languages

A dynamical system (X, f) is a continuous mapping $f : X \rightarrow X$, where X is a compact metric space. A homomorphism $H : (X, f) \rightarrow (Y, g)$ is a continuous map $H : X \rightarrow Y$ such that $Hf = gH$. A surjective homomorphism is called a factor map, a bijective homomorphism is called conjugacy.

A point $x \in X$ is periodic if $f^n(x) = x$ for some $n > 0$. It is eventually periodic if $f^m(x)$ is periodic for some $m \geq 0$ and it is preperiodic if it is eventually periodic but not periodic. It is aperiodic if it is not eventually periodic. A point $x \in X$ is recurrent if for every neighborhood U of x there exists $n > 0$ such that $f^n(x) \in U$.

A dynamical system (X, f) is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$, if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \varepsilon$ for every $n \geq 0$. A subset $Y \subseteq X$ is an attractor if there exists a closed subset $V \subseteq X$ such that $f(V) \subset \text{int}(V)$, and $Y = \bigcap_{n \geq 0} f^n(V)$.

A closed cover of a space X is a finite collection $\mathcal{V} = \{V_a, a \in A\}$ of its closed subsets whose union is X . If $V_a \cap V_b = \emptyset$ for $a \neq b$, then all V_a are clopen (closed and open). In this case we say that \mathcal{V} is a clopen partition. A closed cover of a real compact interval is called interval cover, if it consists of closed intervals which overlap at most in their end points. The diameter of \mathcal{V} is $\text{diam}(\mathcal{V}) = \max\{\text{diam}(V_a); a \in A\}$. We say that \mathcal{V} is finer than $\mathcal{W} = \{W_b; b \in B\}$, if there exists a function $h : A \rightarrow B$ such that $V_a \subseteq W_{h(a)}$. We say that $(\mathcal{V}_i)_{i \in \mathbb{N}}$ is a separating sequence of closed covers, if \mathcal{V}_{i+1} is finer than \mathcal{V}_i , and $\lim_{i \rightarrow \infty} \text{diam}(\mathcal{V}_i) = 0$.

If A is a finite alphabet $n \in \mathbb{N}$ denote by A^n the set of words over A of length n , $A^* = \bigcup_{n \in \mathbb{N}} A^n$ the set of finite words over A , $A^{\mathbb{N}}$ the set of one-way infinite words, and $\overline{A^*} = A^* \cup A^{\mathbb{N}}$. For $u \in \overline{A^*}$, denote by $|u|$ its length ($0 \leq |u| \leq \infty$). A language over A is any subset $L \subseteq A^*$. A (one-sided) full shift is a dynamical system $(A^{\mathbb{N}}, \sigma)$ on the power space $A^{\mathbb{N}}$ equipped with the product topology, given by $\sigma(u)_i = u_{i+1}$. A subshift (over A) is any subsystem of the full shift, that is, a dynamical system (Σ, σ) , where $\Sigma \subseteq A^{\mathbb{N}}$ is a nonempty closed subset Σ of $A^{\mathbb{N}}$ which is σ -invariant, that is, $\sigma(\Sigma) \subseteq \Sigma$ (the inclusion might be strict).

Let (X, f) be a dynamical system and let $\mathcal{V} = \{V_a; a \in A\}$ be a closed cover of X . For $u \in \overline{A^*}$ put

$$V_u = \{x \in X; (\forall i < |u|)(f^i(x) \in V_{u_i})\} = \bigcap_{i < |u|} f^{-i}(V_{u_i})$$

$$\begin{aligned} \mathcal{L}_{\mathcal{V}}(X, f) &= \{u \in A^*; V_u \neq \emptyset\}, \\ \Sigma_{\mathcal{V}}(X, f) &= \{u \in A^{\mathbb{N}}; V_u \neq \emptyset\}. \end{aligned}$$

Then $\Sigma_{\mathcal{V}}(X, f)$ is a subshift and $L = \mathcal{L}_{\mathcal{V}}(X, f)$ is a right-central language, that is, it is closed under subwords (if $u \in L$ and v is a subword of u then $v \in L$) and extendable to the right (if $u \in L$ then there exists $a \in A$ with $ua \in L$). In fact the right-central languages are in one-to-one correspondence with subshifts (cf., [3]), and for a subshift $\Sigma \subseteq A^{\mathbb{N}}$, and partition $\mathcal{V} = \{[a]; a \in A\}$, $\mathcal{L}_{\mathcal{V}}(\Sigma, \sigma)$ is exactly the language of words occurring in points of Σ .

The topological entropy of a dynamical system (X, f) on a cover \mathcal{V} is

$$h_{\mathcal{V}}(X, f) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \text{card}\{u \in A^n; V_u \neq \emptyset\}$$

and the topological entropy of (X, f) is $h(X, f) = \sup_{\mathcal{V}} h_{\mathcal{V}}(X, f)$ where the supremum is taken over all open covers of X . If X is a zero-dimensional space and if (\mathcal{V}_n) is a separating sequence of clopen partitions, then

$$h(X, f) = \lim_{n \rightarrow \infty} h_{\mathcal{V}_n}(X, f) = \lim_{n \rightarrow \infty} h(\Sigma_{\mathcal{V}_n}(X, f), \sigma).$$

3. Complexity classes of languages

We consider language classes using the Chomsky hierarchy. A language is recursive (REC), if it can be recognized by a (deterministic or nondeterministic) Turing machine. A language is context-sensitive (CS) if it can be recognized by a nondeterministic Turing machine in linear space. A language is regular (REG) if it can be recognized by a (deterministic or nondeterministic) finite automaton.

A (finite) graph is a fourtuple $G = (V, E, s, t)$ where V is a finite set of vertices, E is a finite set of edges, and $s, t : E \rightarrow V$ are the source and target maps such that s is surjective. A path in G is a sequence $u \in \overline{E}^*$ such that $t(u_{i-1}) = s(u_i)$ for every $0 < i < |u|$. A path is simple if it does not contain any edge twice. We say that vertices $a, b \in V$ communicate if there exists a path from a to b and a path from b to a . The communication relation is an equivalence on the set of those vertices which communicate with themselves. The equivalence classes are called communicating sets. A communicating set is a cycle, if for every $a, b \in V$ there is exactly one simple path from a to b . A communicating set is final, if no edge leaves it. Every graph has at least one final communicating set.

A labeled graph over an alphabet A is a graph $G = (V, E, s, t)$ together with a labeling function $l : E \rightarrow A$. A labeling function extends to paths $l : \overline{E}^* \rightarrow \overline{A}^*$. The language $L_{G,l}$ and the subshift $\Sigma_{G,l}$ of a labeled graph are the sets of the labels of its finite or infinite paths respectively. A right-central language L is regular if and only if $L = L_{G,l}$ for some labeled graph (G, l) . A general language L (not necessarily right-central) is regular if and only if there exists a labeled graph (G, l) and an initial vertex $v_0 \in V$, such that L is exactly the set of labels of paths which start in v_0 . In both cases we can assume that the graph is right-resolving. This means that for every vertex $v \in V$ and every letter $a \in A$ there exists at most one edge with source v and label a .

