Some Parameters Characterizing Cellular Automata Rules

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Three parameters that can be used to characterize aspects of cellular automata behavior are considered for binary valued one-dimensional rules. These are the λ parameter, the Z parameter, and the obstruction (Θ) parameter. After a brief review of the λ and Z parameters, the Θ parameter is defined and shown to characterize the degree of nonadditivity of a rule. A derivation of the Z parameter in terms of rule table entries is given. It is shown that the λ parameter and Θ parameter are equal respectively to the area and volume under certain graphs. Finally, the nongenerative 3-site rules are listed in terms of these parameters and their decomposition into additive and nonadditive parts and certain regularities are noted.

1. Introduction

Every classification scheme is a set of dimensions along which the items to be classified may differ. These dimensions are chosen so that some information is gained by the location of an item in one or another of the defined categories. That is, differences along the classifying dimensions must make a difference [1].

In this paper, three parameters that have proved useful in classification of one-dimensional binary valued cellular automata are considered. These are the λ, Z, and Θ (obstruction) parameters proposed respectively by Langton [2, 3], Wuensche and Lesser [4–6], and Voorhees [7, 8].

Expressions for each of these parameters are given in terms of rule table components, and the λ and Θ parameters are shown to be invariants of certain iterated systems. This yields an interpretation of these parameters as respectively the area under the graph of a rule, and the volume under the graph of the obstruction map of that rule. Finally, some interesting patterns of distribution of these parameters are shown for the nongenerative 3-site rules when they are listed in terms of decomposition into additive and nonadditive parts.

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2. Rule components and the obstruction map

Let E be the state space consisting of right half-infinite binary sequences and let $X : E \to E$ be the global map of a k-site binary valued cellular automaton rule. The rule table for X is the list of 2^k possible k-site neighborhoods together with the specification of the value of the rule on each neighborhood. If $i_0 \dots i_{k-1}$ is a particular neighborhood, the corresponding rule table component of X is just $x_i = X(i_0 \dots i_{k-1})$ where the component index i is the denary form of the binary number $i_0 \dots i_{k-1}$. Thus, the rule table for X is the set $\{(i_0 \dots i_{k-1}, x_i) | 0 \leq i \leq 2^k - 1\}$. For example, if X is a 3-site rule, the rule table is

000	001	010	011	100	101	110	111
x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7.

A rule X is additive if, for all μ, μ' in E, $X(\mu + \mu') = X(\mu) + X(\mu')$, where addition is site-wise mod (2). In [7] it is shown that the additivity condition is expressed in terms of rule components as $x_i + x_j = x_{i+j}$ $(0 \leq i, j \leq 2^k - 1)$ where $i + j$ is the denary form of the binary number obtained by site-wise addition of $i_0 \dots i_{k-1}$ and $j_0 \dots j_{k-1}$. For binary rules this is written as $x_i + x_j + x_{i+j} = 0 \bmod (2)$. The $2^k \times 2^k$ matrix $U(X)$ defined by

$$[U(X)]_{ij} = x_i + x_j + x_{i+j} \bmod (2) \tag{2.1}$$

is called the *obstruction matrix* for the rule X.

Lemma 1. [7] Let X and Y be k-site rules with components x_i and y_i respectively. Then:

(a) $U(X) = 0$ if and only if X is additive.

(b) $U(X + Y) = U(X) + U(Y) \bmod (2)$ where $(X + Y)_i = x_i + y_i \bmod (2)$.

On this basis, the 2^{2^k} possible k-site rules are partitioned into $2^{2^k - k}$ distinct additivity classes [7, 8]. The obstruction parameter for a rule X is defined as

$$\Theta(X) = \frac{1}{2^{2k}} \sum_{i,j=0}^{2^k - 1} [U(X)]_{ij}. \tag{2.2}$$

This parameter equals the probability that the rule X will be nonadditive on a randomly chosen pair of neighborhoods. For comparison, the λ parameter for a binary valued rule X is the probability that X maps a randomly chosen neighborhood to 1:

