Chaotic Properties of the Q-state Potts Model on the Bethe Lattice: $Q < 2$

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The Q-state Potts model on the Bethe lattice is investigated for $Q < 2$. The magnetization of this model exhibits complicated behavior including both period doubling bifurcation and chaos. The Lyapunov exponents of the Potts–Bethe map are considered as order parameters. A scaling behavior in the distribution of Lyapunov exponents in the fully developed chaotic case is found. Using the canonical thermodynamic formalism of dynamical systems, the nonanalytic behavior in the distribution of Lyapunov exponents is investigated and the phase transition point on the "chaotic free energy" is located.

1. Introduction

The Q-state Potts model is one generalization of the Ising model constructed for investigating phase transitions [1]. This model was initially defined for an integer Q but has many applications for noninteger Q. The exact solution of a two-dimensional Potts model for general Q has been obtained only at the self-dual point by mapping it into a two-dimensional inhomogeneous six vertex model [2].

One can obtain another exact solution of the Potts model for general Q and coordination number on the Bethe lattice. It is interesting to note that similar results have been found for the $N \times N$ Hermitian matrix model on random graphs in the $N \to 1$ limit [3].

A rigorous treatment of the properties of ferromagnetic and antiferromagnetic Potts models in a magnetic field have been conducted on the Bethe lattice by means of recursion relations [4].

The Q-state Potts model on the Bethe lattice was recently investigated for $Q < 2$ [5]. Many physical processes can be formulated in terms of

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the Q-state Potts model when $Q < 2$, for example, the resistor network, dilute spin glass, percolation, and self organizing critical systems [1,6–8]. It was shown that when $Q < 2$ the Q-state Potts model exhibits a large variety of phase transitions leading to specially modulated and chaotic phases. These phases are similar to those obtained in the axial next nearest Ising, chiral Potts, and three-site interacting Ising models. It is interesting to note that contrary to these models the phases in the Potts model with $Q < 2$ are obtained without frustrations [9–11].

In this paper we investigate the Q-state Potts model on the Bethe lattice for $Q < 2$ using canonical thermodynamic formalism. The phase transitions in this model take place in the chaotic regime because the attractors of the Potts–Bethe mapping used for calculating average quantities is a complicated function of the parameters of the Potts model. Being one-dimensional the Potts–Bethe map exhibits a period doubling cascade, chaos, and so forth. By using the canonical thermodynamic formalism one can calculate the distribution of local Lyapunov exponents of the Potts–Bethe map in the case of fully developed chaos. The Lyapunov exponent is not only a good order parameter for the transition to chaos, but also completely characterizes the system in chaotic states. The general aim of this paper is to find the scaling in the distribution of local Lyapunov exponents of the Potts–Bethe map. The fractal approach allows one to map the computation of Lyapunov exponents onto the thermodynamics of the one-dimensional spin model and interpret the scaling in the distribution of Lyapunov exponents as a phase transition in the one-dimensional spin model [13,15–17]. We obtain the phase transition temperature by means of numerical calculations of the free energy of this model.

This paper is organized as follows. The Potts model on the Bethe lattice and its recursion relation is given in section 2. In section 3 we discuss the phase structure of the Potts model on the Bethe lattice. In section 4 we investigate the Potts–Bethe map for the case of fully developed chaos. By using the canonical thermodynamic formalism, the distribution of the local Lyapunov exponents is obtained and the phase transition is analyzed in terms of the one-dimensional spin model. Finally, in section 5 we summarize our results and comment on their implications for the study of other systems.

2. The Potts model on the Bethe lattice and its recursion relation

The Potts model in the magnetic field is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) - H \sum_i \delta(\sigma_i, 1) \qquad (1)$$

where $\delta(\sigma_i, \sigma_j) = 1$ for $\sigma_i = \sigma_j$ and 0 otherwise, σ_i takes the values $1, 2, \ldots, Q$, the first sum is over the nearest-neighbor sites, and the second sum is simply over all sites on the lattice. Additionally, we use the notation $K = J/kT$ and $h = H/kT$.

The partition function and single site magnetization is given by

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\mathcal{H}/kT}$$
$$M = \langle \delta(\sigma_0, 1) \rangle = \mathcal{Z}^{-1} \sum_{\{\sigma\}} \delta(\sigma_0, 1) e^{-\mathcal{H}/kT} \tag{2}$$

where the summation goes over all configurations of the system.

