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Breakdown Processes of Conservative Systems

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Abstract. This paper deals with the regenerative reliability system in terms of the birth and death process $n(t)$. Using the first transition time between two fixed states, the conservative breakdown process $a(t)$ of the system is defined in such a way that the system falls into the breakdown state when $n(t)$ attains the upper limit state and falls into the working state when $n(t)$ attains the lower limit state. The conservative breakdown process and the corresponding "k-out-of-n" breakdown process are compared. It is concluded that the conservative process is more stable when considering the coefficient of variation of the working or breakdown time and also in the sense of forming the clusters of "on" and "off" changes in time t.

1. Conservative breakdown processes

Let $n(t)$, $t \geq 0$, denote the birth and death process defined on the state set $\{0, 1, \dots, N\}$, (N finite or infinite) with the birth rate λ_n and the death rate μ_n when $n(t) = n$. We consider that each state of the process is transitive, that is, we assume that $\lambda_i > 0$, $\mu_{i+1} > 0$, and $i = 0, 1, \dots, N-1$. If N is infinite then the conditions for stationarity and ergodicity of the process are assumed (see [2]):

$$\sum_{n=0}^{\infty} \theta_n < \infty, \qquad \sum_{n=0}^{\infty} \frac{1}{\lambda_n \theta_n} \sum_{i=0}^{n} \theta_i = \infty$$

where

$$\theta_0 = 1, \qquad \theta_n = \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}.$$

Let us consider the process to be right-continuous, $n(0) = 0$, and let $0 \leq L < U \leq N$ be fixed integers. Let us define the conservative alternating process $a(t)$, $t \geq 0$, in the following way.

1. $a(0) = 1$.
2. If $n(\tau) < U$ for $0 < \tau < t$, $n(t) = U$, then $a(\tau) = 1$ for $0 < \tau < t$, $a(t) = 0$.
3. If $a(t_0) = 0$, $n(\tau) > L$ for $t_0 < \tau < t$, $n(t) = L$, then $a(\tau) = 0$ for $t_0 < \tau < t$, $a(t) = 1$.
4. If $a(t_0) = 1$, $n(\tau) < U$ for $t_0 < \tau < t$, $n(t) = U$, then $a(\tau) = 1$ for $t_0 < \tau < t$, $a(t) = 0$.

The realization of a birth and death process and the conservative alternating process corresponding to it for $L = 1$ and $U = 4$ are given in Figure 1.

The reliability interpretation of our processes is as follows. Let us consider a system of N elements, let $n(t)$ denote the number of failed elements in the system and let $a(t)$ denote the state of the system in time t. Note that if $L = U - 1$ then the system has the "U-out-of-N" structure. Let $T_{k,n}$, $k \neq n$, denote the first transition time from state k to state n. In the conservative alternating process (excluding the working time of the system starting at $t = 0$) the working times are a sequence of probabilistic copies of a random variable $T_{L,U}$, the breakdown times are a sequence of probabilistic copies of a random variable $T_{U,L}$. The properties of the birth and death process also guarantee that consecutive working and breakdown times are mutually independent.

The reliability system is characterized by probability distribution functions of the working and breakdown times. Two reliability characteristics of the distributions are considered: the moments of the probability distribution functions and the probability of a system failure in the initial phase of the working time of the system. The moments are useful when describing the regenerative system. The other characteristics are useful when the system has to work in fixed time only and its further existence is not of interest [4].

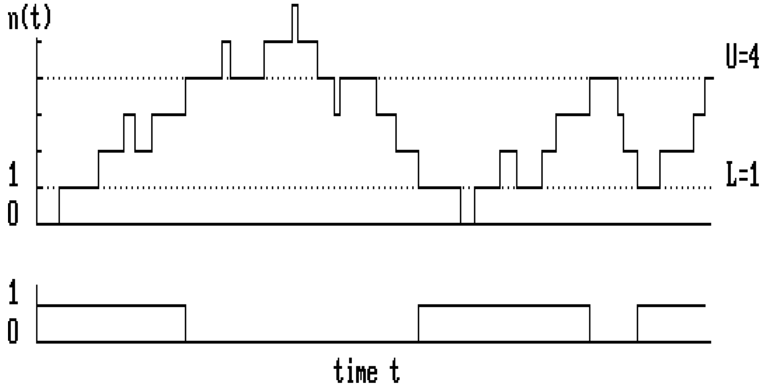

Figure 1: The realization of a birth and death process and the alternating conservative process.

