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Solving a Dynamic Traveling Salesman Problem with an Adaptive Hopfield Network

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Abstract. This paper mathematically solves a dynamic traveling salesman problem (DTSP) with an adaptive Hopfield network (AHN). The DTSP is an extension of the conventional TSP where intercity distances are variable parameters that depend on time; while the AHN, in contrast with current deterministic networks, has the capability to adapt itself to changes in the outside environment. The result is stated as a theorem that the AHN always produces as its final states locally-minimum approximate, feasible solutions to the DTSP. It is also expected that the theorem can provide a solid theoretical basis for developing engineering devices of the AHN to solve the DTSP.

1. Introduction

The traveling salesman problem (TSP) is the most typical NP-hard combinatorial optimization problem that has long been under challenging study in mathematics and engineering [15]. One might be surprised to find, however, that even the TSP is too simplified to encompass any time-dependent parameters that the salesman is often forced to take into account in the real world. For instance, it can take more time to arrive in a destination city if driving along a jammed road.

There are many such real-world problems that abound in our everyday life: for instance, in our conversation about road-traffic, stock market, train-control, weather forecast, or planetary orbit. In general, such a problem can be realistically formulated as a dynamic optimization problem, combinatorial and/or continuous, where both an objective function and variable-constraints can contain time-dependent parameters.

In spite of the urgent need for its study, the dynamic optimization problem so far has not attracted the interest it deserves. We thus propose that researchers should start its study from the beginning; theoretical studies are particularly important in this infancy.

This paper, as a starter, focuses on a dynamic TSP (DTSP) that represents a class of simple dynamic optimization problems in the real world. The

DTSP here is an extension of the TSP where intercity distances are variable parameters that can depend on time.

There has been various excellent research carried out on the optimization problem across broad fields including optimization theory, dynamical systems, nonlinear programming, and operations research. Among others, tremendous progress has been made with the neural network (e.g., [1–9, 12–14, 16, 18]). A variety of neural networks have been developed to solve combinatorial (or continuous) optimization problems efficiently and as accurately as possible. We thus employ a network approach to the DTSP.

The network approach to the optimization problem, the TSP in particular, was originally pioneered by Hopfield and Tank [12]. They made the Hopfield network (HN) with an analog circuit and demonstrated that it produced solutions to the TSP quite efficiently when compared to previous conventional methods. It was also reported, however, that the HN has certain major defects such as the occasional convergence to nonoptimal locally-minimum solutions [1, 17].

Though admitting its defects, we present an appropriate rationale substantiating that the HN is the best for solving the TSP. The TSP is expressed as an optimization problem with variable-constraint. Golden's framework [10] suggests that general optimization problems with variable-constraint can best be solved with networks that make use of the simultaneous cooperation of neuron dynamics and synapse dynamics. This is because most current networks consist only of a dynamical system for either neuron activation levels or synapse weights. The HN is the only current network that inherently exemplifies a general configuration of the dynamical system; its synapse dynamical system can be forced to be inactive, excessively tuned only to the TSP. We are expecting that a certain adaptive dynamical system added to the HN can provide enough flexibility to support both variable-constraints and time-dependent intercity distances of the DTSP. Hence this paper proposes to develop a method for solving the DTSP based on an extension to the HN for the adaptive dynamical system.

2. The Hopfield method to solve the traveling salesman problem

This section adapts the Hopfield method [12] for our purpose.

2.1 The traveling salesman problem

The TSP is to find the shortest closed tour by which every city out of a set of M cities is visited once and only once.

Let us denote by X and Y representative cities and express by a real constant α_{XY} an intercity distance between X and Y. Let us also represent any tour by an $M \times M$ matrix $\boldsymbol{V} \equiv (V_{Xm})$ where $V_{Xm} = 1$ or 0, which represents that city X is or is not visited at the mth order $(X, m = 1, \ldots, M)$.

Now the TSP can be more formally stated: to minimize an objective function $E_2(\boldsymbol{V})$ subject to a constraint equation $E_1(\boldsymbol{V}) = 0$ where the two functions $E_1(\boldsymbol{V})$ and $E_2(\boldsymbol{V})$ are defined as follows:

$$E_1(\boldsymbol{V}) \equiv (A/2)\Sigma_X\Sigma_m\Sigma_{n\neq m}V_{Xm}V_{Xn} + (B/2)\Sigma_m\Sigma_X\Sigma_{Y\neq X}V_{Xm}V_{Ym} + (C/2)(\Sigma_X\Sigma_m V_{Xm} - M)^2 \quad (1)$$

$$E_2(\boldsymbol{V}) \equiv (D/2)\Sigma_X\Sigma_{Y\neq X}\Sigma_m\alpha_{XY}V_{Xm}(V_{Y,m+1} + V_{Y,m-1}) \quad (2)$$

where A, B, C, and D denote positive real parameters. The function $E_2(\boldsymbol{V})$ represents the total path length of a tour, while the constraint equation $E_1(\boldsymbol{V}) = 0$ provides a necessary and sufficient condition for feasible solutions of the TSP.

2.2 The Hopfield network

The HN is physically a mesh-shaped interconnected network comprised of $N \equiv M^2$ neurons; and in addition, logically a composite system that consists of a transformation system and a dynamical system. In this section, the two logical constituent systems are specified.

We first introduce the following notation.

- t: neural-time, $t \in [0, \infty)$.
- $\boldsymbol{V}(t) \equiv (V_{Xm}(t))$: activation level of and output from neuron Xm at t.
- $\boldsymbol{T}(t) \equiv (T_{Xm,Yn}(t))$: synapse weight from neuron Yn to neuron Xm at t.
- $\boldsymbol{u}(t) \equiv (u_{Xm}(t))$: input to neuron Xm at t.

