Cellular Automata in the Cantor, Besicovitch, and Weyl Topological Spaces

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The Besicovitch and Weyl pseudometrics on the space $A^{\mathbb{Z}}$ of biinfinite sequences measure the density of differences in either the central or arbitrary segments of given sequences. The Besicovitch and Weyl spaces are obtained from $A^{\mathbb{Z}}$ by factoring through the equivalence of zero distance. Cellular automata are considered as dynamical systems on the Besicovitch and Weyl spaces and their topological and dynamical properties are compared with those they possess in the Cantor space.

1. Introduction

A cellular automaton (CA) consists of a biinfinite array of cells containing letters from a finite alphabet, which are updated according to a local interaction rule. CAs have been of considerable interest as models of physical and biological phenomena, and in symbolic dynamics as homomorphisms of the shift [8]. They display a large spectrum of dynamical behaviors ranging from stable to chaotic dynamics and could also support universal computation. For a survey, see [2, 5, 13].

Until recently whenever a CA was conceived of as a dynamical system, the space of biinfinite sequences was equipped with the product topology, which makes it homeomorphic to the Cantor space. Often people would criticize this topology, though hardly ever in articles. Intuitively it has several weak points. For example, it endows the shift map with strong chaoticity properties: sensitivity to initial conditions, topological transitivity, and dense periodic points. However, regarded as a shift of the observation point, the shift does not change the configuration at all. Another instance is the importance of the 0 coordinate for any distance defining this topology: two configurations are close to each other if their coordinates coincide in a big interval $[-n, +n]$, even if they disagree completely outside. No such distance suits the usual image of a perturbation. A perturbation ought to affect an asymptotically small proportion of the coordinates, while for fixed n it does not matter much whether many of those in $[-n, +n]$ are changed.

With these reasons in mind the shift-invariant Besicovitch pseudometric was recently introduced [4], which was previously used for the study of almost periodic functions (e.g., [1]). The Besicovitch pseudometrics measure the density of differences in the central part of two given sequences. One obtains the Besicovitch space by factoring the space of biinfinite sequences by the equivalence of zero distance. Obviously the distance in the Besicovitch space represents physical perturbations as described previously in a more fitting way.

We introduce a variant, the Weyl pseudometric, which measures the density of differences between two given sequences in arbitrary segments of given length. It is also shift-invariant, but in the definition of the Weyl space the origin does not play any part at all.

Here our purposes are to further investigate properties of the Besicovitch space, and investigate those of the Weyl space, which are often but not always close to those of the former; and, most of all, compare the topological and dynamical properties of CA in the three spaces. CAs are continuous with respect to both the Besicovitch and Weyl pseudometrics, so they define dynamical systems in the corresponding spaces.

In the first direction it is shown in [6] that the Weyl space is pathwise connected and incomplete. The Besicovitch space is also pathwise connected but complete. Both spaces are infinite-dimensional and neither is separable nor locally compact. This is the matter of section 3.

In section 4 we compare properties of CAs in the different spaces. From the dynamical point of view, when passing from the Cantor metric to one of the shift-invariant pseudometrics there is a drift towards stability. For a CA to have some chaoticity property like topological transitivity or sensitivity in one of the Besicovitch or Weyl spaces it is necessary that it has the same property in the Cantor space (but in general equicontinuity or equicontinuous points are not preserved). On the other hand, chain transitivity and stability properties like equicontinuity, existence of equicontinuity points, and stability of periodic points in the Cantor space imply the same properties in the Besicovitch or Weyl spaces; this is not true for expansivity or even sensitivity. For instance, the shift transformation, which is sensitive to initial conditions in the usual setting, is as far as possible from this property in the Besicovitch space, since there it becomes an isometry! But other CAs lose some local or global sensitivity properties, and the interpretation is not always as easy.

Finally, the shift-invariant pseudometrics proved to have an important unexpected feature. Hedlund's theorem states that for the Cantor topology the class of CAs coincides with that of continuous shift-commuting maps; this is not true in the Weyl and Besicovitch spaces: there are indeed many continuous shift-commuting maps that are not given by any local rule. We do not know yet how to characterize this natural class of maps. It is a matter of fact that the class of continuous functions is larger than the one in Cantor spaces. In particular, it contains the whole set of nonuniform CAs, that is, CAs in which different cells may have different local rules. This class of systems is receiving growing interest among scientists which employ them for solving hard problems issuing from physics, computer science, and biology. Nonuniform CAs have never been considered from the dynamical systems point of view since, in general, they are not continuous in Cantor spaces. We hope that our study may give a new impulse in this direction.

2. Dynamical systems

Since we adopt the point of view of dynamical systems we give a few definitions.

A dynamical system is a continuous map $f : X \to X$ of a nonempty metric space X to itself. The nth iteration $f^n : X \to X$ of f is defined by $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. A point $x \in X$ is fixed, if $f(x) = x$. It is periodic, if $f^n(x) = x$ for some $n > 0$. The least positive n with this property is called the period of x. The orbit of x is the set $\mathfrak{o}(x) = \{f^n(x) : n \geq 0\}$. A set $Y \subseteq X$ is positively invariant, if $f(Y) \subseteq Y$.