When the Bethe lattice is cut apart at the central point, it is separated into γ identical branches. The partition function can be written as follows:

$$\mathcal{Z}_n = \sum_{\{\sigma_0\}} \exp\left\{h\delta(\sigma_0, 1)\right\} [g_n(\sigma_0)]^\gamma \tag{3}$$

where σ_0 is the central spin and $g_n(\sigma_0)$ is the contribution of each lattice branch. The latter is expressed through $g_{n-1}(\sigma_1)$, that is, the contribution of the same branch containing $n - 1$ generations starting from the site belonging to the first generation:

$$g_n(\sigma_0) = \sum_{\{\sigma_1\}} \exp\left\{K\delta(\sigma_0, \sigma_1) - h\delta(\sigma_1, 1)\right\} [g_{n-1}(\sigma_1)]^{\gamma-1}. \tag{4}$$

Introducing the notation

$$x_n = \frac{g_n(\sigma \neq 1)}{g_n(\sigma = 1)} \tag{5}$$

one can obtain the Potts–Bethe map

$$x_n = f(x_{n-1}, K, h)$$
$$f(x, K, h) = \frac{e^h + (e^K + Q - 2)x^{\gamma-1}}{e^{K+h} + (Q - 1)x^{\gamma-1}}. \tag{6}$$

The magnetization of the central site for the Bethe lattice with the nth generation can be written as

$$M_n = \langle \delta(\sigma_0, 1) \rangle = \frac{e^h}{e^h + (Q - 1)x_n^\gamma}. \tag{7}$$

3. The phase structure of the Potts model

Let us consider the magnetization of the central site. In order to achieve the thermodynamic limit we set the number of generations to infinity ($n \to \infty$). The recursion relation of equation (6) converges to stable fixed points at every value of parameters h, K in a ferromagnetic case

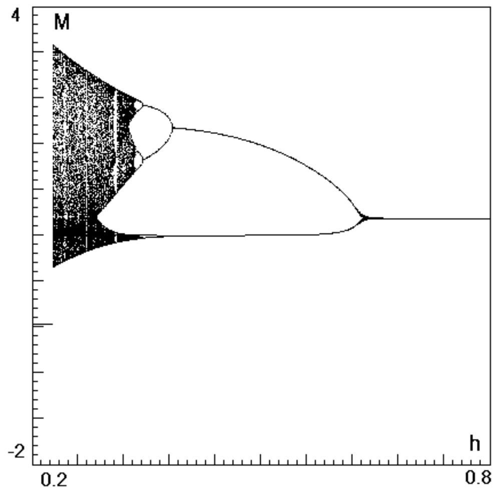

Figure 1. Plot of M (magnetization) versus h (external magnetic field). $K = -0.5$, $Q = 0.8$, and $\gamma = 3$.

($K > 0$), and has only one doubling period in an antiferromagnetic case ($K < 0$). This corresponds to a rise of antiferromagnetic order in different sublattices for $Q \geq 2$ [4]. The situation changes drastically for $Q < 2$. For systems with $Q < 2$ and with antiferromagnetic interactions or for systems with $Q < 1$ and with ferromagnetic interactions one obtains bifurcation diagrams for M *versus* h within the full range of period doubling cascade, chaos, and so forth [5]. Figure 1 shows plots of M versus h for the antiferromagnetic case with $K = -0.5$, $Q = 0.8$, and $\gamma = 3$.

The Potts model has many specially modulated and chaotic phases when $Q < 2$. The presence of phase transitions is in obvious contradiction to the universality hypothesis. The transition to chaos is provided by the Feigenbaum exponents and is well known to be a one-dimensional map. It is interesting to note that a similar transition has been found in the Ising model with three-site interaction [11, 18]. The Potts model ($Q < 2$) and the three-site interacting Ising model have the same universal Feigenbaum exponents.

As mentioned previously, the Lyapunov exponents are not only good order parameters for the transition to chaos but also completely characterize the system in chaotic states. In section 4 we calculate the distribution of Lyapunov exponents by using the canonical thermodynamic formalism. The recursion relations of critical phenomena of the Potts model and those of dynamical systems are similar. The canonical thermodynamic formalism connects the thermodynamical quantities of a one-dimensional spin model and dynamical properties of strange attractors [12,13,15–17].