Then the transformation system transforms at each neuron Xm an input to an output in accordance with a transformation function $g_{Xm}(u_{Xm}(t))$ specified as follows:

$$V_{Xm}(t) = g_{Xm}(u_{Xm}(t)) \equiv (1/2)\{1 + \tanh(u_{Xm}(t)/u^0_{Xm})\} \quad (3a)$$

$$u_{Xm}(t) \equiv \Sigma_Y\Sigma_n T_{Xm,Yn}(0)V_{Yn}(t) - \theta^0_{Xm} \quad (3b)$$

where u^0_{Xm} and θ^0_{Xm} are both real constants.

On the other hand, the dynamical system is specified by a state-change equation for each neuron and synapse [10] as follows:

$$dV_{Xm}/dt = -2V_{Xm}(1 - V_{Xm})\{\partial E(\boldsymbol{V})/\partial V_{Xm}\} \quad (4)$$

$$dT_{Xm,Yn}(t)/dt = 0 \quad (5a)$$

$$T_{Xm,Yn}(0) \equiv -A\delta_{XY}(1 - \delta_{mn}) - B\delta_{mn}(1 - \delta_{XY}) - C - D\alpha_{XY}(1 - \delta_{XY})(\delta_{n,m+1} + \delta_{n,m-1}). \quad (5b)$$

Here, an energy function $E(\boldsymbol{V})$ is defined by

$$E(\boldsymbol{V}) \equiv E_1(\boldsymbol{V}) + E_2(\boldsymbol{V}). \quad (6)$$

Equation (5a) implies that $T_{Xm,Yn}(t) \equiv T_{Xm,Yn}(0)$ for all $t \leq 0$. Also, Equation (5b) produces the synapse symmetry: $T_{Xm,Yn}(0) = T_{Yn,Xm}(0)$ for all X, Y, m, and n. We refer to the systems in equations (4) and (5) as a neuron dynamical system and a synapse dynamical system, respectively.

Consequently, the logical constituents of the HN have been specified by equations (3) through (5).

2.3 The Hopfield method

2.3.1 Review

The Hopfield method systematically constructs locally-minimum approximate solutions to the TSP in three steps.

1. The TSP is formally expressed as an optimization problem with variable-constraint that consists of the objective function $E_2(\boldsymbol{V})$ and the constraint equation $E_1(\boldsymbol{V}) = 0$.

2. The HN is constructed; it was also implemented with an electric circuit. Specifically, the transformation system of equation (3) is determined from the design of the electric circuit, independently of the TSP. The dynamical system of equations (4) and (5), on the other hand, is derived from the TSP in such a way that the function $E(\boldsymbol{V})$ from equation (6) can be an energy function of the HN. Note that feasible TSP solutions require very careful tuning of the parameters A, B, C, and D in equation (5).

3. It is demonstrated that the HN asymptotically converges on its stable states, local minima of $E(\boldsymbol{V})$, that correspond to 2^N corners of the hypercube $[0,1]^N$. Final neuron states $\boldsymbol{V}(\infty)$ of the HN are thus expected to produce locally-minimum approximate solutions to the TSP, starting from any initial states $\boldsymbol{V}(0)$ in $[0,1]^N$. Which solution is selected from among the 2^N corners depends on the human choice of $\boldsymbol{V}(0)$ as well as on A, B, C, and D.

2.3.2 Analysis

We now make an analysis of the Hopfield method with the aid of classic dynamical system theory (e.g., [11]). An energy function is a typical Lyapunov function that is generally defined in dynamical system theory as follows.

Suppose that $(\boldsymbol{V}^E, T^E)$ is any equilibrium point of a dynamical system (such as equations (4) and (5)). Assume also that $L(\boldsymbol{V}, \boldsymbol{T})$ is a function of variables $(\boldsymbol{V}, \boldsymbol{T})$. Then, $L(\boldsymbol{V}, \boldsymbol{T})$ is called a Lyapunov function at $(\boldsymbol{V}^E, \boldsymbol{T}^E)$ if it satisfies the following three conditions.

- L0. $L(\boldsymbol{V}, \boldsymbol{T})$ is continuous on some neighborhood U of $(\boldsymbol{V}^E, \boldsymbol{T}^E)$ and differentiable on $U - (\boldsymbol{V}^E, \boldsymbol{T}^E)$.

- L1. $L(\boldsymbol{V}, \boldsymbol{T}) > L(\boldsymbol{V}^E, \boldsymbol{T}^E)$ for all $\boldsymbol{V} \in U - (\boldsymbol{V}^E, \boldsymbol{T}^E)$.

- L2. $dL(\boldsymbol{V}, \boldsymbol{T})/dt < 0$ along all trajectories $(\boldsymbol{V}, \boldsymbol{T})$ of the dynamical system contained in $U - (\boldsymbol{V}^E, \boldsymbol{T}^E)$.

Furthermore, the Lyapunov stability theorem in dynamical system theory ensures that if a dynamical system has some Lyapunov function at $(\boldsymbol{V}^E, \boldsymbol{T}^E)$, then the trajectory $(\boldsymbol{V}, \boldsymbol{T})$ asymptotically converges on $(\boldsymbol{V}^E, \boldsymbol{T}^E)$. Conversely, one can construct some asymptotically convergent dynamical system from any given Lyapunov function $L(\boldsymbol{V}, \boldsymbol{T})$. That is,

$$dV_i/dt = -\rho_i(\boldsymbol{V}, \boldsymbol{T})\{\partial L(\boldsymbol{V}, \boldsymbol{T})/\partial V_i\} \qquad (i = 1, \ldots, N) \tag{7a}$$

$$dT_{ij}/dt = -\sigma_{ij}(\boldsymbol{V}, \boldsymbol{T})\{\partial L(\boldsymbol{V}, \boldsymbol{T})/\partial T_{ij}\} \qquad (i, j = 1, \ldots, N) \tag{7b}$$

where the functions $\rho_i(\boldsymbol{V}, \boldsymbol{T})$ and $\sigma_{ij}(\boldsymbol{V}, \boldsymbol{T})$ are positive-valued and continuously differentiable.