$$\lambda(X) = \frac{1}{2^k} \sum_{i=0}^{2^k - 1} x_i. \tag{2.3}$$

The state space E maps to the interval $[0,1]$ by

$$\mu \to \sum_{i=1}^{\infty} \frac{\mu_i}{2^i} = \mu^* \tag{2.4}$$

where $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ and μ^* denotes the corresponding element of $[0,1]$. Thus, any rule $X : E \to E$ induces a map $X : [0,1] \to [0,1]$ by $[X(\mu)]_i = X(\mu_i \ldots \mu_{i+k-1})$. Likewise, $U(X)$ induces a map $U(X) : [0,1] \times [0,1] \to [0,1]$ by

$$[U(X)(\mu^*, \nu^*)]_i = x_{i(\mu)} + x_{i(\nu)} + x_{i(\mu+\nu)} \bmod (2) \tag{2.5}$$

where $x_{i(\mu)} = X(\mu_i \ldots \mu_{i+k-1})$. This is called the *obstruction map* of X. Properties of this map, which defines a multifractal surface over the unit square, are studied in [7, 8]. Generalization to rules defined over nonbinary alphabets are studied in [9], where it is shown that the generated surface is self-similar of Hausdorff dimension 2.

3. Computation of the *Z* parameter

The de Bruijn diagram for a k-site rule X is a labeled directed graph with 2^{k-1} vertices and 2^k edges. Vertices are labeled by $i_0 \ldots i_{k-2}$ with $i_s \in \{0,1\}$ and there is an edge directed from vertex $i_0 \ldots i_{k-2}$ to vertex $j_0 \ldots j_{k-2}$ if and only if $j_s = i_{s+1}$ for $0 \le s \le k-3$. That is, if and only if $i_0 \ldots i_{k-2}$ and $j_0 \ldots j_{k-2}$ are respectively the first and last $k-1$ digits of a k-site neighborhood. This neighborhood will be denoted by i^*j. The corresponding edge of the de Bruijn diagram is then labeled by $X(i^*j)$.

The adjacency matrix for the de Bruijn diagram of a k-site rule X is the $2^{k-1} \times 2^{k-1}$ matrix

$$d(X) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}. \tag{3.1}$$

This matrix splits naturally into the sum of two matrices, $d_0(X)$ and $d_1(X)$, each of which can be written in terms of the components of the rule X. With $x_i' = 1 + x_i \bmod (2)$

$$d_0(X) = \begin{pmatrix} x'_0 & x'_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x'_2 & x'_3 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ x'_{2^{k-1}} & x'_{2^{k-1}+1} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & x'_{2^k-2} & x'_{2^k-1} \end{pmatrix} \tag{3.2}$$

$$d_1(X) = \begin{pmatrix} x_0 & x_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_2 & x_3 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ x_{2^{k-1}} & x_{2^{k-1}+1} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & x_{2^k-2} & x_{2^k-1} \end{pmatrix}. \tag{3.3}$$

The matrices $d_0(X)$ and $d_1(X)$ are called *de Bruijn fragments*. They have been used by Wolfram [10] in a study of the relation between cellular automata and formal languages, by Jen [11, 12] in computation of preimages, and their applications are exhaustively reviewed by McIntosh [13]. The following two theorems of particular interest are given in [7].

Theorem 1. The number of preimages of a sequence $s_1 \ldots s_n$ under a k-site rule X is given by

$$\sum \prod_{i=1}^{n} d_{s_i}(X) \tag{3.4}$$

where the sum is over all entries in the matrix product.

Theorem 2. A k-site rule X is surjective if and only if the free semigroup with generators $d_0(X)$ and $d_1(X)$ does not contain the 0 matrix.

Definition 1. Let X be a k-site rule with de Bruijn fragments $d_0(X)$ and $d_1(X)$. The *reduced fragment matrices* $d_0(X, k-r)$ and $d_1(X, k-r)$ are constructed as follows.

1. $d_0(X, k) = d_0(X)$ and $d_1(X, k) = d_1(X)$.
2. For $0 < r \leq k-2$ and $s \in \{0, 1\}$, $d_s(X, k-r)$ is iteratively generated from $d_s(X, k-r+1)$ by the procedure below.
   (a) Partition the $2^{k-r-1} \times 2^{k-r-1}$ matrix $d_s(X, k-r+1)$ into 2×2 blocks.
   (b) Substitute a 0 for each 2×2 block consisting of all 0 entries.
   (c) For all other blocks, substitute the product of the rule components contained in that block.