Note that for the birth and death type random walk the first transition time to the given state set has a large variability. Nevertheless, in the sequence of consecutive sojourn times in the given state, sets are stochastically independent but in time t clusters of "on" and "off" changes are observed. The examples show that the conservative system is more stable than the corresponding "k-out-of-n" system when considering the coefficient of variation of the first transition times and also in the sense of the formation of clusters.

2. Basic mathematical equations

Let $n > 0$ be a fixed integer. Using a semi-Markov process as the structure of a birth and death process the following equalities are satisfied in distributions:

$$\begin{aligned} &T_{0,n} \stackrel{\mathrm{d}}{=} e_0 + T_{1,n}, \\ &T_{k,n} \stackrel{\mathrm{d}}{=} e_k + (1-\delta_k)T_{k-1,n} + \delta_k T_{k+1,n}, \qquad k = 1, 2, \ldots, n-1, \end{aligned} \tag{1}$$

where $T_{n,n} = 0$, all random variables $(e_k,\ \delta_k,\ T_{k-1,n},\ T_{k+1,n})$ are mutually independent, e_k is exponentially distributed with parameter $\lambda_k + \mu_k$, and δ_k is the binary random variable, $P(\delta_k = 1) = p_k = \lambda_k/(\lambda_k + \mu_k)$, $P(\delta_k = 0) = q_k = 1 - p_k$.

System (1) enables the analysis of moments of the first transition times, the distribution functions of the preceding random variables, and estimation of the distribution function.

2.1 Laplace–Stieltjes transform of the distribution

Let $f^*_{k,n}(s) = \mathrm{E}(\exp(-sT_{k,n}))$, $\mathrm{Re}(s) > 0$, denote the Laplace–Stieltjes transform of the distribution $F_{k,n}(x) = P(T_{k,n} \leq x)$, $x \geq 0$. From system (1) we find the following (see also [1, 2]).

Proposition 1. If $k < n$, then

$$\begin{aligned} (s+\lambda_0)f^*_{0,n}(s) - \lambda_0 f^*_{1,n}(s) &= 0, \\ -\mu_k f^*_{k-1,n}(s) + (s+\lambda_k+\mu_k)f^*_{k,n}(s) - \lambda_k f^*_{k+1,n}(s) &= 0, \end{aligned} \tag{2}$$

$k = 1, 2, \ldots, n-1$, where $f^*_{n,n}(s) = 1$.

2.2 Moments

Let us denote $m_{k,n} = \mathrm{E}(T_{k,n})$, $\sigma^2_{k,n} = \mathrm{Var}(T_{k,n})$.

Theorem 1. *If $k < n$, then*

$$m_{k,n} = \sum_{j=k}^{n-1} d_j, \qquad \sigma^2_{k,n} = \sum_{j=k}^{n-1} c_j, \qquad k = 0, 1, \ldots, n-1;$$

where

$$\begin{aligned} d_j &= \frac{1}{\lambda_j \theta_j} \sum_{i=0}^{j} \theta_i, \\ c_j &= \frac{1}{\lambda_j \theta_j} \sum_{i=0}^{j} \theta_i u_i, \\ u_j &= \mu_j d_{j-1}^2 + \lambda_j d_j^2, \qquad j = 0, 1, \ldots . \end{aligned}$$

2.3 Estimation of the distribution function

Let $f_{k,n} = F'_{k,n}$ denote the density of probability distribution function $F_{k,n}$.