One can obtain the HN dynamical system of equations (4) and (5) as a particular case of equation (7) by stating that $L(\boldsymbol{V}, \boldsymbol{T}) \equiv E(\boldsymbol{V})$. Hence it turns out that the essential technique of the Hopfield method lies in the construction of the HN Lyapunov function $E(\boldsymbol{V})$ of equation (6) where the TSP is represented by the objective function $E_2(\boldsymbol{V})$ and the constraint equation $E_1(\boldsymbol{V}) = 0$.

3. A dynamic traveling salesman problem

This section formally specifies a DTSP that the present paper focuses on. In the real-world environment surrounding the salesman in the TSP, the intercity road traffic usually varies every moment during working hours. It is thus indispensable for the TSP to consider the traffic as time-dependent. We substantiate this time-dependence of the traffic by extending all the time-constant intercity distances α_{XY} of the TSP to time-variables so that the intercity distance can vary dynamically depending on traffic situations.

We now proceed to express this extension of the TSP more rigorously. We denote with τ the real-world time that the salesman refers to, distinguishing it from the neural-time t. The time τ is supposed to vary increasingly from a time τ_1 to another τ_2 where a closed interval $[\tau_1, \tau_2]$ represents available working hours; for instance, $\tau_1 = 9{:}00$ AM and $\tau_2 = 5{:}00$ PM. We then extend every constant intercity distance α_{XY} of the TSP to a distance function $\alpha_{XY}(\tau)$ that is continuously differentiable and positive-valued:

$$\alpha_{XY}(\tau) > 0 \text{ for all } \tau \in [\tau_1, \tau_2]. \tag{8}$$

Using the distance function $\alpha_{XY}(\tau)$, we extend the TSP to our DTSP as specified in Definition 1.

Definition 1. A DTSP is to minimize an objective function $E_2(\boldsymbol{V}, \tau)$ subject to the constraint equation $E_1(\boldsymbol{V}) = 0$ where the functions $E_1(\boldsymbol{V})$ and $E_2(\boldsymbol{V}, \tau)$ are defined as follows:

$$\begin{aligned} E_1(\boldsymbol{V}) \equiv{}& (A/2)\Sigma_X\Sigma_m\Sigma_{n\neq m}V_{Xm}V_{Xn} + (B/2)\Sigma_m\Sigma_X\Sigma_{Y\neq X}V_{Xm}V_{Ym} \\ &+(C/2)(\Sigma_X\Sigma_m V_{Xm} - M)^2 \end{aligned} \tag{9}$$

$$E_2(\boldsymbol{V}, \tau) \equiv (D/2)\Sigma_X\Sigma_{Y\neq X}\Sigma_m\alpha_{XY}(\tau)V_{Xm}(V_{Y,m+1} + V_{Y,m-1}). \tag{10}$$

Notes

1. The objective function $E_2(\boldsymbol{V}, \tau)$ from equation (10) of the DTSP is a simple extension of $E_2(\boldsymbol{V})$ from equation (2) of the TSP. Also, the constraint function $E_1(\boldsymbol{V})$ from equation (9) of the DTSP is the same as $E_1(\boldsymbol{V})$ from equation (1) of the TSP; it is reproduced here for easy reference.

2. We have assumed in Definition 1 that it takes no time for the salesman to travel between every pair of cities and conduct business in every city. This assumption has made our concept of the DTSP quite simple at the core though we recognize that it is not realistic. We have to leave it to a further study to remove this unrealistic restriction on the DTSP of Definition 1.

This paper develops a method for solving the DTSP with a certain network that is intended to be used by salesmen in the real world who want to solve it for their daily work. Thus, we now postulate some usual environment for their solving of the DTSP.

First, every intercity distance usually varies every day. We then postulate that the distance function $\alpha_{XY}(\tau)$ is specified as one of five functions $\alpha_{XY}^{\text{day}}(\tau)$ where the index day indicates a working day of the week; namely, from Monday through Friday in some season.

Second, we also postulate that each function $\alpha_{XY}^{\text{day}}(\tau)$ can be obtained beforehand by the salesman, who then supplies the values of the function $\alpha_{XY}(\tau) \equiv \alpha_{XY}^{\text{day}}(\tau)$ dynamically to the DTSP-solving network.

4. An adaptive Hopfield network

This section constructs from the DTSP an extension to the HN, which we call an adaptive Hopfield network (AHN), that is capable of solving the DTSP.

4.1 A method of extending the Hopfield network for the dynamic traveling salesman problem

Our main purpose for extending the HN is to add to it the adaptive capability to process dynamically every function $\alpha_{XY}(\tau)$ that is supplied by the salesman. Incidentally, we also extend the HN for the subsidiary purpose of overcoming one of the major defects in that the HN cannot always produce feasible TSP solutions [1, 9, 17, 18]. This is because both of the extensions can be derived from the same root technique for flexibility of the network.

We now describe specifically our method of extending the HN for the DTSP.

4.1.1 Adapting the distance function $\alpha_{XY}(\tau)$

Like most current deterministic networks, the state-change behavior of the HN in equations (3) through (5) can be deterministically known once the initial state $\boldsymbol{V}(0)$ is chosen. Thus the HN, as it stands, cannot provide any dynamics to process the distance function $\alpha_{XY}(\tau)$ included in the objective function $E_2(\boldsymbol{V},\tau)$ from equation (10) of the DTSP.

The analysis of the HN in section 2.3.2 suggests to us that the most natural way to process $\alpha_{XY}(\tau)$ dynamically is to extend the Lyapunov function $E(\boldsymbol{V})$ from equation (6) so that it can contain all the distance functions $\alpha_{XY}(\tau)$ of $E_2(\boldsymbol{V},\tau)$. From this extended Lyapunov function of $E(\boldsymbol{V})$, we then construct an adaptive dynamical system by use of the classic method in equation (7).