For dynamical purposes, two subfamilies of the family of regular languages seem to be useful. We say that a regular language is periodic (PER), if it can be presented by a graph whose every communicating set is a cycle. We say that a language is bounded periodic (BPER), if every communicating set of the graph is also a final cycle.

Proposition 1. *A subshift has bounded periodic languages if and only if it is finite.*

Proof. Let (G, l) be a labeled graph where every communicating set is a final cycle. Let p be the least common multiple of the length of all cycles, and let m be the maximal length of all paths outside of the communicating sets. Then for every infinite path u of G , $u_i = u_{i+p}$ for all $i > m$. Thus u is determined by its initial substring of length $m + p$, so the number of infinite paths is finite, and $\Sigma_{G,l}$ is finite too. Conversely if (X, f) is a finite dynamical system, construct a graph $G = (X, X, s, f)$, where s is the identity function,

with a labeling function $l : X \rightarrow X$ which is identity also. Then G has the required properties and (X, f) is conjugate to $(\Sigma_{G,l}, \sigma)$. ■

Proposition 2. *A subshift with regular languages has periodic languages if and only if it is countable. In this case it has a nonzero finite number of periodic orbits.*

Proof. Let (G, l) be a labeled graph with every communicating set being a cycle. Let W be the set of simple paths with sources and targets that coincide. Let U be the set of simple paths with sources and targets that lie in different communicating sets. Then W and U are finite. If x is an infinite path in G , then $x = u_0 w_0^{k_0} \dots u_{m-1} w_{m-1}^{k_{m-1}} \overline{w_m}$ for some $u_i \in U$ and $w_i \in W$. This is a countable set, so $\Sigma_{G,l}$ is also countable. It is clear that the set of periodic points is finite. Conversely, suppose that $\Sigma \subset A^{\mathbb{N}}$ is a countable subshift with regular languages. Let (G, l) be a minimal right-resolving presentation for Σ (see [12]). We show that every communicating set W of G is a cycle. Indeed if not, then there exist different paths u, v, w such that $s(u) = s(v) = t(v)$ and $t(u) = t(v) = s(w)$. Since (G, l) is right-resolving, $l(u) \neq l(v)$. Then any sequence $x = x_0 l(w) x_1 l(w) x_2 \dots$ where $x_i \in \{l(u), l(v)\}$, belongs to $\Sigma = \Sigma_{G,l}$. Thus Σ is not countable. ■

Let A_1 and A_2 be finite alphabets. A map $h : A_1 \rightarrow A_2$ extends to a monoid homomorphism $h^* : A_1^* \rightarrow A_2^*$ by $h^*(u)_i = h(u_i)$. We say that a language $L_2 \subseteq A_2^*$ is a factor of a language $L_1 \subseteq A_1^*$ if there exists a map $h : A_1 \rightarrow A_2$ such that $L_2 = h^*(L_1)$. (In formal language theory a more general concept of λ -free homomorphism is used, e.g., [9].) For the corresponding subshifts Σ_1 and Σ_2 we get a (“block one”) factor map $h^{\mathbb{N}} : (\Sigma_1, \sigma) \rightarrow (\Sigma_2, \sigma)$, defined by $h^{\mathbb{N}}(u)_i = h(u_i)$.

Definition 1. A family \mathfrak{L} of languages is closed under factors if it contains all factors of all its members. A family of languages \mathfrak{L} is closed under concatenations if for every language $L \subseteq A^*$ of class \mathfrak{L} , and every $n > 0$, the language $L^n = \{(u_0 \dots u_{n-1})(u_1 \dots u_n) \dots (u_{k-1} \dots u_{k+n-2}) \in (A^n)^*; u_0 \dots u_{k+n-2} \in L\}$ belongs to \mathfrak{L} .

All families of languages considered above are closed under both factors and concatenations. In the sequel, any abstract family of languages considered is supposed to have these properties.

Definition 2. Let \mathfrak{L} be a class of right-central languages closed under factors. We say that a zero-dimensional system (X, f) is of class \mathfrak{L} if there exists a separating sequence of clopen partitions \mathcal{V}_i of X , such that for every i , $\mathcal{L}_{\mathcal{V}_i}(X, f)$ is in \mathfrak{L} . We say that a dynamical system (I, f) on a real interval is of class \mathfrak{L} if there exists a separating sequence of interval covers \mathcal{V}_i of X , such that for every i , $\mathcal{L}_{\mathcal{V}_i}(X, f)$ is in \mathfrak{L} .

Proposition 3. *Every dynamical system (I, f) of class \mathfrak{L} is a factor of a zero-dimensional system of class \mathfrak{L} .*

The proof works for a general dynamical system (X, f) possessing a separating sequence of closed covers. Let $\mathcal{V}_i = \{V_a; a \in A_i\}$ be an increasing sequence of closed covers and let $h_i : A_{i+1} \rightarrow A_i$ satisfy $V_a \subseteq V_{h_i(a)}$ for every $a \in A_{i+1}$. We have factor maps $h_i^N : (Y_{i+1}, \sigma) \rightarrow (Y_i, \sigma)$ where $Y_i = \Sigma_{\mathcal{V}_i}(X, f)$ and $h_i^N(u)_i = h(u_i)$. Let $Y = \{y \in \prod_i Y_i; h_i^N(y_{i+1}) = y_i\}$ be their inverse limit, define $g : Y \rightarrow Y$ by $g(y)_i = \sigma(y_i)$ and $H : Y \rightarrow X$ by $H(y) = \bigcap_i V_{y_{i0}}$. Then (Y, g) is a zero-dimensional system of class \mathfrak{L} and $H : (Y, g) \rightarrow (X, f)$ is a factor map.

Proposition 4. *Every dynamical system with bounded periodic languages is equicontinuous.*

Proof. Since every factor of an equicontinuous system is equicontinuous (see [1]), it suffices to prove the theorem for zero-dimensional spaces. Given $\varepsilon > 0$, let $\mathcal{V} = \{V_a; a \in A\}$ be a clopen partition of X with $\text{diam}(\mathcal{V}) < \varepsilon$ such that $\mathcal{L}_{\mathcal{V}}(X, f)$ is bounded periodic. There exists δ such that whenever $d(x, y) < \delta$, then x and y belong to the same set of the partition, and therefore for all i , $d(f^i(x), f^i(y)) < \varepsilon$. ■

Proposition 5. *Every dynamical system with periodic languages has zero topological entropy.*

Proof. It is again sufficient to prove the theorem for zero-dimensional systems. Let $(\mathcal{V}_i)_{i \in \mathbb{N}}$ be a separating sequence of clopen partitions of a zero-dimensional space X . By the Bowen theorem (see [4]), the topological entropy of a dynamical system is concentrated on its nonwandering set, so the topological entropy of every $(\Sigma_{\mathcal{V}_n}(X, f), \sigma)$ is zero. It follows that the topological entropy of (X, f) is also zero. ■

4. Symbolic dynamics of unimodal systems

With slight modifications we follow the exposition of unimodal systems in [5, 14].

Definition 3. An S -unimodal system is a dynamical system (I, f) on a closed interval $I = [a, b]$ with negative schwarzian derivative (and therefore continuous third derivative) and negative second derivative such that there is exactly one critical point $c \in (a, b)$ with $f'(c) = 0$, and $f(a) = f(b) = a$.