4. Potts–Bethe map in the case of fully developed chaos

In this section we apply canonical thermodynamic formalism to the Potts–Bethe mapping and impose two restrictions on the parameters to achieve fully developed chaotic behavior. First, we consider only odd coordination numbers γ, as in this case the Potts–Bethe mapping $f(x, K, h)$ becomes an even function of x. Second, we place the following condition on $f(x)$:

$$f(0, K, h, \gamma) = -f(f(0, K, h, \gamma), K, h, \gamma). \tag{8}$$

This results in the following restriction on h and K:

$$\exp(h) = \frac{1 - \exp(2K) + 2\exp(K) - \exp(K)Q - Q}{2\exp(\gamma K)}. \tag{9}$$

In this case the range of the function (Potts–Bethe mapping) $f(x, K, h, \gamma)$ is $[-e^{-K}, e^{-K}]$. This is equal to the range of definition of x $I : [-e^{-K}, e^{-K}]$.

These assumptions are made for the computational purpose of finding such a possible set in the phase plane (h, K) where the Potts–Bethe map exhibits fully developed chaotic behavior.

For a crisis map (equations (6) and (9)) we want to describe the scaling properties of an attracting set for the sequences x_n, which in this case is the interval $I : [-e^{-K}, e^{-K}]$ (Figure 2). For an index n, I is partitioned into 2^n intervals or n-cylinders, these being the segments with identical symbolic–dynamics sequences of length n taken with respect to the maximum point (we follow [16]). The inverse function of equation (6), $h = f^{-1}$, has two branches, h_{-1} and h_1 as shown in Figure 2 and the n-cylinders are all the nth-order preimages of I. The length of the cylinders is denoted by $l_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} \equiv h_{\epsilon_1} \circ h_{\epsilon_2} \circ \cdots \circ h_{\epsilon_n}(I)$ where $\epsilon \in \{-1, 1\}$.

Let us consider a one-dimensional Ising-like model. The energy of the given configuration $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ is equal to $|\ln l_{\epsilon_1,\epsilon_2,\ldots\epsilon_n}|$. The partition function $Z(\beta)$ is defined as [13,15–17]:

$$Z_n(\beta) = \sum_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} l^{\beta}_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} = \sum_{\epsilon_1,\epsilon_2,\ldots\epsilon_n} e^{-\beta|\ln l_{\epsilon_1,\epsilon_2,\ldots\epsilon_n}|} \tag{10}$$

where $\beta \in (-\infty, \infty)$ is a free parameter—the inverse "temperature." In the limit $n \to \infty$ the sum behaves as

$$Z(\beta) = e^{-n\beta F(\beta)} \tag{11}$$

which defines the free energy, $F(\beta)$. The partition function $Z(\beta)$ can be alternatively written as [13]:

$$Z(\beta) = \int d\lambda e^{nS(\lambda) - n\lambda\beta}. \tag{12}$$

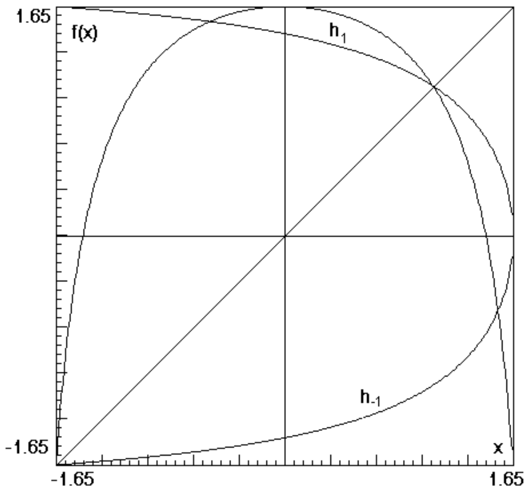

Figure 2. A plot of the function of equation (6) for $K = -0.5$, $Q = 0.8$, $h = 0.23$, and $\gamma = 3$.

The entropy $S(\lambda)$ is the Legendre transform

$$S(\lambda) = -\beta F(\beta) + \lambda\beta \quad (13)$$

where the relation between λ and β is obtained from

$$\lambda = \frac{d}{d\beta}(\beta F(\beta))$$
$$\beta(\lambda) = S'(\lambda). \quad (14)$$

This means that in the limit $n \to \infty$, $e^{nS(\lambda)}$ is the number of cylinders with length $l = e^{-n\lambda}$ or, in the same way, cylinders with local Lyapunov exponent λ. The Hausdorff dimension of the set of points in I having local Lyapunov exponent λ is $S(\lambda)/\lambda$.

We point out that this one-dimensional Ising-like model has no direct physical meaning and is used here to compute the spectrum $S(\lambda)$ of the local Lyapunov exponents λ of the map (equations (6) and (9)).

Using equations (6) and (9) through (11) we numerically calculate the free energy at the point $K = -0.5$, $Q = 0.8$, and $\gamma = 3$ (Figure 3). One can see in Figure 3 that the free energy has a nonanalytic behavior around $\beta_c \approx -1$, that testifies to the existence of the first order phase transition in this region of β.