Consequently, this extension of the adaptive dynamical system is the most distinguishing characteristic when compared to a typical deterministic network.

4.1.2 Feasible solutions

Infeasible solutions of the TSP are produced unless the HN can satisfy the constraint equation $E_1(\boldsymbol{V}) = 0$ at its final state. This can occasionally happen since it is quite difficult for humans to select appropriate values of the constant positive parameters A, B, and C in $E_1(\boldsymbol{V})$ beforehand.

We have found that the parameters A, B, and C are contained in every synapse weight $T_{Xm,Yn}(0)$ of equation (5b) and besides, the HN synapse dynamical system of equation (5a) is currently inactive. We thus extend the HN synapse dynamical system by making the parameters A, B, and C variable so that they can adapt to the state-change of $\boldsymbol{V}$ to satisfy $E_1(\boldsymbol{V},A,B,C) = 0$ at the final state.

It now remains to make this extension of the adaptive synapse dynamical system work effectively for the DTSP since the constraint equation $E_1(\boldsymbol{V}) = 0$ is shared by both the TSP and the DTSP.

4.2 Construction of an adaptive Hopfield network

We construct an AHN on the basis of the method in section 4.1.

4.2.1 Construction of a Lyapunov function

First of all, we introduce three real variables $\boldsymbol{Z} \equiv (Z_1, Z_2, Z_3)$. In addition, we define positive variables $\boldsymbol{Z}^* \equiv (Z_1^*, Z_2^*, Z_3^*)$ by

$$\boldsymbol{Z}_k^* \equiv 1 + \exp(Z_k) > 0 \qquad (k = 1, 2, 3). \tag{11}$$

We then extend every constant synapse weight $T_{Xm,Yn}(0)$ of equation (5b) to a variable $T_{Xm,Yn}(\boldsymbol{Z},\tau)$ specified by

$$\begin{aligned} T_{Xm,Yn}(\boldsymbol{Z},\tau) \equiv &-Z_1^*\delta_{XY}(1-\delta_{mn}) - Z_2^*\delta_{mn}(1-\delta_{XY}) - Z_3^* \\ &-D\alpha_{XY}(\tau)(1-\delta_{XY})(\delta_{n,m+1}+\delta_{n,m-1}). \end{aligned} \tag{12}$$

We consider $(\boldsymbol{Z},\tau)$ independent variables while $T_{Xm,Yn}(\boldsymbol{Z},\tau)$ are dependent variables.

Next, we extend the HN Lyapunov function $E(\boldsymbol{V})$ of equation (6) to a function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$; that is, a Lyapunov function of an AHN, specified as follows:

$$E(\boldsymbol{V}, \boldsymbol{Z}, \tau) \equiv E_1(\boldsymbol{V}, \boldsymbol{Z}) + E_2(\boldsymbol{V}, \tau). \tag{13}$$

Here, $E_1(\boldsymbol{V}, \boldsymbol{Z})$ is the same as equation (9) with the constants A, B, and C replaced by the variables Z_1, Z_2, and Z_3:

$$E_1(\boldsymbol{V}, \boldsymbol{Z}) \equiv (Z_1^*/2)\Sigma_X \Sigma_m \Sigma_{n \neq m} V_{Xm} V_{Xn} + (Z_2^*/2)\Sigma_m \Sigma_X \Sigma_{Y \neq X} V_{Xm} V_{Ym} + (Z_3^*/2)(\Sigma_X \Sigma_m V_{Xm} - M)^2. \tag{14}$$

Also, the function $E_2(\boldsymbol{V}, \tau)$ was previously specified by equation (10):

$$E_2(\boldsymbol{V}, \tau) \equiv (D/2)\Sigma_X \Sigma_{Y \neq X} \Sigma_m \alpha_{XY}(\tau) V_{Xm}(V_{Y,m+1} + V_{Y,m-1}). \tag{15}$$

4.2.2 Specification of an adaptive Hopfield network

We apply the Hopfield method to the Lyapunov function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ from equation (13) and then obtain our target network specified in Definition 2.

Definition 2. An AHN is physically a mesh-shaped interconnected network comprised of $N \equiv M^2$ neurons; and in addition, logically it is a composite system that consists of a transformation system and a dynamical system. Following, the logical constituent systems are specified.

An AHN transformation system is specified as:

$$V_{Xm}(t) = g_{Xm}(u_{Xm}(t)) \equiv (1/2)\{1 + \tanh(u_{Xm}(t)/u_{Xm}^0)\} \tag{16a}$$

$$u_{Xm}(t) \equiv \Sigma_Y \Sigma_n T_{Xm,Yn}(\boldsymbol{Z}(t), \tau(t)) V_{Yn}(t) - \theta_{Xm}^0. \tag{16b}$$

An AHN neuron dynamical system is specified as:

$$dV_{Xm}/dt = -2V_{Xm}(1 - V_{Xm})\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial V_{Xm}\}. \tag{17}$$

An AHN synapse dynamical system is specified by the following differential equation system that consists of four equations; the first three are for the unknown function $\boldsymbol{Z}(t)$ and the last is for the unknown function $\tau(t)$.

$$dZ_1/dt = -\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_1 \tag{18a}$$

$$\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_2\}(dZ_2/dt) + \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_3\}(dZ_3/dt) = -\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_2\}^2 - \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_3\}^2 \tag{18b}$$

$$\Sigma_X \Sigma_m \{\partial F_1(\boldsymbol{V}, \boldsymbol{Z})/\partial V_{Xm}\}(dV_{Xm}/dt) + \Sigma_k \{\partial F_1(\boldsymbol{V}, \boldsymbol{Z})/\partial Z_k\}(dZ_k/dt) = -F_1(\boldsymbol{V}, \boldsymbol{Z}) \tag{18c}$$

$$d\tau/dt = -(d\tau/d\mu)^2 \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial \tau\}. \tag{18d}$$

Here, $F_1(\boldsymbol{V}, \boldsymbol{Z})$ is specified by

$$F_1(\boldsymbol{V}, \boldsymbol{Z}) \equiv E_1(\boldsymbol{V}, \boldsymbol{Z})[1 + (1/2)\{1 + \tanh(Z_2)\}]. \tag{19}$$

Also, the unknown function $\tau(t)$ is related to an auxiliary unknown function $\mu(t)$ as follows:

$$\tau \equiv (1/2)\{(\tau_2 + \epsilon) - (\tau_1 - \epsilon)\}[1 + \tanh\{\mu/\mu_0\}] + (\tau_1 - \epsilon) \tag{20}$$

where μ_0 indicates a positive real constant while ϵ is a small positive real number.