Let A be a finite alphabet and $u, v \in \overline{A^*}$. We write $u \sqsubseteq v$ if u is an initial segment of v , and $u \sqsubset v$ if u is a proper initial segment of v . The initial substring of a word u of length n is denoted by $u|_n = u_0 \dots u_{n-1}$. We denote by λ the word of zero length, and by convention, put $u_i = \lambda$ if $i \geq |u|$, where $|u|$ is the length of u . For binary alphabet $\mathbf{2} = \{0, 1\}$ and $u \in \mathbf{2}$ put $\hat{u} = 1 - u$. For $u \in \mathbf{2}^*$ with $|u| = n > 0$ put $\hat{u} = u_0 \dots u_{n-2} \widehat{u_{n-1}}$.

If (I, f) is an S -unimodal system and $x \in I$, define its itinerary $\mathcal{I}(x) \in \overline{\mathbf{2}}^*$ by

$$\begin{aligned} \mathcal{I}(x)_i &= 0 & \text{if } f^i(x) < c, \\ \mathcal{I}(x)_i &= 1 & \text{if } f^i(x) > c \end{aligned}$$

for all $i < |\mathcal{I}(x)| = \min\{i \geq 0; f^i(x) = c\} \leq \infty$. The kneading sequence of (I, f) is $\mathcal{K}(f) = \mathcal{I}(f(c))$. We have always $\mathcal{I}(a) = \overline{0}$, $\mathcal{I}(c) = \lambda$, and $\mathcal{I}(b) = \overline{10}$. For $u, v \in \overline{\mathbf{2}}^*$ define $u \prec v$ if and only if for $k = \max\{i; u_i = v_i\}$ either

$$\begin{aligned} u_0 + \dots + u_{k-1} \bmod 2 &= 0 & \text{and } u_k \prec v_k, & \text{ or} \\ u_0 + \dots + u_{k-1} \bmod 2 &= 1 & \text{and } u_k \succ v_k. \end{aligned}$$

Here, if u_k or v_k is λ , which happens when one of the words is a proper initial substring of the other, we put $0 \prec \lambda \prec 1$. We also write $u \underset{\square}{\prec} v$ if $u \prec v$ or $u \sqsubset v$, and $u \underset{\square}{\succ} v$ if $u \succ v$ or $u \sqsupset v$. Of course $u \preceq v$ means $u \prec v$ or $u = v$. The order \prec corresponds to the order on the real line so that $\mathcal{I}(x) \prec \mathcal{I}(y)$ implies $x < y$, and $x < y$ implies $\mathcal{I}(x) \preceq \mathcal{I}(y)$.

There is the canonical interval cover $\mathcal{V}_c = \{V_i; i \in \mathbf{2}\}$ with $V_0 = [a, c]$ and $V_1 = [c, b]$. Since the intervals V_u overlap, we have a weaker form of the order preserving property in Proposition 6.

Proposition 6. *Suppose that $u, v \in \overline{\mathbf{2}}^*$, $|u| = |v|$, $u \prec v$, $x \in V_u$, and $y \in V_v$. Then $x \leq y$.*

Proof. By induction on the common length of u and v . ■

An S -unimodal system has at most one (one-sided) stable periodic orbit, and this happens exactly when its kneading sequence $\mathcal{K}(f)$ is finite or periodic. If $\mathcal{K}(f)$ is infinite aperiodic (but possibly preperiodic), then \mathcal{V}_c is a generator so (I, f) is a factor of $(\Sigma_{\mathcal{V}_c}(I, f), \sigma)$. This yields an upper estimate on the complexity of (I, f) .

Theorem 1. *Let \mathfrak{L} be a class of languages closed under factors and concatenations, and let (I, f) be an S -unimodal system whose kneading sequence is neither finite nor periodic. If $\mathcal{L}_{\mathcal{V}_c}(I, f)$ belongs to \mathfrak{L} then (I, f) is of class \mathfrak{L} .*

Proof. For $n > 0$ put $A_n = \mathcal{L}_{\mathcal{V}_c}(I, f) \cap \mathbf{2}^n$. By Theorems II.5.4 and II.6.2 in [6], different points have different itineraries, so $\mathcal{V}_n = \{V_u; u \in A_n\}$ is a separating sequence of interval covers. Since \mathfrak{L} is closed under concatenations, $\mathcal{L}_{\mathcal{V}_n}(I, f) = \mathcal{L}_{\mathcal{V}}^n(I, f)$ belongs to \mathfrak{L} for every n . ■

We now characterize subshifts generated by the canonical covers. For S -unimodal system (I, f) define the upper kneading sequence $\overline{\mathcal{K}}(f) \in \mathbf{2}^{\mathbf{N}}$ as the maximal $u \in \mathbf{2}^{\mathbf{N}}$ for which $f(c) \in V_u$. If the kneading sequence is infinite, then $\overline{\mathcal{K}}(f) = \mathcal{K}(f)$. If $\mathcal{K}(f) = w_0 \dots w_{n-2}$ is finite, and $w_{n-1} =$

$w_0 + \dots + w_{n-2} + 1 \pmod 2$, then $\overline{\mathcal{K}}(f) = w_0 \dots w_{n-2} w_{n-1} \overline{w_0 \dots w_{n-2} w_{n-1}}$. For $w \in \mathbf{2}^{\mathbb{N}}$ put

$$L_w = \{u \in \mathbf{2}^*; (\forall i > 0)(\sigma^i(u) \preceq w)\},$$

$$\Sigma_w = \{u \in \mathbf{2}^{\mathbb{N}}; (\forall i > 0)(\sigma^i(u) \preceq w)\}.$$

A sequence $w \in \mathbf{2}^{\mathbb{N}}$ is called maximal if $w \in \Sigma_w$. If w is maximal, then it can be reconstructed from Σ_w . Indeed, if $v \in \mathbf{2}^{\mathbb{N}}$ is the maximal sequence with the property that $0v \in \Sigma_w$, then $v = w$.

Theorem 2. *Let (I, f) be an S -unimodal system. Then $\mathcal{L}_{V_c}(I, f) = L_{\overline{\mathcal{K}}(f)}$ and $\Sigma_{V_c}(I, f) = \Sigma_{\overline{\mathcal{K}}(f)}$.*

Proof. Suppose that $u \in \mathcal{L}_{V_c}(I, f)$ with $|u| = n$. There exists $x \in V_u$, so $f^i(x) \in V_{\sigma^i(u)}$. If $\sigma^i(u) \not\preceq \overline{\mathcal{K}}(f)$ were not satisfied, then there would exist $j < n$ such that $w_0 \dots w_{j-i} \prec u_i \dots u_j$. By Proposition 6 $f(c) \leq f^i(x)$, so $f^i(x) = f(c)$ and $f(c) \in V_{u_i \dots u_j}$. This is however a contradiction, since $w_0 \dots w_{j-i}$ is the maximal string of length $j - i + 1$ for which $f(c) \in V_{w_0 \dots w_{j-i}}$. Suppose that $u \in L_{\overline{\mathcal{K}}(f)}$. We proceed by induction on the length $n = |u|$. If $n = 1$, then clearly $u \in \mathcal{L}_{V_c}(I, f)$. Suppose that the claim holds for $n - 1$. Then the condition is satisfied for $\sigma(u)$, so it belongs to the language. For $i = 1$ we get $\sigma(u) \preceq \overline{\mathcal{K}}(f)$. If $\sigma(u) \sqsubset \overline{\mathcal{K}}(f)$, then $c \in V_u$. If $\sigma(u) \prec \overline{\mathcal{K}}(f)$, then every $y \in V_{\sigma(u)}$ satisfies $y \leq f(c)$, so there exists $x \in V_u$ with $f(x) = y$. In both cases we have $u \in \Sigma_{V_c}(I, f)$. The equality of subshifts follows by compactness. ■