A point $x \in X$ is equicontinuous ($x \in \mathcal{E}(f)$) if the family of maps f^n is equicontinuous at X, that is, $x \in \mathcal{E}(f)$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B_\delta(x))(\forall n > 0)(d(f^n(y), f^n(x)) < \varepsilon).$$

Here $B_\delta(x) = \{y \in X : d(y, x) < \varepsilon\}$. The map f is equicontinuous if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)(\forall y \in B_\delta(x))(\forall n > 0)(d(f^n(y), f^n(x)) < \varepsilon).$$

For an equicontinuous system, $\mathcal{E}(f) = X$. Conversely if $\mathcal{E}(f) = X$ and X is compact, then f is equicontinuous; this need not be true in the noncompact case. A system (X, f) is sensitive (to initial conditions), if and only if

$$(\exists \varepsilon > 0)(\forall x \in X)(\forall \delta > 0)(\exists y \in B_\delta(x))(\exists n > 0)(d(f^n(y), f^n(x)) \geq \varepsilon).$$

A sensitive system has no equicontinuous points; however, there exist systems with no equicontinuous points that are not sensitive. A system

(X, f) is (positively) expansive if and only if

$$(\exists \varepsilon > 0)(\forall x \neq y \in X)(\exists n \geq 0)(d(f^n(x), f^n(y)) \geq \varepsilon).$$

A positively expansive system on a perfect space (i.e., a space with no isolated points) is sensitive. A system (X, f) is said to be (topologically) transitive if for any nonempty open sets $U, V \subseteq X$ there exists $n \geq 0$ such that $f^{-n}(U) \cap V \neq \emptyset$. If X is perfect and if the system has a dense orbit, then it is transitive. Conversely, if (X, f) is topologically transitive and if X is compact, then (X, f) has a dense orbit. Indeed, the set $\{x \in X : \overline{\mathfrak{o}(x)} = X\}$ is residual in this case. An ε-chain (from x_0 to x_n) is a sequence of points $x_0, \ldots, x_n \in X$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i < n$. A system (X, f) is chain transitive if for any $\varepsilon > 0$ and any $x, y \in X$ there exists an ε-chain from x to y.

A fixed point $x \in X$ is stable if it is equicontinuous and there exists a neighborhood $U \ni x$ such that for every $y \in U$, $\lim_{n\to\infty} f^n(y) = x$. A periodic point x with period n is stable if it is stable for f^n.

3. Cantor, Weyl, and Besicovitch spaces

In this section we introduce the usual metric (here called the Cantor metric) and the Weyl and Besicovitch pseudometrics on the configuration space $A^{\mathbb{Z}}$. Each pseudometric induces a topology on the corresponding quotient space of $A^{\mathbb{Z}}$; we study the properties of the two corresponding topological spaces, calling them the Weyl and Besicovitch spaces. Finally, a few properties of dynamical systems on these spaces are given.

Let A be a finite alphabet with at least two letters. The binary alphabet is denoted by $\mathbf{2} = \{0, 1\}$. For $n \in \mathbb{N}$, denote by A^n the set of words over A of length n, and by $A^* = \bigcup_{n\geq 0} A^n$ the set of finite words over A. We also consider words $u \in A^{[j,k]}$ indexed by an interval of integers $[j, k]$. Denote by $A^{\mathbb{Z}}$ the set of biinfinite sequences of letters of A. The ith coordinate of a point $x \in A^{\mathbb{Z}}$ is denoted by x_i, and $x_{[j,k]} = x_j \ldots x_k \in A^{[j,k]}$ is the segment of x between indices j and k. For $u \in A^{[j,k]}$, u^∞ is the infinite repetition of u, that is, $(u^\infty)_{m+n(k-j+1)} = u_m$ for $n \in \mathbb{Z}$ and $m \in [j, k]$. The cylinder of $u \in A^{[j,k]}$ is the set

$$[u] = \{x \in A^{\mathbb{Z}} : x_{[j,k]} = u\}.$$

The Cantor metric on $A^{\mathbb{Z}}$ is defined by

$$d_{\mathrm{C}}(x, y) = 2^{-k} \quad \text{where} \quad k = \min\{i \geq 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

so $d_{\mathrm{C}}(x, y) < 2^{-k}$ if and only if $x_{[-k,k]} = y_{[-k,k]}$. The cylinders are clopen (closed and open) sets for d_{C}. It is well known that all Cantor spaces (with different alphabets) are homeomorphic. The Cantor space is compact, totally disconnected, and perfect.

The Weyl pseudometric on $A^{\mathbb{Z}}$ is given by the formula

$$d_{\mathrm{W}}(x,y) = \limsup_{l\to\infty} \max_{k\in\mathbb{Z}} \frac{\#\{j \in [k+1, k+l] : x_j \neq y_j\}}{l}.$$

Here # means the number of elements of a set, so $d_{\mathrm{W}}(x,y) < \varepsilon$ if and only if

$$(\exists l_0 \in \mathbb{N})(\forall l \geq l_0)(\forall k \in \mathbb{Z})(\#\{j \in [k+1, k+l] : x_j \neq y_j\} < l\varepsilon).$$

For $x \in A^{\mathbb{Z}}$ denote by $\tilde{x} = \{y \in A^{\mathbb{Z}} : d_{\mathrm{W}}(y,x) = 0\}$ and denote by $X_{\mathrm{W}} = \{\tilde{x} : x \in A^{\mathbb{Z}}\}$ the Weyl space over the alphabet A. Clearly every two Weyl spaces (with different alphabets) are homeomorphic.

The Weyl pseudometric could be considered also on the set $A^{\mathbb{N}}$ of unilateral sequences. For $x, y \in A^{\mathbb{N}}$ put

$$d_{\mathrm{W}}(x,y) = \limsup_{l\to\infty} \max_{k\in\mathbb{N}} \frac{\#\{i \in [k+1, k+l] : x_i \neq y_i\}}{l}.$$

The map $\varphi : A^{\mathbb{Z}} \to A^{\mathbb{N}}$ defined by $\varphi(x) = x_0 x_{-1} x_1 x_{-2} x_2 \ldots$ is a homeomorphism between the unilateral and bilateral Weyl spaces. In fact φ is uniformly continuous, so it preserves completeness.