Notes

1. The AHN transformation system of equation (16) is an extension of the one in equation (3) for the HN. Specifically, the variable synapse weight $T_{Xm,Yn}(\boldsymbol{Z}(t), \tau(t))$ in equation (16b) is extended from the constant $T_{Xm,Yn}(0)$ in equation (3b).

2. The AHN neuron dynamical system of equation (17) is an extension of the one in equation (4) for the HN. Specifically, the variables Z_1, Z_2, Z_3, and $\alpha_{XY}(\tau)$ in equation (17) coupled with equations (13) through (15) are extended from the constants A, B, C, and α_{XY} in equation (5).

3. The AHN synapse dynamical system of equation (18) is constructed from scratch. The equations (18a) through (18c) are intended to fulfill the DTSP constraint equation $E_1(\boldsymbol{V}, \boldsymbol{Z}) = 0$ and equation (18d) is intended to process dynamically all the distance functions $\alpha_{XY}(\tau)$ in the DTSP objective function $E_2(\boldsymbol{V}, \tau)$ of equation (15).

The construction of the AHN synapse dynamical system in equation (18) coupled with the coefficient-function specification of equations (19) and (20) constitutes the technical core of solving the DTSP using the AHN, which is the main subject of this paper. We thus expound the details of the construction techniques of equation (18) in the Appendix.

5. Solving the dynamic traveling salesman problem with the adaptive Hopfield network

In the Hopfield method, the HN produces locally-minimum approximate solutions to the TSP, though not necessarily feasible [17]. In our extended method on the other hand, the AHN always produces locally-minimum approximate, feasible solutions to the DTSP. This section demonstrates that fact rigorously. First we define a few technical terms.

Definition 3. Let $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ denote any point in $(0,1)^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$.

(a) The point $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is called a *solution* to the DTSP if and only if it is produced as an asymptotically stable point of the AHN.

(b) A solution $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is said to be *feasible* if and only if $(\boldsymbol{V}, \boldsymbol{Z})$ satisfies the DTSP constraint equation; namely, $E_1(\boldsymbol{V}, \boldsymbol{Z}) = 0$.

(c) A feasible solution $(\boldsymbol{V}^S, \boldsymbol{Z}^S, \tau^S)$ is said to be *locally-minimum approximate* if and only if it is a locally-minimum point of the Lyapunov function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ from equations (13) through (15); in addition, the point τ^S is some locally-minimum point of the DTSP objective function $E_2(\boldsymbol{V}^S, \tau)$ from equation (15) that is considered a function of the single variable $\tau \in [\tau_1, \tau_2]$ with the variable $\boldsymbol{V}$ fixed to the valid tour $\boldsymbol{V}^S$ of the DTSP.

Notes

1. If the point $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is a solution to the DTSP, then $V_{Xm} = 0$ or 1 for all X, $m = 1, \ldots, M$ where $\boldsymbol{V} = (V_{Xm})$ [12].

2. If a solution $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is feasible, then $\boldsymbol{V}$ represents a valid tour around the cities of the DTSP.

We now proceed to state and prove Theorem 1.

Theorem 1. *Suppose that the DTSP of Definition 1 is given. Suppose also that the AHN of Definition 2 starts at any given initial state* $(\boldsymbol{V}(0), \boldsymbol{Z}(0), \tau(0))$ *in* $(0,1)^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$. *Then the AHN produces as its final state a locally-minimum approximate, feasible solution* $(\boldsymbol{V}(\infty), \boldsymbol{Z}(\infty), \tau(\infty))$ *to the DTSP.*

Proof. It is sufficient to confirm that the following three conditions are simultaneously satisfied.

C1. Solutions to the DTSP consist exactly of the set of all locally-minimum points of the Lyapunov function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ from equations (13) through (15).

C1, coupled with Definition 3, states specifically that the set of all locally-minimum points $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L) \in [0,1]^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$ of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is identical with the set of all asymptotically stable points $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A) \in (0,1)^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$ of the AHN. Thus C1 can be written out equivalently as the following conditions.

C1a. Any locally-minimum point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ is some asymptotically stable point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$.

This condition, coupled with the Lyapunov stability theorem, means that $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ is some equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E) \in [0,1]^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$ of the AHN; and moreover, $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ satisfies the Lyapunov conditions at this point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ as follows.

C1a-L0. $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is continuous on some neighborhood U of $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ and differentiable on $U - (\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$.

C1a-L1. $E(\boldsymbol{V}, \boldsymbol{Z}, \tau) > E(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ for all $(\boldsymbol{V}, \boldsymbol{Z}, \tau) \in U - (\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$.

C1a-L2. $dE(\boldsymbol{V}, \boldsymbol{Z}, \tau)/dt < 0$ along all trajectories $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ of the AHN contained in $U - (\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$.

C1b. Any asymptotically stable point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ of the AHN is some locally-minimum point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$.

C2. The solution $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L) = (\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ produced by the AHN is feasible.

