Cyclic Evolution of Neuronal Automata with Memory When All the Weighting Coefficients are Strictly Positive

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In this paper we study the sequences generated by a neuronal equation of the form $x_n = \mathbf{1}[\sum_{i=1}^{k} a_i x_{n-i} - \theta]$, where k is the size of the memory. We show that in the case where all the parameters $(a_i)_{1\le i\le k}$ are strictly positive reals and k is a multiple of 6 (i.e., $k = 6m$), there exists a subset $B = \{w_1, w_2, \ldots, w_\alpha\}$ of $\{1, 2, \ldots, m-1\}$ and for which there exists a neuronal equation of memory length αk (i.e., $k = 6m$) with strictly positive weighting coefficients that generates a sequence of period $\alpha * \mathrm{lcm}(3m - w_1, \ldots, 3m - w_\alpha)$. As an application; first, for all integer m greater than or equal to 3, we exhibit a neuronal recurrence equation of memory length $2k$ with weighting coefficients that are all strictly positive and which generates a sequence of period $2(3m-1)(3m-2)$; and second, for all odd integer m greater than or equal to 5, we exhibit a neuronal equation of memory length $3k$ with strictly positive weighting coefficients that generates a sequence of period $3(3m-1)(3m-2)(3m-4)$.

1. Introduction

In [3] it is suggested that the dynamic behavior of a single neuron with memory that does not interact with other neurons can be modeled by the following recurrence equation:

$$x_n = \mathbf{1}\left[\sum_{i=1}^{k} a_i x_{n-i} - \theta\right] \qquad (1)$$

where we have the following.

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- x_n is a boolean variable representing the state of the neuron at $t = n$.
- k is the memory length, that is to say, the state of the neuron at time $t = n$ depends on the state assumed by the neuron at the k previous steps $t = n - 1, \ldots, n - k$.
- $a_i (i = 1, \ldots, k)$ are real numbers called the *weighting coefficients*, a_i represents the influence of the state of the neuron at time $n - i$ on the state of the neuron at time n. The influence of the state of the neuron at time $n - i$ on the state of the neuron at time n is said to be *excitatory* if $a_i > 0$, *inhibitory* if $a_i < 0$, and *null* if a_i is equal to zero.
- θ is a real number called the *threshold*.
- $\mathbf{1}[u] = 0$ if $u < 0$, and $\mathbf{1}[u] = 1$ if $u \geq 0$.

The system obtained by interconnecting several neurons is called a *neural network*. Such networks were introduced by McCulloch and Pitts in [9], and are quite powerful. Indeed, it can be shown that they can be used to simulate any Turing machine. More recently, neural networks have been studied extensively as tools for solving various problems such as classification, speech recognition, and image recognition [14]. The application field for the threshold functions is large. Take, for example, the spin moment in solid state physics and other physical models. In electricity, for instance, a threshold function represents a transistor; in social science a threshold function is often used to represent a vote law. Readers interested in the application field for threshold functions may refer to [6, 7, 13]. We draw attention to the fact that in [16] a threshold iteration is interpreted as the computation of the subdifferential of a convex functional.

Let p and T be two positive integers such that $p > 0$ and $T \geq 0$. Equation (1) is said to be of period p and transient T if and only if:

- $\forall\ T, t' \in \mathbb{N}, t' \in \{k - 1, k, \ldots, p + T - 1\}, t \neq t'$ implies that $Y_t \neq Y_{t'}$
- $Y_{p+T} = Y_T$

where $Y(t) = (x_t, x_{t-1}, \ldots, x_{t-k+2}, x_{t-k+1})$. The period and transient of sequences generated by a neuron are good measures of the complexity of the behavior of the neuron.

Let us denote $LP(k)$ as the longest period that can be generated by a neuronal equation with memory length k. In [4], it was conjectured that if $(a_i)_{1 \leq i \leq k} \in \mathbb{R}$, then $LP(k) \leq 2k$. This conjecture has been disproved in [5, 11, 15, 17], where neuronal recurrence equations that generate sequences of periods $O(k^3)$, $O(e^{\sqrt{k}})$, $O(e^{\sqrt{k} \log k})$, and $2k + 6$ respectively have been exhibited.