The Besicovitch pseudometric on $A^{\mathbb{Z}}$ is defined as follows:

$$d_{\mathrm{B}}(x,y) = \limsup_{l\to\infty} \frac{\#\{j \in [-l, l] : x_j \neq y_j\}}{2l+1}$$

so $d_{\mathrm{B}}(x,y) < \varepsilon$ if and only if

$$(\exists l_0)(\forall l \geq l_0)(\#\{j \in [-l, l] : x_j \neq y_j\} < (2l+1)\varepsilon).$$

For $x \in A^{\mathbb{Z}}$ again put $\tilde{x} = \{y \in A^{\mathbb{Z}} : d_{\mathrm{B}}(y,x) = 0\}$ and $X_{\mathrm{B}} = \{\tilde{x} : x \in A^{\mathbb{Z}}\}$ the Besicovitch space over the alphabet A; the notation is the same for the Weyl and Besicovitch equivalence classes but they will always be easy to distinguish according to context. Clearly any two Besicovitch spaces (with different alphabets) are homeomorphic and they are also homeomorphic to the unilateral Besicovitch space obtained from the pseudometric

$$d_{\mathrm{B}}(x,y) = \limsup_{l\to\infty} \frac{\#\{i \in [0, l-1] : x_i \neq y_i\}}{l}, \quad x, y \in A^{\mathbb{N}}.$$

Since $d_{\mathrm{B}}(x,y) \leq d_{\mathrm{W}}(x,y)$, the identity on $A^{\mathbb{Z}}$ resolves into a continuous surjective map $I : X_{\mathrm{W}} \to X_{\mathrm{B}}$.

Both pseudometrics are shift-invariant, for instance, $d_{\mathrm{W}}(\sigma x, \sigma y) = d_{\mathrm{W}}(x,y)$. In other words σ, considered as a continuous transformation on the Weyl or Besicovitch space, is an isometry.

Both the Weyl and Besicovitch spaces are homogenous. For any $u \in \mathbf{2}^{\mathbb{Z}}$, the map $f : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ defined by $f(x)_i = x_i + u_i \mod 2$ is a homeomorphism, which sends 0^∞ to u. Using Toeplitz sequences, it is shown in [6] that the Weyl space is pathwise connected. In the same way we show this for the Besicovitch space and we also show that both spaces are infinite-dimensional.

A sequence $x \in A^{\mathbb{N}}$ is a Toeplitz sequence if each of its subwords occurs periodically, that is, if

$$(\forall n \in \mathbb{N})(\exists p > 0)(\forall j \in \mathbb{N})(x_{n+jp} = x_n).$$

Toeplitz sequences are constructed by filling in periodic parts successively. For an alphabet A put $\tilde{A} = A \cup \{*\}$. For $x, y \in \tilde{A}^{\mathbb{N}}$, $T(x, y) \in \tilde{A}^{\mathbb{Z}}$ is the point obtained by replacing the successive occurrences of stars in x by the letters of y. Let t_i be the increasing sequence of all integers for which $x_{t_i} = *$. Then put

$$\begin{aligned} T(x,y)_i &= x_i \quad \text{if} \quad x_i \neq * \\ T(x,y)_i &= y_j \quad \text{if } i = t_j \text{ for some } j. \end{aligned}$$

Consider a map $f : \{0,1\}^* \to \tilde{A}^{\mathbb{N}}$ defined by induction: $f(\lambda) = *^\infty$, then

$$\begin{aligned} f(x_0 \ldots x_{n+1}) &= T(f(x_0 \ldots x_n), (0*)^\infty) \quad \text{if} \quad x_n = 0 \\ f(x_0 \ldots x_{n+1}) &= T(f(x_0 \ldots x_n), (*1)^\infty) \quad \text{if} \quad x_n = 1. \end{aligned}$$

Thus

$$\begin{aligned} f(0) &= 0*0*0*0*0*0*\ldots \\ f(1) &= *1*1*1*1*1*1\ldots \\ f(00) &= 000*000*000*\ldots \\ f(01) &= 0*010*010*01\ldots \\ f(10) &= 01*101*101*1\ldots \\ f(11) &= *111*111*111\ldots. \end{aligned}$$

For a real number $x \in [0,1]$ with binary expansion $x = \sum_{i=1}^{\infty} x_i 2^{-i}$ put $f(x) = \lim_{n\to\infty} f(x_1 \ldots x_n)$. If $2^n x$ is never an integer for $n \in \mathbb{N}$, then x has a unique expansion and $f(x) \in \{0,1\}^{\mathbb{N}}$. If $2^n x$ is an integer for some n, then x has two binary expansions, and $f(x)$ is the same for both expansions. It contains at most one star, which can be filled in so that $f(x)$ is periodic. If $|x - y| < 2^{-m}$, then $x_{[1,m]} = y_{[1,m]}$; therefore $d_{\mathrm{W}}(x, y) < 2^{-m+1}$ and $f : [0,1] \to X_{\mathrm{W}}$ is continuous. Moreover, it is one-to-one, since

$$d_{\mathrm{W}}(f(x), 0^\infty) = d_{\mathrm{B}}(f(x), 0^\infty) = x.$$

Proposition 1. The Weyl and Besicovitch spaces are pathwise connected and infinite-dimensional.

Proof. Consider the continuous map $f : [0,1] \to X_W$ constructed previously. Given $u \in \mathbf{2}^{\mathbb{Z}}$ the map $g : [0,1] \to \mathbf{2}^{\mathbb{Z}}$ defined by $g(x)_i = u_i f(x)_i$ is continuous. Thus every point $u \in X_W$ can be joined by a continuous path to 0^∞, and X_W is pathwise connected. Since the identity yields a continuous surjection $I : X_W \to X_B$, X_B is pathwise connected too. Since $f : [0,1] \to X_W$ is one-to-one, X_W is at least one-dimensional. For any $n > 1$ construct an embedding $f_n : [0,1]^n \to X_W$ of an n-dimensional cube by $f_n(x_0, \ldots, x_{n-1})_{kn+j} = f(x_j)_k$, so

$$f_n(x_1, \ldots, x_n) = f(x_1)_0 \ldots f(x_n)_0 f(x_1)_1 \ldots f(x_n)_1 \ldots.$$

Thus X_W is at least n-dimensional and therefore infinite-dimensional. ■

The following proof is adapted from [12].

Proposition 2. The Besicovitch space is complete.