The DTSP constraint function $E_1(\boldsymbol{V}, \boldsymbol{Z})$ from equation (14) vanishes at all the asymptotically stable points $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ of the AHN. That is,

$$E_1(\boldsymbol{V}^A, \boldsymbol{Z}^A) = 0 \quad \text{for all the asymptotically stable points } (\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A). \tag{21}$$

C3. The feasible solution $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L) = (\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ produced by the AHN is locally-minimum approximate.

Let the locally-minimum point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ be the asymptotically stable point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ that the AHN produces starting from the given initial state $(\boldsymbol{V}(0), \boldsymbol{Z}(0), \tau(0)) \in (0,1)^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$. Then, the point τ^L is some locally-minimum point of the DTSP objective function $E_2(\boldsymbol{V}^L, \tau)$ from equation (15) that is considered a function of the single variable $\tau \in [\tau_1, \tau_2]$ with the variable $\boldsymbol{V}$ fixed to the valid tour $\boldsymbol{V}^L$ of the DTSP.

We now confirm these three conditions one-by-one in the same order.

Confirmation of C1a. To begin with, let $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ be any locally-minimum point of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$. Then, $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ necessarily satisfies the following condition for the extreme point (locally-maximum or locally-minimum) of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$:

$$\partial E(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)/\partial V_{Xm} = \partial E(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)/\partial Z_k = \partial E(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)/\partial \tau = 0 \quad \text{for all } X, m = 1, \ldots, M \text{ and } k = 1, 2, 3. \tag{22}$$

Moreover, we have from equations (11) and (14) that $E_1(\boldsymbol{V}, \boldsymbol{Z}) \geq 0$ for all the values of $(\boldsymbol{V}, \boldsymbol{Z})$. Hence we have that

$$E_1(\boldsymbol{V}^L, \boldsymbol{Z}^L) = 0. \tag{23}$$

The reason for this follows from equations (11) and (14) in that if $E_1(\boldsymbol{V}^L, \boldsymbol{Z}^L) > 0$, then the partial derivative $\partial E(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)/\partial Z_k$ must be positive for at least one k $(1 \leq k \leq 3)$. This contradicts the condition of equation (22).

In the meantime, any equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ of the AHN complies, by definition, with the condition

$$dV_{Xm}/dt = dZ_k/dt = d\tau/dt = 0 \quad \text{for all } X, m = 1, \ldots, M \text{ and } k = 1, 2, 3. \tag{24}$$

The condition of equation (24), coupled with equations (17) through (20), is equivalently rewritten as:

$$2V_{Xm}^E(1 - V_{Xm}^E)\{\partial E(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)/\partial V_{Xm}\} = 0 \text{ for all } X, m = 1, \ldots, M \tag{25a}$$

$$\partial E(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)/\partial Z_k = 0 \text{ for all } k = 1, 2, 3 \tag{25b}$$

$$E_1(\boldsymbol{V}^E, \boldsymbol{Z}^E) = 0 \tag{25c}$$

$$(d\tau/d\mu)^2\{\partial E(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)/\partial t\} = 0. \tag{25d}$$

Consequently the conditions in equations (22) and (23) for $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ are sufficient for the condition of equation (25) for $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$. We thus have proved that $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ is some equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ of the AHN.

For this locally-minimum and equilibrium point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L) = (\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$, we now examine the particular Lyapunov conditions C1a-L0 through C1a-L2.

First of all, the condition C1a-L0 proves satisfied. This follows from the assumptive condition of the continuous differentiability of $\alpha_{XY}(\tau)$ and the specification of equations (11) and (13) through (15) of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ that the function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ is continuously differentiable on the whole domain $[0,1]^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2]$. In addition, the condition C1a-L1 obviously holds since $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L) = (\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ was taken as a locally-minimum point of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$.

Next, we proceed to the third Lyapunov condition C1a-L2. First, differentiate the function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ of equation (13) with respect to t where $(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ indicates any trajectory of the AHN. Then we have the following equation:

$$\begin{aligned} dE(\boldsymbol{V}, \boldsymbol{Z}, \tau)/dt = {} & \Sigma_X \Sigma_m \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial V_{Xm}\}(dV_{Xm}/dt) \\ & + \Sigma_k \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_k\}(dZ_k/dt) \\ & + \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial t\}(d\tau/dt). \end{aligned} \tag{26}$$

Now, we rewrite each term on the right-hand side of equation (26). It follows from equation (17) that the first term is transformed into

$$\begin{aligned} & \Sigma_X \Sigma_m \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial V_{Xm}\}(dV_{Xm}/dt) = \\ & \quad -\Sigma_X \Sigma_m 2V_{Xm}(1 - V_{Xm})\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial V_{Xm}\}^2. \end{aligned} \tag{27}$$

The conditions from equations (18a) through (18c) then transform the second term into

$$\Sigma_k \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_k\}(dZ_k/dt) = -\Sigma_k \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_k\}^2. \tag{28}$$

Owing to the condition of equation (18d), the third term is transformed into

$$\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial t\}(d\tau/dt) = -(d\tau/d\mu)^2 \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial \tau\}^2. \tag{29}$$

Then we substitute the right-hand sides of equations (27) through (29) for the corresponding terms in equation (26). This operation produces the following evaluation of $dE(\boldsymbol{V}, \boldsymbol{Z}, \tau)/dt$:

$$\begin{aligned} dE(\boldsymbol{V}, \boldsymbol{Z}, \tau)/dt = {} & -\Sigma_X \Sigma_m 2V_{Xm}(1 - V_{Xm})\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial V_{Xm}\}^2 \\ & - \Sigma_k \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_k\}^2 \\ & - (d\tau/d\mu)^2 \{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial \tau\}^2 \\ \le {} & 0. \end{aligned} \tag{30}$$

The last inequality holds because of $\boldsymbol{V} \in [0,1]^N$, which follows from equation (16).