Example 1. If X is a 3-site rule then

$$d_0(X,3) = \begin{pmatrix} x'_0 & x'_1 & 0 & 0 \\ 0 & 0 & x'_2 & x'_3 \\ x'_4 & x'_5 & 0 & 0 \\ 0 & 0 & x'_6 & x'_7 \end{pmatrix}$$

$$d_1(X,3) = \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_3 \\ x_4 & x_5 & 0 & 0 \\ 0 & 0 & x_6 & x_7 \end{pmatrix}$$

$$d_0(X,2) = \begin{pmatrix} x'_0x'_1 & x'_2x'_3 \\ x'_4x'_5 & x'_6x'_7 \end{pmatrix}$$

$$d_1(X,2) = \begin{pmatrix} x_0x_1 & x_2x_3 \\ x_4x_5 & x_6x_7 \end{pmatrix}. \tag{3.5}$$

Wuensche [5, 6] gives a computational method for calculating the Z parameter. One computes two parameters, Z_l and Z_r and defines Z as the larger of the two. Both Z_l and Z_r arise as probabilities in the construction of preimages for given sequences. Z_l is the probability that the next (unknown) cell to the right in a partial preimage has a uniquely determined value and Z_r is the probability that the next (unknown) cell to the left in a partial preimage has a uniquely determined value.

If the next cell (right or left) in construction of a preimage has a uniquely determined value, no bifurcation can occur at that point in the construction. Thus, Z gives a measure of a degree of restriction on the number of preimages. Wuensche's procedure is as follows.

To compute Z_l consider pairs of r-definite neighborhoods

$$\begin{array}{ll} (i_0 \dots i_{k-2}0, i_0 \dots i_{k-2}1) & r = 2 \\ (i_0 \dots i_{k-r}0s_0 \dots s_{r-3}, i_0 \dots i_{k-r}1s_0 \dots s_{r-3}) & 3 \le r \le k \\ (0s_0 \dots s_{k-2}, 1s_0 \dots s_{k-2}) & r = k+1 \end{array}$$

where $i_0 \dots i_{k-r}$ is fixed and $s_0 \dots s_{r-3}$ is arbitrary. Let n_{k-r+2} be the number of r-definite neighborhoods in the rule table that are deterministic, that is, such that

$$\begin{aligned} &X(i_0 \dots i_{k-r}0s_0 \dots s_{r-3}) = t \\ &X(i_0 \dots i_{k-r}1s_0 \dots s_{r-3}) = t + 1 \bmod (2). \end{aligned} \tag{3.6}$$

The probability that the next cell to the right is determined by equation (3.6) is

$$R^{(l)}_{k-r+2} = \frac{n_{k-r+2}}{2^k}$$

and Z_l is the union of these probabilities for $2 \le r \le k+1$. Formally

000	001	010	011	100	101	110	111

(a)

000	100	001	101	010	110	011	111

(b)

Figure 1. Pairs of neighborhoods compared for computation of (a) Z_l and (b) Z_r.

this is given by

$$Z_l = R_k^{(l)} + \sum_{s=1}^{k-1} R_{k-s}^{(l)} \left[\prod_{j=k-s+1}^{k} (1 - R_j^{(l)}) \right]. \tag{3.7}$$

A similar procedure, going from right to left, yields

$$Z_r = R_k^{(r)} + \sum_{s=1}^{k-1} R_{k-s}^{(r)} \left[\prod_{j=k-s+1}^{k} (1 - R_j^{(r)}) \right]. \tag{3.8}$$

Figure 1 gives a graphical representation of the comparison process involved in computing Z_l and Z_r for a 4-site rule. (Note that the rule components are rearranged for ease of presentation for Z_r.)