Proof. We prove this for the unilateral Besicovitch space. Let $x^{(n)} \in A^{\mathbb{N}}$ be a Cauchy sequence. There exists a subsequence $x^{(n_j)}$ such that

$$d_B(x^{(n_{j+1})}, x^{(n_j)}) < 2^{-j-1}.$$

Choose a sequence l_j of positive integers such that $l_{j+1} \geq 2l_j$ and for every $l \geq l_j$

$$\#\{i \in [0,l) : x_i^{(n_{j+1})} \neq x_i^{(n_j)}\} < l \cdot 2^{-j-1}.$$

Then for $k > j$ and $l \geq l_k$

$$\#\{i \in [0,l) : x_i^{(n_k)} \neq x_i^{(n_j)}\} < l \cdot (2^{-j-1} + \cdots + 2^{-k}) < l \cdot 2^{-j}.$$

Define $x \in A^{\mathbb{N}}$ by $x_t = x_t^{(n_j)}$ if $l_j \leq t < l_{j+1}$ and x_t arbitrarily if $t < l_0$. If $k > j$ and $l_k \leq l < l_{k+1}$, then

$$\begin{aligned}\#\{i \in [0,l) : x_i \neq x_i^{(n_j)}\} &\leq \#\{i \in [0,l_j) : x_i \neq x_i^{(n_j)}\} \\ &\quad + \#\{i \in [l_{j+1}, l_{j+2}) : x_i^{(n_{j+1})} \neq x_i^{(n_j)}\} \\ &\quad + \cdots + \#\{i \in [l_{k-1}, l_k) : x_i^{(n_{k-1})} \neq x_i^{(n_j)}\} \\ &\quad + \#\{i \in [l_k, l) : x_i^{(n_k)} \neq x_i^{(n_j)}\} \\ &\leq l_j + (l_{j+2} + \cdots + l_k + l)2^{-j} \leq l_j + 3l \cdot 2^{-j}.\end{aligned}$$

It follows that $d_B(x, x^{(n_j)}) \leq 3 \cdot 2^{-j}$, so $x^{(n_j)}$ converges to x. Since $x^{(n)}$ is a Cauchy sequence, it converges to x as well. ■

To show further properties of the Weyl and Besicovitch spaces, we use Sturmian sequences (e.g., [3, 11]). For an irrational $x \in (0,1)$ define

$S(x) \in \mathbf{2}^{\mathbb{N}}$ by

$$S(x)_n = 0 \quad \text{if} \quad 0 < nx - k < 1 - x \quad \text{for some} \quad k \in \mathbb{N};$$
$$S(x)_n = 1 \quad \text{otherwise.}$$

$S(x)$ is called a Sturmian sequence with density x. We again have

$$d_{\mathrm{W}}(S(x), 0^{\infty}) = d_{\mathrm{B}}(S(x), 0^{\infty}) = x.$$

Lemma 1. If $x, y \in (0,1)$ and x/y are all irrational, then

$$d_{\mathrm{W}}(S(x), S(y)) = d_{\mathrm{B}}(S(x), S(y)) = x(1-y) + (1-x)y.$$

Proof. Consider the rotation

$$T(a, b) = (a + x \mod 1, b + y \mod 1)$$

defined on the torus $\mathbb{R}^2/\mathbb{Z}^2$; T is uniquely ergodic and its invariant measure is the Lebesgue measure. One has $S(x)_n \neq S(y)_n$ if and only if

$$T^n(0,0) \in [0, 1-x] \times [1-y, 1] \cup [1-x, 1] \times [0, 1-y],$$

this set has Lebesgue measure $x(1-y) + y(1-x)$, by unique ergodicity this is exactly the density of the set of coordinates where $S(x)_n$ and $S(y)_n$ disagree. ■

Proposition 3. The Weyl and Besicovitch spaces are neither separable nor locally compact.

Proof. For any $0 < a < b < 1$ there exists an uncountable set $E_{ab} \subseteq (a, b)$ such that for all $x, y \in E_{ab}$, x, y, and x/y are all irrationals. For every $x, y \in E_{a,b}$ one has

$$a(1-b) < d_{\mathrm{W}}(S(x), S(y)) = d_{\mathrm{B}}(S(x), S(y)) < b(1-a).$$

It follows that neither X_{W} nor X_{B} is separable (i.e., they do not have a countable base). Since $b(1-a)$ can be arbitrarily small, and since both X_{W} and X_{B} are homogeneous, neither is locally compact. ■

Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a W- or B-continuous map. Then $f(\tilde{x}) \subseteq \widetilde{f(x)}$, so $\tilde{f} : X_{\mathrm{W}} \to X_{\mathrm{W}}$ defined by $\tilde{f}(\tilde{x}) = \widetilde{f(x)}$ is continuous and $(X_{\mathrm{W}}, \tilde{f})$ (or $(X_{\mathrm{B}}, \tilde{f})$) is a dynamical system. We refer to a dynamical property of a map f that is continuous in at least one of the Cantor, Besicovitch, and Weyl spaces by prefixing the letter C, B, or W: for instance, sensitivity in the Weyl space is called W-sensitivity. Since $d_{\mathrm{B}}(x, y) \leq d_{\mathrm{W}}(x, y)$ the following statements are true.

Proposition 4. Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be both W- and B-continuous. Then we have the following.

1. If f is W-transitive, then it is B-transitive.

2. If f is W-chain transitive, then it is B-chain transitive.
3. If f is B-expansive, then it is W-expansive.

Since the Besicovitch and Weyl spaces are not separable, no dynamical system on them can have a dense orbit. Nevertheless transitive dynamical systems do exist on them.

Example 1. The map f defined by $f(x)_i = x_{2i}$ is W- and B-transitive.

This map is obviously continuous (but not shift-commuting) on both spaces. To check that it is transitive choose two points x and y, and define z by putting $z_{k \cdot 2^n} = y_k$ and $z_i = x_i$ for $i \neq k \cdot 2^n$; thus z is at a distance of at most 2^{-n} from x, and $f^n(z) = y$.