Large deviations of the fluctuations of local Lyapunov exponents can be described by means of $S(\lambda)$. To consider the given results in terms of the entropy function $S(\lambda)$, we now discuss the general view of the entropy

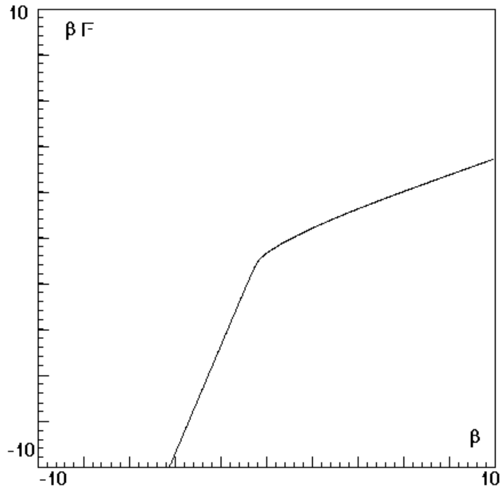

Figure 3. A plot of $F(\beta)$ for $K = -0.5$, $Q = 0.8$, $h = 0.23$, and $\gamma = 3$.

function. First of all, it should be positive in the interval $[\lambda_{\text{min}}, \lambda_{\text{max}}]$. The value $\lambda = \ln 2$ must be within this interval, as follows from the fact that the sum of the lengths of all cylinders on a given level is 1. Second, it is often found that the values of λ_{min} and λ_{max} are given by logarithms of the slopes at the origin.

The precise form of the entropy function is not easy to obtain with high accuracy. The existence of the first order phase transition implies that there should be a straight line segment in $S(\lambda)$ with the slope of the line equal to β_c. This scenario is shown in Figure 4. The curve in the figure corresponds to $n = 13$. Of course, with the finite-size data, it is impossible to determine the straight line segment in $S(\lambda)$ and the straight line will increase with n.

Note that by using Feigenbaum's formulas [20] this scaling can also be interpreted in terms of generalized dimensions of microcanonical thermodynamic formalism [13–15].

5. Conclusion

In this paper we investigated the Q-state Potts model on the Bethe lattice in an external magnetic field. A close relation to the results of the theory of dynamical systems including chaos has been pointed out for $Q < 2$.

The local Lyapunov exponents are introduced as order parameters for characterizing the large variety of phase transitions that occur in the Potts model. For certain values of parameters a distribution of local Lyapunov exponents are obtained by using the thermodynamic formal-

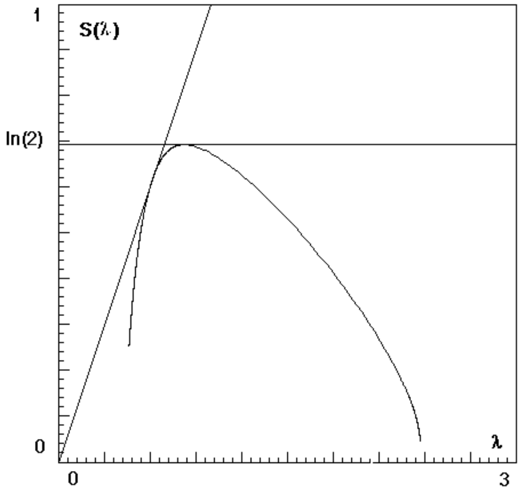

Figure 4. A plot of $S(\lambda)$ corresponding to $n = 13$ for $K = -0.5$, $Q = 0.8$, $h = 0.23$, and $\gamma = 3$.

ism of multifractals. The scaling in the distribution of local Lyapunov exponents is interpreted as a phase transition in the thermodynamics of a one-dimensional Ising-like model. This phase transition is analyzed in terms of "temperature" and local Lyapunov exponents λ.

It is noteworthy that similar behavior has been found in the three-site interacting Ising model in the Husimi three [18, 19].

Note that a dense Mandelbrot set of Fisher's zeroes for the noninteger valued Potts model can be obtained as the three-site antiferromagnetic interaction Ising model [22]. The noninteger ($Q < 1$) valued Potts model is connected to the gelatation and vulcanization of branched polymers [21]. A few polymer monolayers are described by higher-dimensional maps [23]. The thin films of branched polymers can be regarded as the critical behavior of period p-tuplings in coupled one-dimensional maps [24]. The investigation of modulated phases and chaotic properties of polymers will be discussed in future publications.

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