Consider the computation of $R_k^{(l)}$. As illustrated in Figure 1, pairs of rule components (x_{2i}, x_{2i+1}) are compared for $0 \leq i \leq 2^{k-1} - 1$. Further, the rule X is deterministic on a pair (x_{2i}, x_{2i+1}) if and only if these two components have distinct values, that is, if and only if $x_{2i+1} = x_{2i} + 1 \bmod (2)$. Thus

$$x_{2i}x'_{2i+1} + x'_{2i}x_{2i+1} = \begin{cases} 0 & X \text{ is nondeterministic on } (x_{2i}, x_{2i+1}) \\ 1 & X \text{ is deterministic on } (x_{2i}, x_{2i+1}) \end{cases} \tag{3.9}$$

and hence

$$n_{k-2}^{(l)} = \sum_{i=0}^{2^{k-1}-1} (x_{2i}\, x'_{2i+1} + x'_{2i}\, x_{2i+1}). \tag{3.10}$$

Similar considerations eventually yield results for all the n_{k-r+2} and give the following theorem.

Theorem 3. For $0 \leq r \leq k-1$

$$R_{k-r}^{(l)} = \frac{1}{2^{k-r-1}} \sum_{s=0}^{2^{k-r-1}-1} \left[\prod_{j=0}^{2^r-1} x_{2^{r+1}s+j}\, x'_{2^{r+1}s+2^r+j} + \prod_{j=0}^{2^r-1} x'_{2^{r+1}s+j}\, x_{2^{r+1}s+2^r+j} \right]$$

$$R_{k-r}^{(r)} = \frac{1}{2^{k-r-1}} \sum_{s=0}^{2^{k-r-1}-1} \left[\prod_{j=0}^{2^r-1} x_{2^r s+j} x'_{2^r s+2^{k-1}+j} + \prod_{j=0}^{2^r-1} x'_{2^r s+j} x_{2^r s+2^{k-1}+j} \right]. \tag{3.11}$$

Examining the form of equation (3.11), and comparing it to equations (3.2) and (3.3) yields formulas in terms of the reduced de Bruijn fragments.

Theorem 4. For a k-site rule X with $0 \leq r \leq k-2$

$$R_{k-r}^{(l)} = \frac{1}{2^{k-r-1}} \sum_{i,j} [d_1^T(X, k-r) d_0(X, k-r)]_{ij}$$

$$R_{k-r}^{(r)} = \frac{1}{2^{k-r-1}} \sum_{i,j} [d_1(X, k-r) d_0^T(X, k-r)]_{ij} \tag{3.12}$$

where T denotes transpose. If $r = k-1$ then

$$R_1^{(l)} = R_1^{(r)} = x_0 \ldots x_{2^{k-1}-1} x'_{2^{k-1}} \ldots x'_{2^k-1} + x'_0 \ldots x'_{2^{k-1}-1} x_{2^{k-1}} \ldots x_{2^k-1}. \tag{3.13}$$

4. The λ and Θ parameters as invariants

In [8] it is shown that the graphs of the maps $X : [0,1] \to [0,1]$ and $U(X) : [0,1]^2 \to [0,1]$ are generated as the limits of an iterative process of concatenation substitution. The algorithm outlined next gives the definition of this process for $X : [0,1] \to [0,1]$.

Algorithm for the graph of $X : [0,1] \to [0,1]$

If X is a k-site rule then the neighborhood set $\{i_0 \ldots i_{k-1}\}$ partitions $[0,1]$ into 2^k equal segments $[i/2^k,\ (i+1)/2^k]$, each of length 2^{-k}. The 0th order approximation to the graph of X over $[0,1]$ is the histogram with each of these segments as a bin and the height of bin i equal to x_i.

The $n+1$ order approximation is obtained iteratively from the nth order approximation by dividing each of the nth order bins in half. If the generic nth order bin is labeled $s_0 \ldots s_{k+n-1}$ then the two corresponding $n+1$ order bins are labeled $s_0 \ldots s_{k+n-1} s_{k+n}$ where s_{k+n} is 0 for the first bin and 1 for the second. The height assigned to each of these bins is

$$h_{n+1}(s) = \left(\frac{2^{n+1}}{2^{n+1}-1} \right) \sum_{i=0}^{n+1} \frac{x_{i(s)}}{2^{i+1}} \tag{4.1}$$

where $x_{i(s)} = X(s_i \ldots s_{i+k-1})$. That is, $h_{n+1}(s)$ is the decimal value of the periodic binary given by the expansion $.\overline{x_{0(s)} \ldots x_{(n+1)(s)}}$. In the limit $n \to \infty$ this histogram converges to the graph of X over $[0,1]$.