If $\mathcal{K}(f)$ is finite, then $\overline{\mathcal{K}}(f)$ is preperiodic and there exists a unimodal system (I, g) with $\mathcal{K}(g) = \overline{\mathcal{K}}(f)$. The two systems are, however, quite different: the former has a superstable periodic orbit, while the latter is topologically transitive on a finite union of closed intervals.

We now explore the relation between the complexity of an infinite maximal sequence $w \in \mathbf{2}^{\mathbb{N}}$ and its corresponding subshift Σ_w . Every maximal w can be split in a unique manner into a sequence of blocks $w = \Delta_0 \Delta_1 \dots$ where $|\Delta_0| = 1$, and for every $k > 0$, Δ_k is the initial segment of w up to and including the first difference, so $\Delta_k = w_0 \dots w_{j-1} \widehat{w}_j$ for some $j \geq 0$. If for some k there is no such j , then the last block is infinite, $\Delta_k = w$, and the number of blocks $\|w\| = k + 1$ is finite. Otherwise the number of blocks is infinite, $\|w\| = \infty$. If $w_0 = 0$ then $w = \overline{0}$, so the splitting is $w = 0.\overline{0}$ and $\|w\| = 2$. In all other cases $\Delta_0 = 1$. For every $k > 0$ we have $\Delta_k \prec w$, so Δ_k is even and since $\Delta_0 = 1$, $\Delta_0 \dots \Delta_k$ is odd. Proposition 7 shows that every block Δ_k ends between two consecutive blocks.

Proposition 7. *Let $w \in \mathbf{2}^{\mathbb{N}}$ be a maximal sequence. Then for every $0 < k < \|w\|$ there exists $Q(k) < k$ such that $\Delta_k = \Delta_0 \dots \Delta_{Q(k)-1} \widehat{\Delta_{Q(k)}}$. Put $Q(0) = -1$. Q is called the kneading function and it completely determines the kneading sequence.*

Proof. Suppose that $\Delta_k = \Delta_0 \dots \Delta_{p-1} \Delta$ where $0 < |\Delta| < |\Delta_p|$ for some p . Then $\Delta = w_0 \dots w_{i-1} \widehat{w}_i$ for some i and since $\Delta \prec w$, Δ is even. On the other hand, since $\Delta_k \prec w$, we get $\Delta_0 \dots \Delta_{p-1} \Delta \prec \Delta_0 \dots \Delta_{p-1} w_0 \dots w_i$. Since $\Delta_0 \dots \Delta_{p-1}$ is odd, we get $\Delta \succ w_0 \dots w_i$, which is a contradiction. ■

Example 1 (Feigenbaum) The least kneading sequence of a unimodal system is $0.\overline{0}$. The corresponding unimodal system has either a unique fixed point, or two fixed points both less than the critical point. When the second fixed point passes the critical point, the kneading sequence becomes $1.\overline{1}$, and in a series of period doubling bifurcations we get systems with kneading sequences $1.0.\overline{10}$, $1.0.11.\overline{1011}$, \dots . All these systems have finite splittings and a kneading function $Q(k) = k - 1$. At the limit of these systems there is the Feigenbaum system with infinite splitting and the kneading function $Q(k) = k - 1$, so

$$\mathcal{K}(f) = 1.0.11.1010.10111011.1011101010111010\dots$$

Example 2 (Fibonacci) The Fibonacci unimodal system has the kneading function $Q(0) = -1$, $Q(1) = 0$, $Q(k) = k - 2$ for $k > 1$, so

$$\mathcal{K}(f) = 1.0.0.11.101.10010.10011100.1001110110011\dots$$

Proposition 8. Let $w = \Delta_0 \Delta_1 \dots$ be a maximal sequence with infinite splitting $\|w\| = \infty$. Let G be an infinite graph with vertex set $V_G = \{-1, 0, 1, \dots\}$ having labeled edges $-1 \xrightarrow{0} 0$, $-1 \xrightarrow{1} 0$, $k \xrightarrow{\Delta_k} k + 1$, and $k \xrightarrow{\widehat{\Delta}_k} Q(k) + 1$ for $k \geq 0$. Then Σ_w consists exactly of the labels of infinite paths in G which start at vertex -1 .

Proof. Note that for every $u \in \mathbf{2}^{\mathbb{N}}$ there is at most one path with label u starting at -1 . Let $u \in \Sigma_w$. Then u can be split in a unique way into blocks $u = \Gamma_0 \Gamma_1 \dots$ such that $\Gamma_0 = 10^k$ or $\Gamma_0 = 0^{k+1}$ for some $k \geq 0$, $1 \in \Gamma_1$, and for every $k > 0$, Γ_k is the initial sequence of w up to the last block where there is no difference encountered. In other words there exists $R(k) > 0$ such that $\Gamma_k = \Delta_0 \dots \Delta_{R(k)-1}$ and $\Delta_{R(k)} \not\sqsubseteq \Gamma_{k+1}$. If for some k no difference is encountered, then the number of blocks is $k + 1$ and Γ_k is infinite. We show that $\widehat{\Delta_{R(k)}} \sqsubseteq \Gamma_{k+1}$. Indeed, suppose that $\Delta_0 \dots \Delta_{R(k)-1} \Delta \sqsubset \Gamma_k \Gamma_{k+1} \dots$ with $\Delta = w_0 \dots w_{i-1} \widehat{w}_i$ for some i with $0 < |\Delta| < |\Delta_{R(k)}|$. Then $\Delta \prec w$, but $\Delta_0 \dots \Delta_{R(k)-1} \Delta \prec w$, and $\Delta_0 \dots \Delta_{R(k)-1}$ is odd, so $\Delta \succ w$ and we get a contradiction. It follows that $\widehat{\Delta_{R(k)}} = \Delta_0 \dots \Delta_{Q(R(k))} \sqsubseteq \Gamma_{k+1}$, so there is a path

$$R(k) \xrightarrow{\widehat{\Delta_{R(k)}}} Q(R(k)) + 1 \xrightarrow{\Delta_{Q(R(k))+1}} \dots \xrightarrow{\Delta_{R(k+1)-1}} R(k + 1)$$

and $R(k) \xrightarrow{\Gamma_{k+1}} R(k + 1)$. Put $R(0) = 0$. Since $-1 \xrightarrow{\Gamma(0)} 0 \xrightarrow{\Gamma(1)} R(1)$, $u = \Gamma_0 \Gamma_1 \dots$ corresponds to an infinite path in G . Conversely every infinite path u in G can be split into blocks $u = \Gamma_0 \Gamma_1 \dots$ such that Γ_0 is the longest path with