Proposition 5. Let (X, f) be a dynamical system on a nonseparable space X. If (X, f) is transitive, then it is sensitive.

Proof. By the assumption, there exist $\varepsilon > 0$ and an uncountable set $E \subseteq X$ such that for every $x, y \in E$, $x \neq y$ one has $d(x, y) > 4\varepsilon$. We show that ε is a sensitivity constant for (X, f).

Let $x \in X$. For every $n \geq 0$ there is at most one $z \in E$ whose distance from $f^n(x)$ is less than 2ε. Since E is uncountable there exists $z \in E$ such that $d(f^n(x), z) > 2\varepsilon$ for all $n \geq 0$. By transitivity, in every neighborhood U of x there exists $y \in U$ such that $d(f^n(y), z) < \varepsilon$ for some n; hence

$$d(f^n(x), f^n(y)) \geq d(f^n(x), z) - d(z, f^n(y)) \geq 2\varepsilon - \varepsilon = \varepsilon. \blacksquare$$

4. Cellular automata

A CA is a C-continuous map $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ that commutes with the shift $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$. Every CA is defined by some local rule $F : A^{2r+1} \to A$ with radius $r \geq 0$ by

$$f(x)_i = F(x_{i-r} \ldots x_{i+r}).$$

It follows that any CA is continuous for the Weyl and Besicovitch pseudometrics [4].

We now compare topological and dynamical properties of CAs in the Cantor, Weyl, and Besicovitch spaces. After a result on surjectivity we address stability properties and then chaoticity properties. Then we give a series of illustrative examples and counterexamples, finishing with a B- and W-continuous, shift-commuting map on $A^{\mathbb{Z}}$ that is not a CA. Recall, for instance, that a CA is "B-sensitive" if it acts sensitively on the Besicovitch space.

Proposition 6. A CA $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is surjective if and only if it is W-surjective and if and only if it is B-surjective (i.e., if $\tilde{f} : X_{\mathrm{W}} \to X_{\mathrm{W}}$ or $\tilde{f} : X_{\mathrm{B}} \to X_{\mathrm{B}}$ is surjective).

Proof. Clearly if f is surjective so is $\tilde{f}$. Suppose that $\tilde{f} : X_{\mathrm{W}} \to X_{\mathrm{W}}$ is surjective. By a theorem of Hedlund in [8], f is surjective if and only if every block $u \in A^*$ has a preimage. Consider the periodic point $x = u^\infty$. By the assumption, there exists $y \in A^{\mathbb{Z}}$ such that $d_{\mathrm{W}}(f(y), x) = 0$. It follows that in y one can find blocks that are mapped to u. The proof for d_{B} is identical. ∎

Proposition 7. If a CA f is C-equicontinuous, then it is both W- and B-equicontinuous.

Proof. By the assumption for $\varepsilon = 1$ there exists $\delta = 2^{-m}$ such that for every $x, y \in A^{\mathbb{Z}}$ if $x_{[-m,m]} = y_{[-m,m]}$, then $f^n(x)_0 = f^n(y)_0$ for every $n \geq 0$. Therefore, if $x_{[j-m,k+m]} = y_{[j-m,k+m]}$, then $f^n(x)_{[j,k]} = f^n(y)_{[j,k]}$ for every $n \geq 0$. Given $\varepsilon > 0$ put $\delta = \varepsilon/(2m+2)$ and suppose that $d_{\mathrm{W}}(x, y) < \delta$, so there exists l_0 such that for all $l \geq l_0$ and all $k \in \mathbb{Z}$

$$\#\{i \in [k+1, k+l] : x_i \neq y_i\} < l\delta.$$

Thus in the interval $[k+1, k+l]$, $f^n(x)$ may differ from $f^n(y)$ only in one of the end intervals $[k+1, k+m]$, $[k+l-m+1, k+l]$ or in an interval $[i-m, i+m]$ for some i with $x_i \neq y_i$. It follows that

$$\mathrm{card}\{i \in [k+1, k+l] : f^n(x)_i \neq f^n(y)_i\} < l\delta(2m+1) + 2m.$$

If $l\delta > 2m$, then

$$\frac{\#\{i \in [k+1, k+l] : f^n(x)_i \neq f^n(y)_i\}}{l} < \delta(2m+2) = \varepsilon$$

so $d_{\mathrm{W}}(f^n(x), f^n(y)) < \varepsilon$. Thus f is W-equicontinuous. The proof of B-equicontinuity is analogous. ∎

Proposition 8. If a CA f has a C-equicontinuity point, then it has a W-equicontinuity point and a B-equicontinuity point.

Proof. Let r be the radius of f and $z \in A^{\mathbb{Z}}$ be a C-equicontinuity point of f. For $\varepsilon = 2^{-r}$ there exists $\delta = 2^{-m}$ such that whenever $y_{[-m,m]} = z_{[-m,m]} = u \in A^{[-m,m]}$, then $f^n(y)_{[-r,r]} = f^n(z)_{[-r,r]}$ for all $n \geq 0$. We show that $x = u^\infty$ is a W-equicontinuity point. For given $\varepsilon > 0$ put $\delta = \varepsilon/(4m - 2r + 1)$. If $d_{\mathrm{W}}(x, y) < \delta$, then there exists l_0 such that for all $l \geq l_0$ and all $k \in \mathbb{Z}$

$$\#\{i \in [k+1, k+l] : x_i \neq y_i\} < l\delta.$$

Every change in one of the blocks $x_{[k+1,k+2m+1]} = u$ with $k = j(2m+1)$ may change only this block or $m - r$ positions in any of its two neighboring blocks, that is, at most $4m - 2r + 1$ positions. Thus

$$\frac{\#\{i \in [k+1, k+l] : f^n(x)_i \neq f^n(y)_i\}}{l} < \delta(4m - 2r + 1) = \varepsilon$$

and $d_W(f^n(x), f^n(y)) < \varepsilon$. The proof is practically the same in the Besicovitch space. ■

The following result is implicit in [9].