If $A(n,X)$ is the area under the nth order approximation to the graph of X, then $\lim_{n\to\infty} A(n,X) = A(X)$ is the area under the graph. By definition,

$$A(0,X) = \frac{1}{2^k} \sum_{i=0}^{2^k-1} x_i = \lambda(X). \tag{4.2}$$

Theorem 5. [8] $A(n,X) = A(0,X)$ for all n, hence $\lambda(X) = A(X)$.

A similar, although far more complicated construction, yields a formula for approximations to the graph of $U(X)$, and shows that the volume $V(X)$ under this graph is equal to the obstruction parameter $\Theta(X)$.

5. Discussion

Three parameters that have been shown to characterize aspects of cellular automata behavior have been considered in the case of one-dimensional binary valued rules. A considerable amount of work has been done on the significance of both the λ and Z parameters. Wuensche and Lesser [4] note that the λ parameter is best represented in terms of what they call the "λ-ratio," denoted λ_r which is given by

$$\lambda_r = \begin{cases} 2\lambda & \lambda \le \frac{1}{2} \\ 2(1-\lambda) & \lambda > \frac{1}{2}. \end{cases} \tag{5.1}$$

This number satisfies $Z \le \lambda_r$. Wuensche [5] proposes that λ indicates the probability of the value of Z, and carries out comparisons of Z and λ_r. The Z parameter gives a quantification of the probability that the next cell in a partial preimage is determined. In this way it reflects the preimaging characteristics of a given rule.

McIntosh [13] gives a relation between the maximum eigenvalue of the de Bruijn fragments and the λ parameter. Let $\mu = \max\{\nu | d_s x = \nu x, s = 0,1\}$, $\underline{x}$ be the eigenvector of d_s corresponding to the eigenvalue μ, normalized so that it is a probability vector.

Then, for $s \in \{0,1\}$ the index for the de Bruijn fragment with eigenvalue μ, define the quantities

$$c_i = \sum_{j=0}^{2^{k-1}-1} [d_s]_{ij} \qquad \overline{c} = \frac{1}{2^{k-1}} \sum_{i=0}^{2^{k-1}-1} c_i$$

$$\overline{x} = \frac{1}{2^{k-1}} \sum_{i=0}^{2^{k-1}-1} x_i.$$

Then for a k-site rule

$$\mu = \gamma + \underline{x}^* \cdot \underline{c}^* \tag{5.2}$$

where

$$\gamma = \frac{1}{2^{k-1}} \sum_{i,j} [d_s]_{ij} = \begin{cases} 2\lambda & \mu \text{ is an eigenvalue of } d_1 \\ 2(1-\lambda) & \mu \text{ is an eigenvalue of } d_0 \end{cases} \tag{5.3}$$

and the components of $\underline{c}^*$ and $\underline{x}^*$ are respectively $(c_i - \overline{c})$ and $(x_i - \overline{x})$. For example, for rule 22 the d_0 matrix has maximum eigenvalue $\mu \sim 1.46557$. The λ parameter for rule 22 is 0.375, hence $\gamma = 1.25$. The matrix d_0 is

$$d_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The eigenvector for the maximum eigenvalue μ, normalized to a probability vector, is given by $x = (0, 0.31766, 0.21676, 0.46557)$, while the vector of column sums is $c = (1, 1, 1, 2)$. Thus the vectors of residuals are respectively $(-0.25, 0.06766, -0.03324, 0.21557)$ and $(-0.25, -0.25, -0.25, 0.75)$. The inner product of these two vectors is just 0.21557 and $1.25 + 0.21557 = 1.46557$.

The obstruction parameter Θ characterizes the nonadditivity of a rule. The 128 possible 3-site rules with $000 \to 0$ can be grouped into five classes determined by the Θ value. It is instructive to represent these classes in terms of decomposition of rules into additive and nonadditive parts.