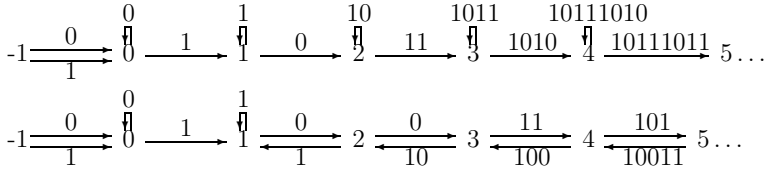


Figure 1: The graphs for the Feigenbaum and Fibonacci systems.

source -1 and target $0 = R(0)$. Γ_1 is the longest path with source $R(0)$ with vertices that increase. Denote by $R(1)$ its target. For every $k > 1$, Γ_k is the longest path with source $R(k - 1)$, going first to $Q(R(k - 1))$ and then continuing to ever increasing vertices $Q(R(k - 1)) + 1, \dots$ until some $R(k)$. Suppose by contradiction that for some $i > 0$ we have $\sigma^i(u) \succ w$. Then there exists k, l , and j such that $\sigma^i(u) \supseteq \Gamma\Gamma_{k+1} \dots \Gamma_l = w_0 \dots w_{j-1} \widehat{w}_j$, where Γ is either a proper final segment of Γ_k or it is empty. But then $\Gamma_k \dots \Gamma_l$ is an initial segment of w , and $\Gamma_{k+1} \dots \Gamma_l \succ w$ which is in contradiction with the maximality of w . The graphs for the Feigenbaum and Fibonacci systems are shown in Figure 1. ■

Proposition 9. *Let $w \in 2^{\mathbb{N}}$ be an infinite maximal sequence that is eventually periodic. Then there exists a finite labeled graph such that Σ_w consists exactly of the labels of infinite paths in G .*

Proof. The splitting of w is either infinite eventually periodic or it is finite. In the former case $w = \Delta_0 \dots \Delta_{q-1} \overline{\Delta_q \dots \Delta_{q+p-1}}$ and the graph constructed in Proposition 8 works for it. (Note that the period $\Delta_q \dots \Delta_{q+p-1}$ is even in this case.) For every $r > q$ we can identify vertices $r, r + p$, and $r + 2p, \dots$ without changing the language. Thus we obtain a finite graph with vertex set $V_G = \{-1, 0, \dots, q + p\}$, and the same edges except that Δ_q leads to $q + 1$ now, $q + p \xrightarrow{\Delta_q} q + 1$. If the splitting of w is finite $w = \Delta_0 \dots \Delta_q$, then the graph constructed in Proposition 8 also works, but there is a unique path from q with infinite label $\Delta_q = w$. (Note that the period $\Delta_0 \dots \Delta_{q-1}$ is odd in this case.) The same language can be obtained by a finite graph with vertex set $V_G = \{-1, 0, \dots, q\}$ and the same edges, except that there is a single edge $q \xrightarrow{\Delta_0 \dots \Delta_{q-1}} q$ starting at q . The graphs for $\overline{101} = 1.0\overline{11}$ and $\overline{100} = 1.0.0.\overline{100}$ are shown in Figure 2. ■

Definition 4. *Let \mathfrak{L} be a class of right-central languages. We say that an infinite sequence $u \in A^{\mathbb{N}}$ is of class \mathfrak{L} , if the language $\{u_n; n \in \mathbb{N}\}$ belongs to \mathfrak{L} .*

Observe that u is regular if and only if it is eventually periodic.

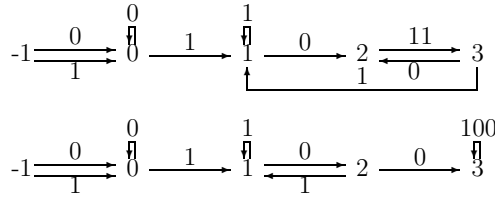


Figure 2: The graphs for $\overline{101} = 1.0.1\overline{1}$ and $\overline{100} = 1.0.0.\overline{100}$.

Theorem 3. Let $w \in 2^N$ be a maximal sequence. For the classes \mathfrak{L} of regular, context-sensitive, and recursive languages, the language L_w is of class \mathfrak{L} if and only if w is of class \mathfrak{L} .

Proof. Note that w can be reconstructed from Σ_w . If v is the maximal sequence with the property that $0v$ belongs to Σ_w , then $v = w$. Suppose that L_w is a regular language. Let G with vertex set $|G|$ be a right-resolving labeled graph and $y_0 \in |G|$ be an initial vertex such that L_w is the set of labels of paths in G which start at y_0 . Construct sequences $y_i \in |G|$, $x_i, v_i \in 2$, $i > 0$ as follows. y_1 is the vertex for which there exists an edge $y_0 \xrightarrow{0} y_1$, and $x_1 = 0$. Suppose that y_i and x_i have already been constructed. Then there exists the unique couple v_i, y_{i+1} such that there exists an edge $y_i \xrightarrow{v_i} y_{i+1}$, and if the other outgoing edge $y_i \xrightarrow{\hat{v}_i} y'$ exists, then either $x_i = 0$ and $\hat{v}_i < v_i$ or $x_i = 1$ and $\hat{v}_i > v_i$. Put $x_{i+1} = x_i + v_i \pmod 2$, thus at each step x_i is the parity of $v_0 \dots v_{i-1}$. Since there are only finitely many possibilities for the pairs (y_i, x_i) , v is either periodic or preperiodic. Since v is the maximal sequence with the property that $0v \in \Sigma_w$, $v = w$, so w is regular. Suppose now that L_w is context-sensitive. Given $u \in 2^*$, construct a word $v \in 2^*$ with length $|u|$ which is maximal with the property that $0v$ belongs to L_w . Since L_w is context-sensitive, this construction might be done in linear space, and $v \sqsubset w$. Then it suffices to verify whether $u = v$. If L_w is recursive, then we get instead a recursive procedure for the recognition of w . Conversely, if w is regular, then Σ_w is regular by Proposition 9. Suppose that w is context-sensitive. We describe a Turing machine that recognizes in linear space whether a given $u \in 2^*$ belongs to L_w . Suppose that u is written on tape 1 of the machine. Using the machine for the recognition of $\{w|_n; n \geq 0\}$ we construct on tape 2 (using as many additional tapes as this machine requires) the initial substring of w of length $|u|$. Consider the alphabet $A = \{0, 1, \bullet\}$ and the word $w^\bullet = \Delta_0 \bullet \Delta_1 \bullet \dots \in A^N$ obtained from the splitting of w by inserting \bullet between the blocks. Construct on tape 3 the initial substring of w^\bullet obtained from tape 2. Its length is at most $2|u|$ and it can be constructed in linear space. Finally, verify whether u corresponds to a path in the graph associated to w using tapes 2 and 3. First skip u_0 and compare $\sigma(u)$ bit by bit with the string on tape 3 ignoring

the bullets. If a discrepancy is encountered in the middle of the block, the input word is rejected. If a discrepancy is encountered at the end of a block Δ_k , the machine finds on tape 3 the beginning of the block Δ_k . Then it compares Δ_k with the initial string on tape 2 (ignoring the bullets) until the first discrepancy is encountered, which is at the end of block $\Delta_{Q(k)}$. Mark this position on tape 2, reposition the pointers on both tapes 2 and 3 to the beginning, and advance them simultaneously, until the marker on tape 2 is encountered. In this situation, the pointer at tape 3 is at the end of block $\Delta_{Q(k)}$, and the interrupted comparison of tapes 1 and 3 can be resumed. If no difference is found, u is accepted. If w is recursive, then we get a recursive procedure for the recognition of Δ_w . ■

5. Periodic languages

In conformity with Definition 2 we now consider general interval covers. Every interval cover of $I = [a, b]$ is determined by an increasing sequence $a = d_0 < \dots < d_n = b$, and we can then write $\mathcal{V} = \{V_i; i \in A\}$, where $A = \{0, 1, \dots, n - 1\}$, and $V_i = [d_i, d_{i+1}]$.