Theorem 2. *If $x \geq 0$, then*

$$F_{n-k,n}(x) = \frac{1}{k!} B_{n-k,n}^{(k)} x^k + \frac{1}{(k+1)!} B_{n-k,n}^{(k+1)} x^{k+1} + o(x^{k+1}), \qquad x \to 0, \tag{3}$$

where

$$B_{n-k,n}^{(k)} = \prod_{j=n-k}^{n-1} \lambda_j, \qquad B_{n-k,n}^{(k+1)} = -B_{n-k,n}^{(k)} \sum_{j=n-k}^{n-1} (\lambda_j + \mu_j).$$

Remark

Let $\succ_{\mathrm{d}}$ denote the ordering of the random variables in distribution (i.e., $X \succ_{\mathrm{d}} Y$ iff $P(X > x) \geq P(Y > x)$). From system (1) it follows that $T_{k,n} \succ_{\mathrm{d}} e_k + \delta_k T_{k+1,n}$, hence $T_{k,n} \succ_{\mathrm{d}} e_k + \delta_k e_{k+1} + \cdots + (\delta_k \cdots \delta_{n-2}) e_{n-1}$, where e_k, δ_k are as defined in system (1). It may be verified that the estimation, equation (3), is a consequence of the above ordering, hence it is an upper estimation.

2.4 Supplementary results

If $k > n$ and N is fixed, then $T_{k,n}$ may be given using the process $\bar{n}(t) = N - n(t)$, $t \geq 0$, for which $\bar{\lambda}_i = \mu_{N-i}$, $\bar{\mu}_i = \lambda_{N-i}$, $i = 0, 1, \ldots, N$. Hence the equality $T_{k,n} \stackrel{\mathrm{d}}{=} \bar{T}_{\bar{n},\bar{k}}$ follows, where $\bar{k} = N - n$, $\bar{n} = N - k$.

Proposition 2. *If $n(t)$, $t \geq 0$, is the stationary birth and death process, then the conservative alternating process $a(t)$, $t \geq 0$, is stationary too and*

$$P(a(t) = 1) = \frac{E(T_{L,U})}{E(T_{L,U}) + E(T_{U,L})} = \left(\sum_{j=L}^{U-1} \frac{1}{\lambda_j p_j} \sum_{i=0}^{j} p_i \right) \left(\sum_{j=L}^{U-1} \frac{1}{\lambda_j p_j} \right)^{-1},$$

where $p_j = P(n(t) = j) = \theta_j (\sum_{j=0}^{\infty} \theta_j)^{-1}$, $j = 0, 1, \ldots, U-1$.

Table 1: The expected values of $T_{k,n}$.

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	0.111	0.250	0.433	0.690	1.097	1.855	3.704	10.676	67.454
1		0.139	0.321	0.579	0.986	1.744	3.593	10.565	67.343
2			0.183	0.440	0.847	1.605	3.454	10.426	67.204
3				0.258	0.664	1.422	3.271	10.244	67.021
4					0.406	1.164	3.013	9.986	66.763
5						0.758	2.607	9.579	66.357
6							1.849	8.821	65.599
7								6.972	63.750
8									56.778

Table 2: The coefficients of variation $\mathrm{Var}^{1/2}(T_{k,n})/\mathrm{E}(T_{k,n})$ of $T_{k,n}$.

$k\backslash n$	1	2	3	4	5	6	7	8	9
0	1.000	0.745	0.654	0.624	0.637	0.692	0.789	0.904	0.982
1		1.077	0.809	0.719	0.700	0.734	0.813	0.914	0.983
2			1.165	0.882	0.796	0.792	0.844	0.926	0.985
3				1.261	0.962	0.881	0.889	0.942	0.988
4					1.355	1.039	0.959	0.966	0.992
5						1.421	1.088	1.005	0.998
6							1.419	1.085	1.009
7								1.320	1.038
8									1.154

3. Example

Let $N > 1$ be a fixed integer, $\lambda_n = (N-n)\lambda$, $\mu_n = n\mu$, $n = 0, 1, \ldots, N$. Then $n(t) \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N} a_i(t)$, where $a_i(t)$, $i = 1, 2, \ldots, N$, are the breakdown processes of independent elements for which the working time is exponentially distributed with parameter λ and the breakdown time is exponentially distributed with parameter μ.

Tables 1 and 2 give the expected value and the coefficient of variation of the random variables $T_{k,n}$ for $N = 9$, $\lambda = \mu = 1$. It is proved [1,3] that the probability distribution function of the first transition times to the given state, under a suitably defined passing to the limit, is exponentially distributed. The sufficient condition for this conjecture is limit 1 for the coefficient of variation.