Every rule X can be written as a sum $X = A + F$ of an additive rule (A) and a nonadditive rule (F), where addition is defined by $x_i = a_i + f_i \bmod (2)$. There are eight additive 3-site rules, listed in Table 1.

A useful set of nonadditive rules for carrying out this decomposition is given in terms of the eight "unit" rules that have only one nonzero component. This set is shown in Table 2.

Rule		Definition
(0)	0	all neighborhoods $\to 0$
(60)	D^-	$010, 011, 100, 101 \to 1$
(90)	δ	$001, 011, 100, 110 \to 1$
(102)	D	$001, 010, 101, 110 \to 1$
(150)	Δ	$001, 010, 100, 111 \to 1$
(170)	σ	$001, 011, 101, 111 \to 1$
(204)	I	$010, 011, 110, 111 \to 1$
(240)	σ^{-1}	$100, 101, 110, 111 \to 1$

Table 1. 3-site nongenerative additive rules.

$\beta^+ : 110 \to 1$	$\beta^- : 011 \to 1$	$\eta^+ : 001 \to 1$
$\eta^- : 100 \to 1$	$\chi : 111 \to 1$	$\theta : 101 \to 1$
$\iota : 010 \to 1$	$\beta^+ + \chi :$ $110, 111 \to 1$	$\beta^- + \chi :$ $011, 111 \to 1$
$\beta^+ + \theta :$ $110, 101 \to 1$	$\beta^- + \theta :$ $011, 101 \to 1$	$\chi + \theta :$ $111, 101 \to 1$
$\chi + \iota :$ $111, 010 \to 1$	$\theta + \iota :$ $101, 010 \to 1$	$\chi + \theta + \iota :$ $111, 101, 010 \to 1$

Table 2. Nonadditive parts of rules in distinct additivity classes.

Θ value	Rule Numbers
0 (additive rules)	0, 60, 90, 102, 150, 170, 204, 240
9/32	2, 4, 8, 16, 22, 26, 28, 32, 38, 42, 44, 52, 56, 62, 64, 70, 74, 76, 82, 88, 94, 98, 100, 110, 112, 118, 122, 124, 128, 134, 138, 140, 146, 148, 158, 162, 168, 174, 176, 182, 186, 188, 196, 200, 206, 208, 214, 218, 220, 224, 230, 234, 236, 242, 244, 248
5/16	12, 30, 34, 48, 68, 86, 106, 120, 136, 154, 166, 180, 192, 210, 238, 252
3/8	6, 10, 18, 20, 24, 36, 40, 46, 54, 58, 66, 72, 78, 80, 92, 96, 108, 114, 116, 126, 130, 132, 142, 144, 156, 160, 172, 178, 184, 190, 198, 202, 212, 216, 222, 226, 228, 232, 246, 250
21/32	14, 50, 84, 104, 152, 164, 194, 254

Table 3. Additivity classes for nongenerative 3-site rules.

Grouping the 128 3-site rules according to their Θ value yields the five classes shown in Table 3.

Finally, each cell in Table 4 contains the rule number; a symbol E, F, L, or N; the λ-ratio; and the pair (Z_l, Z_r). The symbols E, F, L, and N refer to the nature of the intrinsic Garden-of-Eden for the rule. In [7, 14] it is shown that the Garden-of-Eden for a rule X, denoted GE(X), is generated by a seed set $\text{GE}^*(X)$ of finite strings such that if $s \in \text{GE}^*(X)$ then:

1. s has no preimage under X.
2. Every substring of s does have a preimage under X.

Then:

E $\Rightarrow$ $\text{GE}^*(X)$ is empty.

F $\Rightarrow$ $\text{GE}^*(X)$ is finite.

L ⇒ The number of elements in $\text{GE}^*(X)$ having length n is linear in n

N ⇒ The number of elements in $\text{GE}^*(X)$ of length n grows faster than linearly with n.

The structure of the Garden-of-Eden is relevant since the faster the seed set GE^* grows, the fewer sequences are available as preimages with increasing string length.

A number of observations can be made about Table 4.