Theorem 4. *An S -unimodal system (I, f) has bounded periodic languages if and only if it has a unique fixed point.*

Proof. If the only fixed point of (I, f) is a , then $f(x) < x$ for every $x \in I$. If \mathcal{V} is an interval cover with $a \in V_0$, then there exists m such that for every $u \in \Sigma_{\mathcal{V}}(I, f)$ and every $i > m$, $u_i = 0$, so $\mathcal{L}_{\mathcal{V}}(I, f)$ is bounded periodic. On the other hand, if (I, f) has a fixed point different from a then it is not equicontinuous and by Proposition 4 it does not have bounded periodic languages. ■

Theorem 5. *Let (I, f) be an S -unimodal system, then the following conditions are equivalent.*

1. (I, f) has periodic languages.
2. (I, f) has zero topological entropy.
3. The period of every periodic point of (I, f) is a power of 2.
4. $\mathcal{K}(f) \preceq W = 1.0.11.1010 \dots$ (Feigenbaum system).

Proof. 1 \Rightarrow 2: Proposition 5.

2 \Leftrightarrow 3: A theorem of Misiurewicz (Proposition VIII.34 in [4]).

3 \Rightarrow 4: Suppose by contradiction that $\mathcal{K}(f) \succ W$, let $\mathcal{K}(f) = \Delta_0 \Delta_1 \dots$ be its splitting and let Δ_k be the first block different from the splitting of W . Then $Q(k) = j - 1$ for some $j < k$ and there exists a periodic point $\widehat{W_j W_j W_j \dots W_{k-1}}$ of period $2^{k-1} + 2^{j-1}$.

4 \Rightarrow 1: Suppose first $\mathcal{K}(f) \prec W$. Then (I, f) has a finite number of periodic

points and for every $x \in I$, $f^n(x)$ converges to a periodic orbit (see Proposition VI.10 in [4]). Denote by P the set of periodic points of (I, f) . There exists a system of closed disjoint intervals $\{V_p; p \in P\}$ such that if p is stable, then $f(V_p) \subset V_{f(p)}$, and if p is unstable, then $f(V_p) \supset V_{f(p)}$. The system $\{V_p; p \in P\}$ can be extended into an interval cover $\{V_a; a \in A\}$ such that $P \subset A$ and for every $a \in A$ which is not an unstable periodic point there exists a unique $b \in A$ with $f(V_a) \subseteq V_b$. Consider a labeled graph G with vertex set $|G| = A$ and labeled edges $a \xrightarrow{a} b$ if and only if $f(V_a) \cap V_b \neq \emptyset$. Then every communicating set of G is a cycle, and G presents $\mathcal{L}_{\mathcal{V}}(I, f)$. Clearly a separating sequence of interval covers can be constructed in this way. Suppose finally that (I, f) is the Feigenbaum system. Then there exists a system of periodic points $\{p_u; u \in \mathbf{2}^*\}$, a system of closed intervals $\{J_u; u \in \mathbf{2}^*\}$ such that $J_u \cap J_v = \emptyset$ for $|u| = |v|$ and $u \neq v$, $J_u \subset J_v$ for $u \sqsubset v$, $p_u \in J_u - (J_{u_0} \cup J_{v_1})$, and there exists a length preserving function $g: \mathbf{2}^* \rightarrow \mathbf{2}^*$ such that for every n , g restricted to $\mathbf{2}^n$ is a permutation and $f(p_u) = p_{g(u)}$, $f(J_u) \subseteq J_{g(u)}$. Moreover, for every n the union $\cup\{J_u; u \in \mathbf{2}^n\}$ is an attractor and finally for every $u \in \mathbf{2}^{2N}$ the intersection $\cap\{J_v; v \sqsubset u\}$ is a one-point set. Given $n > 0$ we construct an interval cover \mathcal{V}_n as follows. Since $\cup\{J_u; u \in \mathbf{2}^n\}$ is an attractor, there exists a system of disjoint closed intervals $\{V_u; u \in \mathbf{2}^n\}$ such that $J_u \subset \text{int}(V_u)$, $f(V_u) \subset \text{int}V_{g(u)}$. Since for every u with $|u| < n$, p_u is a repelling periodic point, there exists a system of disjoint closed intervals $\{V_u; |u| < n\}$ such that $p_u \in \text{int}(V_u)$, $f(V_u) \supset V_{f(u)}$. We can again extend $\{V_u; |u| \leq n\}$ into an interval cover $\mathcal{V}_n = \{V_a; a \in A_n\}$ so that for every $a \in A_n$ which is not an unstable periodic point there exists unique $b \in A_n$ with $f(V_a) \subset V_b$. Thus $\mathcal{L}_{\mathcal{V}_n}(I, f)$ is a periodic language. It is easy to see that \mathcal{V}_n can be constructed as a separating sequence. ■

6. Regular languages

By Theorems 1, 2, and 3, every S -unimodal system with a preperiodic kneading sequence has regular languages. In the last section we have seen that systems with zero topological entropy have regular languages too. Besides these systems there are many others with stable periodic orbits, and at their limits there are infinitely renormalizable systems. All these systems have regular languages too.

Definition 5 (cf. [14]) An S -unimodal system (I, f) is renormalizable with period $n > 1$, if there exists a proper (restrictive) subinterval $J \subset I$, such that the interiors of $J, f(J), \dots, f^{n-1}(J)$ are disjoint, $f^n(J) \subseteq J$, $f^n(\partial J) \subseteq \partial J$, $f^{n-1}(J)$ contains the critical point, and J is maximal with these properties. An S -unimodal system is infinitely renormalizable if it has restrictive subintervals of arbitrary large periods n .