The diagonal of Table 2 shows that if n increases then the coefficient of variation of the "n-out-of-N" systems tends to 1 rather slowly. The special examples also give this index for the conservative systems and corresponding "k-out-of-n" systems. Let $N = 9$, $L = 4$, $U = 5$, $\lambda = \mu = 1$. For the 5-out-of-9 system we have $T_{4,5} \stackrel{\mathrm{d}}{=} T_{5,4}$, $\mathrm{E}(T_{4,5}) = 0.406$, $\mathrm{Var}^{1/2}(T_{4,5})/\mathrm{E}(T_{4,5}) =$

1.355, and for the conservative system we have $T_{3,6} \overset{\mathrm{d}}{=} T_{6,3}$, $\mathrm{E}(T_{3,6}) = 1.422$, $\mathrm{Var}^{1/2}(T_{3,6})/\mathrm{E}(T_{3,6}) = 0.881$. In both cases the stationary probability of working state of the system is $P(a(t) = 1) = 0.5$.

In the asymmetry case let us consider the k-out-of-n system in which the working time is $T_{5,6}$ and the breakdown time is $T_{3,4}$. We have $\mathrm{E}(T_{5,6}) = 0.758$, $\mathrm{E}(T_{3,4}) = 0.258$, $\mathrm{Var}^{1/2}(T_{5,6})/\mathrm{E}(T_{5,6}) = 1.421$, $\mathrm{Var}^{1/2}(T_{3,4})/\mathrm{E}(T_{3,4}) = 1.261$. For the conservative system let us assume $T_{4,7}$ for the working time and $T_{2,5}$ for the breakdown time. We have $\mathrm{E}(T_{4,7}) = 3.013$, $\mathrm{E}(T_{2,5}) = 0.847$, $\mathrm{Var}^{1/2}(T_{4,7})/\mathrm{E}(T_{4,7}) = 0.959$, $\mathrm{Var}^{1/2}(T_{2,5})/\mathrm{E}(T_{2,5}) = 0.796$.

Now the stationary probability of working state of the system $P(a(t) = 1)$ changes from 0.746 in the first case to 0.781 in the second.

Now we consider the problem of estimation of the probability distribution function of the first passage time. From Theorem 2 we obtain

$$F_{4,5}(x) = \lambda_4 x - \frac{1}{2}\lambda_4(\lambda_4 + \mu_4)x^2 + o(x^2),$$
$$F_{3,6}(x) = \frac{1}{6}\lambda_3\lambda_4\lambda_5 x^3 - \frac{1}{24}\lambda_3\lambda_4\lambda_5(\lambda_3 + \mu_3 + \lambda_4 + \mu_4 + \lambda_5 + \mu_5)x^4 + o(x^4).$$

For the working time of the 4-out-of-9 system we have $F_{4,5}(x) = 5x - 22.5x^2 + o(x^2)$, and for the working time of the conservative system we have $F_{3,6}(x) = 20x^3 - 135x^4 + o(x^4)$. Let $x = 0.0222$. Then $F_{4,5}(x) \cong 0.1$, $F_{3,6}(x) \cong 0.0002$.

The example shows that under a similar probability of the working state of the system the conservative system is more stable in the sense that the changes "on" and "off" are not cluster forming.

Appendix

Proof of Proposition 1. From system (1) there immediately follows

$$f^*_{0,n}(s) = \frac{\lambda_0}{s + \lambda_0} f^*_{1,n}(s),$$
$$f^*_{k,n}(s) = \frac{\lambda_k + \mu_k}{s + \lambda_k + \mu_k}(q_k f^*_{k-1,n}(s) + p_k f^*_{k+1,n}(s)), \qquad k = 1, 2, \ldots, n-1,$$

where $f^*_{n,n}(s) = 1$. Transforming it we find equation (2). ■

Proof of Theorem 1. Taking in system (1) the expected value we find

$$m_{0,n} = \frac{1}{\lambda_0} + m_{1,n},$$
$$m_{k,n} = \frac{1}{\lambda_k + \mu_k} + q_k m_{k-1,n} + p_k m_{k+1,n}, \qquad k = 1, 2, \ldots, n-1,$$

where $m_{n,n} = 0$.