1. For $\Theta = 9/32$ and $21/32$, $Z_l = Z_r$ if the additive part of a rule is symmetric (i.e., A is 0, δ, Δ, or I), and $Z_l \neq Z_r$ if the additive part is skew (i.e., A is D, D^-, σ, or σ^{-1}).
2. If $\Theta = 9/32$ and A is skew left (D^- or σ^{-1}) then $Z_l < Z_r$ while if A is skew right (D or σ) this inequality is reversed.
3. For $\Theta = 9/32$ and A given by 0, σ, I, or σ^{-1} the set $\text{GE}^*(X)$ is finite.
4. For $\Theta = 9/32$ and A given by D^-, δ, D, or Δ 20 of the 28 rules have $\text{GE}^*(X)$ growing faster than linearly in string length. For rules 38, 100, 52, and 44 $\text{GE}^*(X)$ grows linearly in string length. Rules 38 and 100 are $D + \beta^+$ and $D + \eta^+$ respectively, while rules 52 and 44 are $D^- + \beta^-$ and $D^- + \eta^-$. For rules 28, 56, 70, and 98 $\text{GE}^*(X)$ is finite.

	0	D^-	δ	D	Δ	σ	I	σ^{-1}
0 additive rules →	0	60 E 1 1,1	90 E 1 1,1	102 E 1 1,1	150 E 1 1,1	170 E 1 1,1	204 E 1 1,1	240 E 1 1,1
β^+	64 F 0.25 0.25,0.25	124 N 0.75 0.625,0.75	26 N 0.75 0.75,0.75	38 L 0.75 0.75,0.625	214 N 0.75 0.75,0.75	234 F 0.75 0.75,0.25	140 F 0.75 0.625,0.625	176 F 0.75 0.25,0.75
β^-	8 F 0.25 0.25,0.25	52 L 0.75 0.625,0.75	82 N 0.75 0.75,0.75	110 N 0.75 0.75,0.625	158 N 0.75 0.75,0.75	162 F 0.75 0.75,0.25	196 F 0.75 0.625,0.625	248 F 0.75 0.25,0.75
η^+	2 F 0.25 0.25,0.25	62 N 0.75 0.625,0.75	88 N 0.75 0.75,0.75	100 L 0.75 0.75,0.625	148 N 0.75 0.75,0.75	168 F 0.75 0.75,0.25	206 F 0.75 0.625,0.625	242 F 0.75 0.25,0.75
η^-	16 F 0.25 0.25,0.25	44 L 0.75 0.625,0.75	74 N 0.75 0.75,0.75	118 N 0.75 0.75,0.625	134 N 0.75 0.75,0.75	186 F 0.75 0.75,0.25	220 F 0.75 0.625,0.625	224 F 0.75 0.25,0.75
χ	128 F 0.25 0.25,0.25	188 N 0.75 0.625,0.75	218 N 0.75 0.75,0.75	230 N 0.75 0.75,0.625	22 N 0.75 0.75,0.75	42 F 0.75 0.75,0.25	76 F 0.75 0.625,0.625	112 F 0.75 0.25,0.75
θ	32 F 0.25 0.25,0.25	28 F 0.75 0.625,0.75	122 N 0.75 0.75,0.75	70 F 0.75 0.75,0.625	182 N 0.75 0.75,0.75	138 F 0.75 0.75,0.25	236 F 0.75 0.625,0.625	208 F 0.75 0.25,0.75
ι	4 F 0.25 0.25,0.25	56 F 0.75 0.625,0.75	94 N 0.75 0.75,0.75	98 F 0.75 0.75,0.625	146 N 0.75 0.75,0.75	174 F 0.75 0.75,0.25	200 F 0.75 0.625,0.625	244 F 0.75 0.25,0.75

Table 4(a). $\theta = 0$ (top row), $\theta = 9/32$.

	0	D^-	δ	D	Δ	σ	I	σ^{-1}
$\beta^+ + \chi$	192 F 0.5 0.5,0.5	252 F 0.5 0.5,0.5	154 E 1 1,0.5	166 E 1 1,0.5	86 E 1 1,0.5	106 E 1 1,0.5	12 F 0.5 0.5,0.5	48 F 0.5 0.5,0.5
$\beta^- + \chi$	136 F 0.5 0.5,0.5	180 E 1 0.5,1	210 E 1 0.5,1	238 F 0.5 0.5,0.5	30 E 1 0.5,1	34 F 0.5 0.5,0.5	68 F 0.5 0.5,0.5	120 E 1 0.5,1

Table 4(b). $\theta = 5/16$.