Lemma 2. If $x \in A^{\mathbb{Z}}$ is a C-stable periodic point of a CA f, then $\sigma(x) = x$ and $f(x) = x$.

Proof. Let p be the period of x. If $[u] \ni x$ is an attracting neighborhood of x, then $\sigma(x)$ is a stable periodic point with attracting neighborhood $\sigma([u])$. For k large enough $[u] \cap \sigma^k[u]$ and $[u] \cap \sigma^{k+1}[u]$ are both nonempty. For $y \in [u] \cap \sigma^k[u]$ and $z \in [u] \cap \sigma^{k+1}[u]$ we get

$$\sigma^k(x) = \lim_{n\to\infty} f^{np}(y) = x = \lim_{n\to\infty} f^{np}(z) = \sigma^{k+1}(x)$$

so $\sigma(x) = x$ and $x = a^\infty$ for some $a \in A$. Then $f(x) = b^\infty$ for some $b \in A$, and $a^\infty = \lim_{n\to\infty} f^{np}(a^\infty b^\infty) = b^\infty$, so $a = b$ and $p = 1$. ■

Proposition 9. If $x \in A^{\mathbb{Z}}$ is a C-stable periodic point of a CA f, then $\tilde{x}$ is W-stable.

Proof. By Lemma 2, $x = a^\infty$ for some $a \in A$. By the proof of Proposition 8, $\tilde{x}$ is both W- and B-equicontinuous. Since x is C-stable, there is $m > 0$ such that for $a^{2m+1} \in A^{[-m,m]}$, $\lim_{n\to\infty} f^n(y) = x$ for every $y \in [a^{2m+1}]$. Then there is s such that $f^s[a^{2m+1}] \subseteq [a^{2m+3}]$ with $a^{2m+3} \in A^{[-m-1,m+1]}$, so occurrences of a spread at least one coordinate in both directions after s steps. For the Weyl pseudometric consider a neighborhood

$$U = \{y \in A^{\mathbb{Z}} : d_W(y, x) < \tfrac{1}{2m+1}\}.$$

For $y \in U$ there exists l such that for every k

$$\#\{i \in [k+1, k+l(2m+1)] : y_i \neq a\} < l,$$

so every subword of y of length $l(2m+1)$ contains a^{2m+1} as a subword. It follows that for $t > s(l-1)(2m+1)$, $f^t(y) = x$, so x is W-stable. ■

Proposition 10. If a CA f is W- or B-sensitive, it is also C-sensitive.

Proof. If f is W- or B-sensitive, it has no W- or B-equicontinuity point, so by Proposition 8 it has no C-equicontinuity point and by Theorem 3 in [10] it is C-sensitive. ■

The existence of W- or B-transitive CAs is an open question, so though the next statement may be empty, it at least tells us where not to look for counterexamples.

Proposition 11. If a CA f is W- or B-transitive, then it is C-transitive.

Proof. Let f be B-transitive and $u, v \in A^{[-m,m]}$. We show that $[u] \cap f^{-n}[v] \neq \emptyset$ for some $n > 0$. Consider spatially periodic points u^∞, v^∞. By the assumption for $\varepsilon = 1/3(2m+1)$ there exists $x \in A^{\mathbb{Z}}$ and $n > 0$ with $d_{\mathrm{B}}(x, u^\infty) < \varepsilon$ and $d_{\mathrm{B}}(y, v^\infty) < \varepsilon$, where $y = f^n(x)$. It follows that there is $l > 0$ such that in the interval $[-m-(2m+1)l, m+(2m+1)l]$ there are at most $(2m+1)(2l+1)\varepsilon = (2l+1)/3$ differences, that is,

$$\#\{i \in [-m-(2m+1)l, m+(2m+1)l] : x_i \neq (u^\infty)_i\} < \tfrac{2l+1}{3}$$
$$\#\{i \in [-m-(2m+1)l, m+(2m+1)l] : y_i \neq (v^\infty)_i\} < \tfrac{2l+1}{3}.$$

Thus there exists at least one unperturbed block, that is, there is $|l_1| \leq l$ such that for $j = (2m+1)l_1$ one has

$$x_{[j-m,j+m]} = u, \quad f^n(x)_{[j-m,j+m]} = v$$

and $\sigma^j(x) \in [u] \cap f^{-n}([v])$. For W-transitivity apply Proposition 4. ■

Proposition 12. If a CA f is C-chain transitive, then it is W- and B-chain transitive.

Proof. Let $F : A^{[-r,r]} \to A$ be the local rule for f. A sequence $x^{(i)} \in A^{\mathbb{Z}}$ is a 2^{-m}-chain for d_{C} if $x_j^{(n+1)} = F(x_{j-r}^{(n)}, \ldots, x_{j+r}^{(n)})$ for $|j| \leq m$. Since only the sites $|j| \leq m+r$ are involved, we identify 2^{-m}-chains with sequences $x_{[-m-r,m+r]}^{(i)} \in A^{[-m-r,m+r]}$. There exists a letter $a \in A$ such that a^∞ is periodic. Denote its period by p. Given $\varepsilon > 0$ let $m \in \mathbb{N}$ be such that $2r/(2r+2m+1) < \varepsilon$. By the assumption for every $u \in A^{[-m-r,m+r]}$ there exists a 2^{-m}-chain $u^{(1)}, \ldots, u^{(n)} \in A^{[-m-r,m+r]}$ such that $u^{(1)} = a^{2m+2r+1}$ and $u^{(n)} = u$. We can assume that $n > p$. Let $w \in A^{[-b,b]}$ be a word containing all the words $u^{(n-p+1)}, \ldots, u^{(n)}$ as subwords. By the assumption, there is again a 2^{-b+r}-chain from a^{2b+1} to w. Denote by q the length of this chain. If we restrict this chain to positions where $u^{(j)}$ is located, we obtain a 2^{-m}-chain of length l from $a^{2r+2m+1}$ to $u^{(j)}$. It follows that there are 2^{-m}-chains of all lengths $q, q+1, \ldots, q+p-1$ from $a^{2m+2r+1}$ to u and since a^∞ has period p there are chains from $a^{2m+2r+1}$ to u of all lengths greater than l. If we also consider chains from v to a^∞, we obtain that there exists q such that for every pair $u, v \in A^{[-m-r,m+r]}$ there exists a 2^{-m}-chain from u to v, whose length is exactly q.