Let J be a proper restrictive subinterval of period n . Put $J = J_0 = [a_0, b_0]$, $J_i = f^i(J)$, $a_i = f^i(a_0)$, $b_i = f^i(b_0)$ for $i < n$. Put $w_i = \mathcal{K}(f)_i$ for $i < n - 1$, and $w_{n-1} = w_0 + \dots + w_{n-2} + 1 \pmod 2$, so that $w = w_0 \dots w_{n-1}$ is odd. Then for $i < n - 1$, $f: J_i \rightarrow J_{i+1}$ is increasing if $w_i = 0$, and decreasing if $w_i = 1$,

$\partial J_i = \{a_i, b_i\}$, and there exists unique points $c_i \in J_i$ with $f^{n-i-1}(c_i) = c \in J_{n-1}$. If $w_{n-1} = 1$, then $f^{n-1} : J_0 \rightarrow J_{n-1}$ is increasing, so $a_{n-1} < c < b_{n-1}$, and $f^n : J_0 \rightarrow J_0$ is increasing on $[a_0, c_0]$ and decreasing on $[c_0, b_0]$. If $w_{n-1} = 0$, then $f^{n-1} : J_0 \rightarrow J_{n-1}$ is decreasing, so $b_{n-1} < c < a_{n-1}$, and $f^n : J_0 \rightarrow J_0$ is increasing on $[a_0, c_0]$ and decreasing on $[c_0, b_0]$. Thus in both cases (J_0, f^n) is S -unimodal. It follows $\mathcal{I}(a_{n-1}) = \widehat{w_{n-1}}\overline{w}$, $\mathcal{I}(b_{n-1}) = w_{n-1}\overline{w}$, $\mathcal{I}(a_0) = \overline{w}$, $\mathcal{I}(b_0) = w\overline{w}$. For the upper kneading sequences we have

$$\begin{aligned} \overline{\mathcal{K}}(f)_{jn+i} &= w_i && \text{for } j > 0, \quad 0 \leq i < n - 1, \\ \overline{\mathcal{K}}(f)_{jn+n-1} &= \overline{\mathcal{K}}(f^n)_j + w_{n-1} + 1 \pmod 2 && \text{for } j \geq 0. \end{aligned}$$

We consider the renormalization interval cover $\mathcal{W}_r = \{W_i; i \in C\}$, where $C = \{0, c, 1\}$ and

$$\begin{aligned} W_0 &= [a, a_{n-1}] & W_c &= [a_{n-1}, b_{n-1}] & W_1 &= [b_{n-1}, b] & \text{if } a_{n-1} < b_{n-1} \\ W_0 &= [a, b_{n-1}] & W_c &= [b_{n-1}, a_{n-1}] & W_1 &= [a_{n-1}, b] & \text{if } b_{n-1} < a_{n-1}. \end{aligned}$$

Lemma 1. *Let (I, f) be renormalizable with period $n > 1$. Let $u \in C^*$ and put $k = \min\{i < |u|; u_i = c\}$ or $k = |u|$ if $u \in \mathbf{2}^*$. Then $u \in \mathcal{L}_{\mathcal{W}_r}(I, f)$ if and only if the following conditions are satisfied.*

1. $u_{k+nj+i+1} = w_i$ for $j \geq 0, 0 \leq i < n - 1, k + nj + i + 1 < |u|$.
2. $u_{k+nj} \in \{\widehat{w_{n-1}}, c\}$ for $j \geq 0, k + nj + i + 1 < |u|$.
3. There exists $v_k \in \mathbf{2}$ such that for $v = u_0 \dots u_{k-1} v_k w_0 \dots w_{n-2}$ (or for $v = u$ if $k = |u|$) we have $\sigma^i(v) \preceq \mathcal{I}(a_0) = \overline{w}$ for $0 < i < k$.

Proof. Suppose for simplicity that $a_{n-1} < b_{n-1}$. Let $u \in \mathcal{L}_{\mathcal{W}_r}(I, f)$ and pick a point $x \in V_u$. We can assume that $f^i(x) \neq c$ for $i < |u| + n$, since otherwise there exists in the vicinity of x a point which belongs to V_u also. Put $v_k = 0$ if $f^k(x) < c$, and $v_k = 1$ if $f^k(x) > c$. Then $v \sqsubset \mathcal{I}(x)$, so the conditions 1 and 2 are satisfied. Suppose that $\sigma_i(v) \succ \mathcal{I}(a_0)$ for some $0 < i < k$. Then $f^i(x) > a_0$, and therefore $a_{n-1} < f_{i-1}(x) < b_{n-1}$, which is a contradiction. Suppose conversely that the conditions 1, 2, and 3 are satisfied. Assume that there exists a minimal $j < |u|$ such that $\sigma^j(v)$ is an initial substring of either $\mathcal{I}(a_{n-1})$ or $\mathcal{I}(b_{n-1})$. Put $x_j = a_{n-1}$ in the former case and $x_j = b_{n-1}$ in the latter case. Obviously, if $k < |u|$, then $j \leq k$. Then successively for $i = j - 1, j - 2, \dots$ we construct $x_i \in V_{\sigma^i(v)}$ with $f(x_i) = x_{i+1}$. This is always possible since $\sigma^i(v) \prec \mathcal{I}(a_0) \preceq \mathcal{K}(f)$ as otherwise $\sigma^{i-1}(v)$ would be an initial substring of either $\mathcal{I}(a_{n-1})$ or $\mathcal{I}(b_{n-1})$. Thus $x_0 \in V_v$, and since $f^i(x_0) \notin (a_{n-1}, b_{n-1})$ for $0 < i < j, x_0 \in W_u$. If there exists no $j < |u|$ such that $\sigma^j(v)$ is an initial substring of either $\mathcal{I}(a_{n-1})$ or $\mathcal{I}(b_{n-1})$, then for all $i > 0, \sigma^i(v) \prec \mathcal{I}(a_0) \preceq \mathcal{K}(f)$, so $u \in \mathcal{L}_{V_c}(I, f)$, and there exists $x \in V_u$. Again $f^i(x) \notin (a_{n-1}, b_{n-1})$ for $0 < i$, so $x \in W_u$. ■

Proposition 10. *If (I, f) is renormalizable, then $\mathcal{L}_{\mathcal{W}_r}(I, f)$ is a regular language.*

Proof. By Proposition 9 there exists a finite labeled graph G for $\Sigma_{\overline{w}}$. Extend G as follows. Let i be a vertex such that there exists a path with source i and label $\overline{w_{n-1}w_0 \dots w_{n-2}}$ or $\overline{w_{n-1}w_0 \dots w_{n-2}}$. Add to G new vertices v_0, \dots, v_{n-1} and labeled edges $i \xrightarrow{c} v_0 \xrightarrow{w_0} \dots \xrightarrow{w_{n-2}} v_{n-1} \xrightarrow{\overline{w_{n-1}}} v_0$ and $v_{n-1} \xrightarrow{c} v_0$. Then the extended graph is a presentation for $\mathcal{L}_{W_r}(I, f)$. ■

Theorem 6. *Every S -unimodal system whose kneading sequence is either finite or periodic has regular languages.*