Transforming this we find

$$\lambda_0 m_{0,n} - \lambda_0 m_{1,n} = 1,$$
$$-\mu_k m_{k-1,n} + (\lambda_k + \mu_k) m_{k,n} - \lambda_k m_{k+1,n} = 1.$$

We now introduce the notation $d_{k,n} = m_{k,n} - m_{k+1,n}$, $k = 0, 1, \ldots, n-1$. From the above we find

$$d_{0,n} = \frac{1}{\lambda_0}, \qquad \lambda_k d_{k,n} - \mu_k d_{k-1,n} = 1, \qquad k = 1, 2, \ldots, n-1.$$

Hence

$$d_{k,n} = \frac{\sum_{i=0}^{k} \theta_i}{\lambda_k \theta_k}, \qquad m_{k,n} = \sum_{j=k}^{n-1} d_{j,n}.$$

Let $m_{k,n}^{(2)} = \mathrm{E}(T_{k,n}^2)$. Taking the square in system (1) and next taking the expected value, because $\delta_k^2 = \delta_k$, $(1-\delta_k)^2 = 1 - \delta_k$, $\delta_k(1-\delta_k) = 0$, we find

$$m_{k,n}^{(2)} = \mathrm{E}(e_k^2) + q_k m_{k-1,n}^{(2)} + p_k m_{k+1,n}^{(2)} + 2\mathrm{E}(e_k)(q_k m_{k-1,n} + p_k m_{k+1,n}).$$

Introducing the variances we find

$$\begin{aligned}\sigma_{k,n}^2 + m_{k,n}^2 = {} & \frac{2}{(\lambda_k + \mu_k)^2} + q_k(\sigma_{k-1,n}^2 + m_{k-1,n}^2) + p_k(\sigma_{k+1,n}^2 + m_{k+1,n}^2) \\ & + \frac{2}{\lambda_k + \mu_k}(q_k m_{k-1,n} + p_k m_{k+1,n}).\end{aligned}$$

Transforming this we find

$$\begin{aligned}&(\lambda_k + \mu_k)\sigma_{k,n}^2 - \mu_k \sigma_{k-1,n}^2 - \lambda_k \sigma_{k+1,n}^2 \\ &\quad = \frac{2}{\lambda_k + \mu_k} - (\lambda_k + \mu_k) m_{k,n}^2 + \mu_k m_{k-1,n}^2 + \lambda_k m_{k+1,n}^2 \\ &\qquad + \frac{2}{\lambda_k + \mu_k}(\mu_k m_{k-1,n} + \lambda_k m_{k+1,n}) \\ &\quad = \frac{2}{\lambda_k + \mu_k} + \mu_k d_{k-1,n}^2 + \lambda_k d_{k,n}^2 + 2m_{k,n}(\mu_k d_{k-1,n} - \lambda_k d_{k,n}) \\ &\qquad + \frac{2}{\lambda_k + \mu_k}(\mu_k m_{k,n} + \mu_k d_{k-1,n} + \lambda_k m_{k,n} - \lambda_k d_{k,n}) \\ &\quad = \mu_k d_{k-1,n}^2 + \lambda_k d_{k,n}^2.\end{aligned}$$

Let $u_{k,n} = \mu_k d_{k-1,n}^2 + \lambda_k d_{k,n}^2$. Hence

$$-\mu_k \sigma_{k-1,n}^2 + (\lambda_k + \mu_k)\sigma_{k,n}^2 - \lambda_k \sigma_{k+1,n}^2 = u_{k,n}, \qquad k = 1, 2, \ldots, n-1.$$

Similar to before, introducing the notation $c_{k,n} = \sigma_{k,n}^2 - \sigma_{k+1,n}^2$, we find

$$\lambda_k c_{k,n} - \mu_k c_{k-1,n} = u_{k,n}, \qquad k = 1, 2, \ldots, n-1.$$

From the first line of system (1) it follows that $c_{0,n} = \sigma_{0,n}^2 - \sigma_{1,n}^2 = \frac{1}{\lambda_0^2}$, $u_{0,n} = \lambda_0 c_{0,n}$. Hence

$$c_{k,n} = \frac{1}{\lambda_k \theta_k} \sum_{i=0}^{k} \theta_i u_{i,n}, \qquad \sigma_{k,n}^2 = \sum_{j=k}^{n-1} c_{j,n}.$$

Note that $d_{j,n} = d_j$, $u_{j,n} = u_j$, $c_{j,n} = c_j$ do not depend upon n. ■