Given $x, y \in A^{\mathbb{Z}}$ we now construct an ε-chain $x^{(1)}, \ldots, x^{(q)}$ leading from x to y for the Weyl pseudometric. In every interval

$$[b_j, c_j] = [-m-r+j(2m+2r+1), m+r+j(2m+2r+1)]$$

where $j \in \mathbb{Z}$, we construct a 2^{-m}-chain $x_{[b_j,c_j]}^{(n)}$ from $x_{[b_j,c_j]}$ to $y_{[b_j,c_j]}$, so $x^{(1)} = x$ and $x^{(q)} = y$. Moreover $f(x^{(n)})_k = x_k^{(n+1)}$ for every $k \in [b_j+m, c_j-m]$, so $x^{(n)}$ is an ε-chain for d_{W}. ■

Proposition 13. No CA is B-positively expansive.

Proof. Let $f : \mathbf{2}^{\mathbb{Z}} \to \mathbf{2}^{\mathbb{Z}}$ be a CA and fix $\varepsilon > 0$. Choose a positive integer q with $1/(q+1) < \varepsilon$ and consider points $x, y \in \mathbf{2}^{\mathbb{Z}}$ that are symmetric (i.e., $x_{-i} = x_i$ and $y_{-i} = y_i$) with nonnegative coordinates

$$x_{[0,\infty)} = 0^{q^1} 1^{q^0} 0^{q^3} 1^{q^2} 0^{q^5} \ldots$$
$$y_{[0,\infty)} = 1^{q^1} 0^{q^0} 1^{q^3} 0^{q^2} 1^{q^5} \ldots.$$

To evaluate $d_{\mathrm{B}}(0^\infty, x)$, note that the differences are greatest at the ends of blocks of ones. Thus for $l = q^1 + q^0 + \cdots + q^{2n+1} + q^{2n} = (q^{2n+2} - 1)/(q-1)$, we have

$$\{i \in [0, l-1] : x_i \neq 0\} = q^0 + q^2 + \cdots + q^{2n} = \frac{q^{2n+2} - 1}{q^2 - 1}$$

so $d_{\mathrm{B}}(0^\infty, x) = 1/(1+q) < \varepsilon$ and similarly $d_{\mathrm{B}}(1^\infty, y) = 1/(1+q) < \varepsilon$. Let $F : A^{2r+1} \to A$ be the local rule of f. The dynamics of f on the points $x, y, 0^\infty$, and 1^∞ depend only on the value of F on the homogenous blocks. All other blocks have zero density. Thus we distinguish four cases; in each of them one can find a pair of points that contradicts expansivity.

1. $F(0^{2r+1}) = 0$ and $F(1^{2r+1}) = 0$; in this case $d_{\mathrm{B}}(f(x), 0^\infty) = 0$, thus for any $t \in \mathbb{N}$, $d_{\mathrm{B}}(f^t(0^\infty), f^t(x)) < \varepsilon$.

2. $F(0^{2r+1}) = 0$ and $F(1^{2r+1}) = 1$; in this case $d_{\mathrm{B}}(f(x), x) = 0$, thus for any $t \in \mathbb{N}$, $d_{\mathrm{B}}(f^t(0^\infty), f^t(x)) = 1/(1+q) < \varepsilon$.

3. $F(0^{2r+1}) = 1$ and $F(1^{2r+1}) = 1$; in this case $d_{\mathrm{B}}(f(y), 1^\infty) = 0$, and for any $t \in \mathbb{N}$, $d_{\mathrm{B}}(f^t(1^\infty), f^t(y)) < \varepsilon$.

4. $F(0^{2r+1}) = 1$ and $F(1^{2r+1}) = 0$; in this case $d_{\mathrm{B}}(f(0^\infty), 1^\infty) = 0$, $d_{\mathrm{B}}(f(1^\infty), 0^\infty) = 0$, $d_{\mathrm{B}}(f(x), y) = 0$, and $d_{\mathrm{B}}(f(y), x) = 0$, hence $\forall t \in \mathbb{N}$, $d(f^t(0^\infty), f^t(x)) = 1/(1+q) < \varepsilon$. ■

We do not know whether the same is true in the Weyl space. Note that $d_{\mathrm{W}}(x, 0^\infty) = 1$, so the preceding proof does not work for X_{W}.

The next set of observations account (together with Propositions 7, 8, and 10) for the fact that passing from the Cantor to the Besicovitch and Weyl topologies considerably diminishes the set of sensitive CA.

Proposition 14. Let f be a continuous shift-commuting map on the Weyl space. Suppose f is W-equicontinuous, or W-sensitive, or that x is a W-equicontinuity point for f: then $\sigma^n \circ f$ has the same property. The same statements are true in the Besicovitch space.

Proof. These are immediate consequences of the facts that f commutes with the shift and that σ preserves the Weyl and Besicovitch pseudometrics. ■

Now we give some examples showing that the converses of Propositions 7–9, 10, and 12 are false.

Example 2. The identity map $f(x) = x$.

The identity is W-chain transitive (since the Weyl space is connected), but not C-chain transitive (since the Cantor space is totally disconnected). Thus the converse of Proposition 12 is false.

Example 3. The shift map $\sigma(x)_i = x_{i+1}$.