	0	D^-	δ	D	Δ	σ	I	σ^{-1}
$\beta^+ + \theta$	96 F 0.5 0.5,0.5	92 L 1 0.75,0.5	58 L 1 0.75,0.5	6 F 0.5 0.5,0.5	246 F 0.5 0.5,0.5	202 L 1 0.75,0.5	172 L 1 0.75,0.5	144 F 0.5 0.5,0.5
$\beta^- + \theta$	40 F 0.5 0.5,0.5	20 F 0.5 0.5,0.5	114 L 1 0.5,0.75	78 L 1 0.5,0.75	190 F 0.5 0.5,0.5	130 F 0.5 0.5,0.5	228 L 1 0.5,0.75	216 L 1 0.5,0.75
$\chi + \theta$	160 F 0.5 0.5,0.5	156 F 1 0.75,0.75	250 F 0.5 0.5,0.5	198 F 1 0.75,0.75	54 F 1 0.75,0.75	10 F 0.5 0.5,0.5	108 F 1 0.75,0.75	80 F 0.5 0.5,0.5
$\chi + \iota$	132 F 0.5 0.5,0.5	184 F 1 0.5,0.5	222 F 0.5 0.5,0.5	226 F 1 0.5,0.5	18 N 0.5 0.5,0.5	46 L 1 0.5,0.5	72 N 0.5 0.5,0.5	116 L 1 0.5,0.5
$\theta + \iota$	36 F 0.5 0.5,0.5	24 F 0.5 0.5,0.5	126 F 0.5 0.5,0.5	66 F 0.5 0.5,0.5	178 F 1 0.5,0.5	142 F 1 0.5,0.5	232 F 1 0.5,0.5	212 F 1 0.5,0.5

Table 4(c). $\theta = 3/8$.

	0	D^-	δ	D	Δ	σ	I	σ^{-1}
$\chi + \theta + \iota$	164 N 0.75 0.75,0.75	152 N 0.75 0.75,0.625	254 F 0.25 0.25,0.25	194 N 0.75 0.625,0.75	50 F 0.75 0.75,0.75	14 F 0.75 0.25,0.75	104 N 0.75 0.75,0.75	84 F 0.75 0.75,0.25

Table 4(d). $\theta = 21/32$.

5. If $\Theta = 9/32$ and A is symmetric but not the identity rule I then $\lambda_r = Z$ while if $A = I$ then $\lambda_r = 0.75$ and $Z = 0.625$.

6. If $\Theta = 5/16$ then $\text{GE}^*(X)$ is either empty or finite. If it is finite, then $\lambda_r = Z_l = Z_r = 0.5$. If $\text{GE}^*(X)$ is empty then $\lambda_r = 1$ and (Z_l, Z_r) is $(1, 0.5)$ for $\text{F} = \beta^+ + \chi$ or $(0.5, 1)$ for $\text{F} = \beta^- - \chi$.

7. For $\Theta = 3/8$ there are 40 rules, with 28 having $\text{GE}^*(X)$ finite. This class contains two subclasses: rules with $\text{F} = \beta^\pm + \theta$ and rules with F given by $\chi + \theta$, $\chi + \iota$, or $\theta + \iota$. In each of these last cases F is symmetric. If F is $\chi + \theta$ or $\theta + \iota$ then $\text{GE}^*(X)$ is finite, and for all rules in this second subclass $Z_l = Z_r$.

8. For $\Theta = 3/8$ and $\text{F} = \beta^\pm + \theta$ either $\lambda_r = Z_l = Z_r = 0.5$ and $\text{GE}^*(X)$ is finite, or $\lambda_r = 1$ and (Z_l, Z_r) is given by either $(0.75, 0.5)$ ($\text{F} = \beta^+ + \theta$), or by $(0.5, 0.75)$ ($\text{F} = \beta^- + \theta$) and $\text{GE}^*(X)$ grows linearly with sequence length.

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