Proof of Theorem 2. Let us consider the Laurent expansion of $f^*_{k,n}(s)$ in the neighborhood of ∞:

$$f^*_{k,n}(s) = \sum_{j=1}^{\infty} B^{(j)}_{k,n} \frac{1}{s^j}.$$

The function $f^*_{k,n}$, as the quotient of two polynomials, is regular in infinity. Hence the original is determined by transformation and the following equalities are satisfied

$$f_{k,n}(x) = \sum_{j=0}^{\infty} B^{(j+1)}_{k,n} \frac{x^j}{j!}, \qquad F_{k,n}(x) = \sum_{j=1}^{\infty} B^{(j)}_{k,n} \frac{x^j}{j!}.$$

we find

$$\begin{aligned} f^*_{k,n}(s) &= \frac{1}{s+\lambda_k+\mu_k}(\mu_k f^*_{k-1,n}(s) + \lambda_k f^*_{k+1,n}(s)) \\ &= \frac{1}{s}\left(1 - (\lambda_k+\mu_k)\frac{1}{s} + \cdots\right)(\mu_k f^*_{k-1,n}(s) + \lambda_k f^*_{k+1,n}(s)). \end{aligned}$$

This implies that the first term of the expansion is

$$B^{(1)}_{k,n} = 0, \qquad k = 0, 1, \ldots, n-2.$$

Now we recurrently obtain

$$B^{(i)}_{k,n} = 0, \qquad k = 0, 1, \ldots, n-i-1, \qquad i = 1, 2, \ldots, n-1.$$

From

$$f^*_{n-1,n}(s) = \frac{1}{s}\left(1 - (\lambda_{n-1}+\mu_{n-1})\frac{1}{s} + \cdots\right)(\mu_{n-1} f^*_{n-2,n}(s) + \lambda_{n-1}),$$

and using the terms obtained before, we find the first and second nonzero terms of the expansion:

$$\begin{aligned} B^{(1)}_{n-1,n} &= \lambda_{n-1}, \\ B^{(2)}_{n-1,n} &= -\lambda_{n-1}(\lambda_{n-1}+\mu_{n-1}). \end{aligned}$$

From the equation

$$f^*_{n-i,n}(s) = \frac{1}{s}\left(1 - (\lambda_{n-i}+\mu_{n-i})\frac{1}{s} + \cdots\right)(\mu_{n-i} f^*_{n-i-1,n}(s) + \lambda_{n-i} f^*_{n-i+1,n}(s)),$$

we recurrently obtain

$$\begin{aligned} B^{(i)}_{n-i,n} &= \lambda_{n-i} B^{(i-1)}_{n-i+1,n}, \\ B^{(i+1)}_{n-i,n} &= \lambda_{n-i}(B^{(i)}_{n-i+1,n} - (\lambda_{n-i}+\mu_{n-i})B^{(i-1)}_{n-i+1,n}). \end{aligned}$$

Hence

$$B^{(k)}_{n-k,n} = \prod_{j=n-k}^{n-1} \lambda_j, \qquad B^{(k+1)}_{n-k,n} = -(\prod_{j=n-k}^{n-1} \lambda_j) \sum_{j=n-k}^{n-1} (\lambda_j + \mu_j). \blacksquare$$

Proof of Proposition 2. The expected value of $T_{L,U}$, $L < U$, is given in Theorem 1 where $k = L$, $n = U$. Moreover we have

$$\mathrm{E}(T_{U,L}) = \mathrm{E}(\bar{T}_{\bar{L},\bar{U}}) = \sum_{j=N-U}^{N-L-1} \frac{1}{\bar{\lambda}_j \bar{\theta}_j} \sum_{i=0}^{j} \bar{\theta}_i = \sum_{j=L}^{U-1} \frac{1}{\lambda_j \theta_j} \sum_{i=j+1}^{N} \theta_i.$$

The process $a(t)$, $t \geq 0$, is a stationary alternating process for which (e.g., [1]) $P(a(t) = 1) = \frac{\mathrm{E}(T_{L,U})}{\mathrm{E}(T_{L,U}) + \mathrm{E}(T_{U,L})}$. The substitution of expected values ends the proof of Proposition 2. $\blacksquare$

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