The shift map is a W-isometry, so it is W-equicontinuous, though it is C-transitive and C-sensitive. Thus the converses of Propositions 7, 8, and 10 are not true. Observe that $\tilde{\sigma} : X_{\mathrm{W}} \to X_{\mathrm{W}}$ has an infinite number of fixed points. Any sequence k_n of positive integers growing fast enough yields a fixed point

$$x = \dots 1^{k_3} 0^{k_2} 1^{k_1} 0^{k_0} 1^{k_1} 0^{k_2} 1^{k_3} \dots.$$

Example 4. The permutive CA $f(x)_i = x_{i-1} + x_i + x_{i+1}$.

This is B-sensitive (see [4]). We do not know whether it is B-transitive.

Example 5. The multiplication CA $f(x)_i = x_{i-1} x_i x_{i+1}$.

The system has a C-stable fixed point 0^∞, and a W- and B-stable fixed point $\widetilde{0^\infty}$. In X_{B} and X_{W} $\tilde{f}$ has many other fixed points such as $0^\infty 1^\infty$, $1^\infty 0^\infty$, and when the sequence k_n grows fast enough, the point

$$x = \dots 1^{k_3} 0^{k_2} 1^{k_1} 0^{k_0} 1^{k_1} 0^{k_2} 1^{k_3} \dots.$$

Example 6. Gilman's CA $f(x)_i = x_{i+1} x_{i+2}$.

Here the fixed point 0^∞ is W-stable but not C-stable. The converse of Proposition 9 is false.

Example 7 has an important topological, not merely dynamical, meaning. It is well known that in the Cantor topology any continuous shift-commuting map on $A^{\mathbb{Z}}$ is a CA. This is not true for the Besicovitch and Weyl pseudometrics, as shown in the following. The construction is generic. It uses the local rule of the CA "addition of the two nearest neighbors," and in order to obtain another transformation with the same property it is enough to use another local rule instead.

Example 7. Let the application $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, where $A = \{0, 1, s\}$, be defined as follows:

$$\begin{array}{lll} f(x)_i = a + b + c & \text{if} & x_{[i-j-1,i+k+1]} = as^j bs^k c \\ f(x)_i = a + b & \text{if} & x_{[i-j-1,\infty)} = as^j bs^\infty \\ f(x)_i = b + c & \text{if} & x_{(-\infty,i+k+1]} = s^\infty bs^k c \\ f(x)_i = b & \text{if} & x_{(-\infty,\infty)} = s^\infty bs^\infty,\ x_i = b \\ f(x)_i = s & \text{if} & x_i = s \end{array}$$

where $a, b, c \in \mathbf{2}$.

The restriction of this map to $\{0,1\}^{\mathbb{Z}}$ is just the addition of the two nearest neighbors. In $A^{\mathbb{Z}}$ the letter s stays unmodified, and wherever possible f acts on other letters as if it were the addition of the two nearest neighbors, overlooking occurrences of s in between. By definition f commutes with the shift; but a coordinate of $f(x)$ does not depend on any bounded set of neighbors, so f is not a CA. We claim it is both W- and B-continuous. First let $x \in A^{\mathbb{Z}}$ and suppose $x'_i = x_i$ except for $i = 0$; then $f(x')_i \neq f(x)_i$ for at most three values of i: 0, the first occurrence of a 0 or 1 to the left, and the first one to the right. Now consider $y \in A^{\mathbb{Z}}$ and an integer $n > 0$; for each interval of coordinates $[k, k+n-1]$, $k \in \mathbb{Z}$ one has

$$\#\{j \in [k+1, k+l] : f(x)_j \neq f(y)_j\} \leq 3 \cdot \#\{j \in [k+1, k+l] : x_j \neq y_j\} + 2.$$

The first term of the right-hand sum is a very rough majoration of the differences between $f(x)$ and $f(y)$ arising in this interval from differences between x and y in the same interval; the term 2 majorates the number of differences arising in the interval because of differences between x and y outside this interval. Dividing by n and taking the lim sup one obtains $d_{\mathrm{W}}(f(x), f(y)) \leq 3d_{\mathrm{W}}(x,y)$, and the same is obviously true for d_{B}, since one has only to consider one value of k for each odd n; so f is both W- and B-continuous.

This example has an interesting dynamical property: there is a unique W-equicontinuous point for f. One easily shows that the fixed point $\widetilde{s^{\infty}}$ has this property; all other points in the Weyl space do not, because they inherit the sensitivity property of their coordinates on $\{0,1\}$.

5. Discussion

There are many differences between the Cantor space and the two others. The Cantor space is a compact metric, so CAs acting on it fit perfectly into the general theory of topological dynamics as developed during the last 50 years.

Compactness and the Hausdorff property often create clear-cut situations, which is not the case with less powerful topologies; this paper illustrates the contrast in many instances. First, we know that many continuous shift-commuting transformations on d_{B} and d_{W} are not CAs. A CA having no equicontinuous points for d_{C} is sensitive to initial conditions, whereas for d_{B} or d_{W} there may exist counterexamples. A CA having one equicontinuous point for d_{C} has a dense set of them, but we do not know whether this is true for d_{B} and d_{W}; we know of at least one shift-commuting map acting continuously on these spaces, having exactly one equicontinuous point. Many dynamical properties in the Cantor space can be interpreted simply in terms of words and their combinatorics; for instance, surjectivity [8], existence of equicontinuous points [10], expansiveness, and others; in the other two spaces this is the case only for surjectivity, but, apparently, this is an isolated case.

Our idea is not to advertise the Cantor topology especially. We just want to point out that there may still be plenty of hard work for researchers in the Weyl and Besicovitch theories of CAs.

Since our point of view is almost purely mathematical we cannot conclude that one topology is much better than the two others for the study of CAs. The only remark we can make is that if one wishes a small perturbation to concern a set of coordinates of small density rather than any coordinates not in $[-n, +n]$, then one should prefer the Weyl or Besicovitch space.

At present we are not aware of many significant differences between the Weyl and Besicovitch spaces. While the set of intervals used for the definition of the pseudometric d_{B} still privileges the origin, this feature completely disappears in the definition of d_{W}; on the other hand, because of lack of completeness, the Weyl space may be harder to use.

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