The equality in equation (30) holds if and only if the condition of equation (25) is satisfied for the equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$. This is because the condition of equation (25c) can be derived from equation (25b) in the same way that the condition of equation (23) is derived from equation (22). Furthermore, the function specification of equations (13) through (15) of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ enables us to take some neighborhood U of the point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ such that

$$U \cap \{(\boldsymbol{V}, \boldsymbol{Z}, \tau) \in [0,1]^N \times (-\infty, \infty)^3 \times [\tau_1, \tau_2] \mid \text{equation (25) holds for } (\boldsymbol{V}, \boldsymbol{Z}, \tau)\} = \{(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)\}. \tag{31}$$

This completes the confirmation of C1a-L2 and thus C1a.

Confirmation of C1b. Let $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ be any asymptotically stable point of the AHN. Then, $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ is, by definition, some equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ and thus satisfies the condition of equation (25). Furthermore, the condition of equation (25) is equivalent to the extreme point condition in equation (22) that was proved to produce the condition of equation (23) in equation (25c). The reasoning is as follows. First, it follows from the results of Hopfield and Tank [12] that the condition of equation (25a) is equivalent to the first condition in equation (22). In addition, the condition of equation (25d) is equivalent to the third condition in equation (22). This follows from the specification in equation (20) of the transformation between τ and μ that

$$d\tau/d\mu > 0 \text{ for all } \tau \in [\tau_1, \tau_2]. \tag{32}$$

Therefore, the point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ is some extreme point (locally-maximum or locally-minimum) of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$.

Moreover, it follows from equations (30) and (31), and the definition of the asymptotically stable point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$, that the function $E(\boldsymbol{V}(t), \boldsymbol{Z}(t), \tau(t))$ monotonically decreasingly converges on the point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ along all trajectories $(\boldsymbol{V}(t), \boldsymbol{Z}(t), \tau(t))$ starting from any initial states within some neighborhood of $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$. In this consequence, the point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ is some locally-minimum point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$, not any locally-maximum point. This concludes the confirmation of C1b.

Confirmation of C2. Any asymptotically stable point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ is, by definition, some equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ of the AHN. Thus, the equilibrium point $(\boldsymbol{V}^E, \boldsymbol{Z}^E, \tau^E)$ satisfies the condition of equation (25c) as demonstrated in the confirmation of C1a. The condition of equation (25c) is identical to the target condition of equation (21) in C2. This completes the confirmation of C2.

Confirmation of C3. Let the locally-minimum point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ of $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ be the asymptotically stable point $(\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$ of the AHN that starts from the given initial state $(\boldsymbol{V}(0), \boldsymbol{Z}(0), \tau(0))$.

First of all, the point $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)$ satisfies the condition of equation (22) as described previously in the confirmation of C1a. Equation (22), coupled with equations (13) through (15), enables us to evaluate the τ-derivative of the function $E_2(\boldsymbol{V}^L, \tau)$ from equation (15) at the point $\tau = \tau^L$ as follows:

$$\begin{aligned}
&[dE_2(\boldsymbol{V}^L, \tau)/d\tau]_{\tau=\tau^L} \\
&\quad = [(D/2)\Sigma_X \Sigma_{Y\neq X} \Sigma_m \{d\alpha_{XY}(\tau)/d\tau\} V^L_{Xm}(V^L_{Y,m+1} + V^L_{Y,m-1})]_{\tau=\tau^L} \\
&\quad = \lim_{t\to\infty} [(D/2)\Sigma_X \Sigma_{Y\neq X} \Sigma_m \{d\alpha_{XY}(\tau(t))/d\tau\} V_{Xm}(t)\{V_{Y,m+1}(t) + V_{Y,m-1}(t)\}] \\
&\quad = \lim_{t\to\infty} [\partial E_2(\boldsymbol{V}(t), \tau(t))/\partial t] \\
&\quad = \lim_{t\to\infty} [\partial E(\boldsymbol{V}(t), \boldsymbol{Z}(t), \tau(t))/\partial t] \\
&\quad = \partial E(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L)/\partial t \\
&\quad = 0. \qquad (33)
\end{aligned}$$

We have traced here the time limitation of the trajectory $(\boldsymbol{V}(t), \boldsymbol{Z}(t), \tau(t))$ produced by the AHM to its final state $(\boldsymbol{V}^L, \boldsymbol{Z}^L, \tau^L) = (\boldsymbol{V}^A, \boldsymbol{Z}^A, \tau^A)$. The condition of equation (33) shows that the point τ^L is some extreme point of the function $E_2(\boldsymbol{V}^L, \tau)$.

In the meantime, the specification in equation (18d) of the dynamical system for τ, coupled with equations (13) through (15), produces the following evaluation of $d\tau/dt$:

$$\begin{aligned}
d\tau/dt &= -(d\tau/d\mu)^2\{\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial\tau\} \\
&= -(d\tau/d\mu)^2\{\partial E_2(\boldsymbol{V}, \tau)/\partial\tau\}. \qquad (34)
\end{aligned}$$

The combination of equations (33) and (34) then allows us to take some positive real number $\delta > 0$ such that

$$\begin{aligned}
&dE_2(\boldsymbol{V}^L, \tau)/d\tau < 0 \text{ when } d\tau/dt > 0, \text{ i.e., } \tau \in (\tau^L - \delta, \tau^L), \text{ and} \\
&dE_2(\boldsymbol{V}^L, \tau)/d\tau > 0 \text{ when } d\tau/dt < 0, \text{ i.e., } \tau \in (\tau^L, \tau^L + \delta). \qquad (35)
\end{aligned}$$

This condition thus shows that the point τ^L is some locally-minimum point of $E_2(\boldsymbol{V}^L, \tau)$, not any locally-maximum point. This concludes the confirmation of C3. ■

6. Conclusion

This paper mathematically solved the dynamic traveling salesman problem (DTSP) with the adaptive Hopfield network (AHN). The DTSP of Definition 1, extended from the conventional TSP, contains the intercity-distance function $\alpha_{XY}(\tau)$ of the time τ in the objective function $E_2(\boldsymbol{V}, \tau)$. The AHN of Definition 2, extended from the HN, includes the new adaptive synapse dynamical system that not only fulfills precisely the DTSP constraint equation $E_1(\boldsymbol{V}, \boldsymbol{Z}) = 0$ but also processes dynamically the function $\alpha_{XY}(\tau)$ of the DTSP. The result, stated in Theorem 1, is that the AHN always produces as its final state the locally-minimum approximate, feasible solution to the DTSP.