Proof. We can suppose that $\overline{\mathcal{K}}(f)$ is neither of $\lambda, 1, \overline{0}$, nor $\overline{1}$, as in these cases Theorem 5 applies. Let n be the period of $\mathcal{K}(f)$ if $\mathcal{K}(f)$ is periodic, and $n = |\mathcal{K}(f)| + 1$ if it is finite. Then $n > 1$ and there exists a stable or semistable periodic orbit of period n which attracts the critical point. It follows that there exists an interval J containing the critical point such that $f^n(J) \subseteq J$, and $J, f(J), \dots, f^{n-1}(J)$ are disjoint. By Lemma 5.1 in [14], (I, f) is renormalizable and the renormalized unimodal system (W_c, f^n) has a stable fixed point. Given $m > 0$, there exists an interval cover \mathcal{V}_m of W_r , such that $\text{diam}(\mathcal{V}_m) < 2^{-m}$ and $\mathcal{L}_{\mathcal{V}_m}(W_c, f^n)$ is regular. There exists p such that $\mathcal{U}_m = (\mathcal{V}_m \cup \{W_0, W_1\})^p$ has diameter less than 2^{-m} . This follows from a theorem of Misiurewicz (Theorem II.5.2 in [6]), saying that for a unimodal system with stable periodic orbit the only homtervals are those that are eventually mapped into $V_{\mathcal{K}(f)}$. By a choice of \mathcal{V}_m , \mathcal{U}_{m+1} will be a refinement of \mathcal{U}_m and $\mathcal{L}_{\mathcal{U}_m}(I, f)$ is regular. ■

Theorem 7. *Every infinitely renormalizable S -unimodal system has regular languages.*

Proof. We have a decreasing sequence of intervals $I = J_0 \supset J_1 \dots$, such that (J_k, f^{n_k}) is S -unimodal. Let \mathcal{V}_k be an interval cover consisting of J_k and connected components of $J_i - J_{i+1}$ for $i < k$. Then $\mathcal{L}_{\mathcal{V}_k}(I, f)$ is regular. There exists an increasing sequence m_k such that $\mathcal{V}_k^{m_k}$ is a separating sequence of interval covers which all yield regular languages. ■

Corollary 1. *Every S -unimodal system with a finite or eventually periodic kneading sequence has regular languages. Every infinitely renormalizable S -unimodal system has regular languages.*

7. Nonrecurrent critical point

For S -unimodal systems whose kneading sequence is neither finite nor periodic, the language complexity of the kneading sequence yields an upper estimate on the language complexity of the system in question (Theorem 1). For the S -unimodal systems whose critical point is not recurrent, we have a lower estimate as well. This is a variation on Theorem 3.

Theorem 8. *Let (I, f) be an S -unimodal system with a nonrecurrent critical point. Then for the classes \mathfrak{L} of regular, context-sensitive and recursive languages, if (I, f) is of class \mathfrak{L} , then $\mathcal{K}(f)$ is of class \mathfrak{L} also.*

Proof. Let $\varepsilon > 0$ be such that for all $n > 0$ we have $|f^n(c) - c| > \varepsilon$. There exists an interval cover $\mathcal{V} = \{V_i; i \in A\}$ with $\text{diam}(\mathcal{V}) < \varepsilon$ such that $\mathcal{L}_{\mathcal{V}}(I, f)$ is in \mathfrak{L} . If \mathcal{V} contains c among its end points, then it is a refinement of the canonical cover \mathcal{V}_c , so $\mathcal{L}_{\mathcal{V}_c}(I, f)$ is also in \mathfrak{L} . If not, then there exist real numbers $a = d_0 < \dots < d_k < c < d_{k+1} < \dots < d_m = b$, such that $A = \{0, \dots, m - 1\}$, and $V_i = [d_i, d_{i+1}]$. Put $B = A - \{k\}$ and define map $\nu : B \rightarrow \mathbf{2}$ by $\nu(i) = 0$ if $i < k$ and $\nu(i) = 1$ if $i > k$. Define order \prec on \overline{B}^* by $u \prec v$ if and only if for $k = \max\{i; u_i = v_i\}$ either

$$\nu(u_0) + \dots + \nu(u_{k-1}) \bmod 2 = 0 \text{ and } u_k \prec v_k,$$

or

$$\nu(u_0) + \dots + \nu(u_{k-1}) \bmod 2 = 1 \text{ and } u_k \succ v_k.$$

Here $0 \prec \dots \prec k - 1 \prec \lambda \prec k + 1 \prec \dots \prec m - 1$. Let $w \in B^N$ be the maximal sequence satisfying $kw \in \Sigma_{\mathcal{V}}(I, f)$. We claim $\nu(w) = \mathcal{K}(f)$. Indeed, consider alphabet $C = B \cup \{k_0, k_1\}$ and the interval cover $\mathcal{W} = \{W_i; i \in C\}$ which is the common refinement of \mathcal{V} and \mathcal{V}_c . We again have an order \prec on \overline{C}^* which corresponds to the order on the real line. Suppose that $v \in C^N$ is the maximal sequence with the property that both k_0v and k_1v belong to $\Sigma_{\mathcal{W}}(I, f)$. Then $f(c) \in W_v$, and since $f^i(c)$ never visits neither V_{k_0} nor V_{k_1} , we have $v = w \in B^N$, and $\nu(v) = \mathcal{K}(f)$. Suppose that $\mathcal{L}_{\mathcal{V}}(I, f)$ is a regular language. Let G with the vertex set $|G|$ be a right-resolving labeled graph and $y_0 \in |G|$ be an initial vertex such that $\mathcal{L}_{\mathcal{V}}(I, f)$ is the set of labels of paths which start at y_0 . We construct sequences $y_i \in |G|$, $v_i \in B$, and $x_i \in \mathbf{2}$ as follows. y_1 is the vertex for which there exists an edge $y_0 \xrightarrow{k} y_1$, and $x_1 = 0$. Suppose that y_i and x_i have already been constructed. Then there exists the unique couple v_i, y_{i+1} , such that there exists an edge $y_i \xrightarrow{v_i} y_{i+1}$, and if $y_i \xrightarrow{v'}$ y' is another edge coming out from y_i , with $v_i \neq v' \in B$, then either $x_i = 0$ and $v' \prec v_i$, or $x_i = 1$ and $v' \succ v_i$. Put $x_{i+1} = x_i + v_i \bmod 2$. Thus at each step, x_i is the parity of $v_0 \dots v_{i-1}$ and $\mathcal{K}(f) = \nu(v_0v_1\dots)$. Since there are only finitely many possibilities for y_i and x_i , $\mathcal{K}(f)$ is either periodic or preperiodic, and therefore regular. Suppose that $\mathcal{L}_{\mathcal{V}}(I, f)$ is context-sensitive. For given $u \in \mathbf{2}^*$ construct a word $v \in B^*$ with length $|v| = |u|$ that is maximal with the property that kv belongs to $\mathcal{L}_{\mathcal{V}}(I, f)$, and verify whether $\nu(v) = u$. If $\mathcal{L}_{\mathcal{V}}(I, f)$ is recursive, then we get instead a recursive procedure for the recognition of $\mathcal{K}(f)$. ■

Example 3. Every sequence $w = 1.0.0.\Delta_3.\Delta_4\dots$ with $\Delta_i \in \{11, 101\}$ is maximal, and its corresponding unimodal system has a nonrecurrent critical point. Thus there exist S -unimodal systems which are context-sensitive but not regular, those that are recursive but not context-sensitive, and also those that are not recursive.

For the systems with a recurrent critical point that are not infinitely renormalizable, no lower estimate has been obtained. These systems include the Fibonacci systems from Example 2 with a kneading sequence easily shown to be context-sensitive. The regularity of the Fibonacci system remains an open question.

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