The DTSP represents a class of simple dynamic optimization problems in the real world. We thus think that the method for solving the DTSP in this paper has opened up new vistas for the study of the dynamic optimization problem. Current and future studies can benefit from Theorem 1 in two ways, quite different from each other. Scientists can obtain deeper understanding of the method for solving dynamic optimization problems with networks. Engineers, on the other hand, can make use of Theorem 1 as a solid theoretical basis for developing devices of the AHN, similar to the HN electric circuit, that can solve the DTSP.

We now refer to some future studies of this work. First, no analysis of the performance of the AHN to the DTSP was given. Efficiency of the method is vital to the DTSP in the real world; in particular, large-scale, complicated DTSPs such as a 100 city problem. It thus is required, for instance, to evaluate the AHN performance to the 100 city problem by mathematical analysis and/or computer simulation. Second, we have specified the DTSP of Definition 1 on the unrealistic assumption that no time is necessary for the salesman to travel between every pair of cities and conduct business in every city. In addition, the DTSP only covers the formulation where the variable-constraint is expressed as equations but not inequalities; many real-world problems can be expressed in the inequality variable-constraint form. Hence the DTSP specification needs to be extended for a more realistic, general class of dynamic optimization problems.

Appendix A. Construction techniques of the adaptive Hopfield network synapse dynamical system

We now give more details about the construction techniques of the AHN synapse dynamical system in equation (18) coupled with the coefficient-function specification of equations (19) and (20).

Appendix A.1 Construction techniques for the unknown function $Z(t)$

First of all, we apply the classic method in equation (7) of the dynamical system construction to the Lyapunov function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ of equation (13). Then we obtain a dynamical system for the unknown function $\boldsymbol{Z}(t)$ expressed by

$$dZ_k/dt = -\partial E(\boldsymbol{V}, \boldsymbol{Z}, \tau)/\partial Z_k \text{ for all } k = 1, 2, 3. \tag{A1}$$

From among the three equations ($k = 1, 2, 3$) of (A1), we have used $k = 1$ as it stands; this constitutes equation (18a) of the system for $\boldsymbol{Z}(t)$.

Next, it is evident that the DTSP constraint equation $E_1(\boldsymbol{V}) = 0$ in Definition 1 is equivalently rewritten as

$$E_1(\boldsymbol{V}, \boldsymbol{Z}) = 0 \tag{A2}$$

where the constraint function $E_1(\boldsymbol{V}, \boldsymbol{Z})$ was specified previously by equation (14).

Furthermore, it follows from the coefficient-function specification in equation (19) of $F_1(\boldsymbol{V}, \boldsymbol{Z})$ that equation (A2) is precisely substantiated by the following dynamical system at $t = \infty$:

$$dF_1(\boldsymbol{V}, \boldsymbol{Z})/dt = -F_1(\boldsymbol{V}, \boldsymbol{Z}). \tag{A3}$$

The reason follows. First, equation (A3) is equivalently rewritten as

$$F_1(\boldsymbol{V}(t), \boldsymbol{Z}(t)) = F_1(\boldsymbol{V}(0), \boldsymbol{Z}(0)) \exp(-t) \text{ for all } t \in [0, \infty). \tag{A4}$$

Meanwhile, it follows from equation (19) that

$$1 < 1 + (1/2)\{1 + \tanh(Z_2)\} < 2 \text{ for all real values } Z_2. \tag{A5}$$

It thus follows from equations (A4) and (A5), coupled with equation (19), that the DTSP constraint of equation (A2) is precisely satisfied at $t = \infty$ of the dynamical system in equation (A3).

Obviously, equation (A3) is equivalent to equation (18c) of the system for $\boldsymbol{Z}(t)$.

Finally, we are required to decrease the number of the constituent equations of the dynamical system for $\boldsymbol{Z}(t)$. This is because there are four equations (A1) and (A3) we have already constructed for the three unknown functions $\boldsymbol{Z}(t) = (Z_1(t), Z_2(t), Z_3(t))$. We have thus combined the two equations for $k = 2$ and 3 in equation (A1) into one, which constitutes equation (18b) of the system for $\boldsymbol{Z}(t)$. We have intended that this combination must not escape from the Lyapunov condition L2 that is fulfilled by the four equations (A1) and (A3). Specifically, the combination has been made so that the resulting dynamical system can satisfy the following condition:

$$\Sigma_k\{\partial E(V, Z, \tau)/\partial Z_k\}(dZ_{k1}/dt) \leq 0 \text{ for all } t \in [0, \infty). \tag{A6}$$

Consequently, we have obtained the AHN synapse dynamical system of equations (18a) through (18c) for the unknown function $\boldsymbol{Z}(t)$.

Appendix A.2 Construction techniques for the unknown function $\tau(t)$

We have obtained the dynamical system of equation (18d) for the unknown function $\tau(t)$ by applying the classic method in equation (7) of the dynamical system construction to the Lyapunov function $E(\boldsymbol{V}, \boldsymbol{Z}, \tau)$ of equation (13). There, we also devised the auxiliary unknown function $\mu(t)$, transformed into $\tau(t)$ by equation (20), that is much more tractable than the original unknown function $\tau(t)$ in the analysis of the AHN dynamical system. This is because the function $\tau(t)$ is restricted to the range of the closed interval $[\tau_1, \tau_2]$ while the value of the function $\mu(t)$ can vary from $-\infty$ to ∞; we need not pay any attention to the range of $\mu(t)$.

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