When all the weighting coefficients are positive, the influence of the previous state of a neuron (at time $n - k, n - k + 1, \ldots, n - 2, n - 1$) on its state at time n are excitatory, and from a physiological point of view, it is important to know the behavior of the neuron, and also the operator T defined as:

$$(x_{n-k}, x_{n-k+1}, \ldots, x_{n-2}, x_{n-1}) \longrightarrow (x_{n-k+1}, x_{n-k+2}, \ldots, x_{n-1}, x_n)$$

where $x_n = \mathbf{1}[\sum_{i=1}^{k} a_i x_{n-i} - \theta]$ is a monotone map. Monotone functions are used in convex analysis [2]. It was conjectured in [4] that if $\forall\, i$, $i = 1, \ldots, k$, $a_i \in \mathbb{R}^+$ (i.e., $a_i \geq 0$), then $LP(k) \leq k$. This conjecture has been disproved in [12, 17] where on one side a neuronal recurrence equation of memory length 7 with period 11 has been exhibited and on another side a neuronal recurrence equation of memory length 8 with period 18 has been exhibited.

In this work, we show that in the case where all the parameters $(a_i)_{1\leq i\leq k}$ are positive reals and k is a multiple of 6 (i.e., $k = 6m$), there exists a subset $B = \{w_1, w_2, \ldots, w_\alpha\}$ of $\{1, 2, \ldots, m-1\}$ and for which a neuronal equation of memory length αk ($k = 6m$) exists with strictly positive weighting coefficients that generates a sequence of period $\alpha * \operatorname{lcm}(3m - w_1, \ldots, 3m - w_\alpha)$. As an application; first, for all integer m greater than or equal to 3, we exhibit a neuronal recurrence equation of memory length $2k$ with all weighting coefficients strictly positive that generates a sequence of period $2(3m-1)(3m-2)$; and second, for all odd integer m greater than or equal to 5, we exhibit a neuronal equation of memory length $3k$ with all weighting coeffients strictly positive that generates a sequence of period $3(3m-1)(3m-2)(3m-4)$.

2. Construction of neuronal equations with strictly positive weighting coefficients

Tchuenté and Tindo consider in [11] an integer k which is a multiple of 6 (i.e., $k = 6m$), a positive real number $\theta \geq 2m$, and k real coefficients $(a_i)_{1\leq i\leq k}$ defined as follows:

$$a_j = \begin{cases} \theta/2 - i, & \text{if } j = 3m - i,\ 1 \leq i \leq m \\ \theta/2 + i, & \text{if } j = 2(3m - i),\ 1 \leq i \leq m \\ -k(\theta + m), & \text{otherwise.} \end{cases} \tag{2}$$

They defined for each r, $1 \leq r \leq m-1$, the first k terms of the sequence $\{(x_n^r); n \geq 0\}$ as follows:

$$(x_0^r, x_1^r, \ldots, x_{k-1}^r) = \underbrace{0\,0\ldots 0}_{2r}\ \ \underbrace{1\,0\,0\ldots 0}_{3m-r}\ \ \underbrace{1\,0\,0\ldots 0}_{3m-r} \tag{3}$$

and have shown [11] that with the parameters defined in equation (2), the sequence $\{(x_n^r); n \geq 0\}$ generated by neuronal recurrence equation (1) is of period $3m - r$.

$\forall\, \alpha \in \mathbb{N}$, the term $x_\alpha^r, x_{\alpha+1}^r, \ldots, x_{\alpha+k-1}^r$ contributes in the calculation of the term $x_{\alpha+k}^r$. We want to characterize the set of i, $1 \leq i \leq k$ such that

$x^r_{\alpha+k-i} = 1$. Considering that the evolution of the sequence $\{(x^r_n); n \geq 0\}$ is of the form

$$\underbrace{0\,0\ldots0}_{2r}\;\;\underbrace{1\,0\,0\ldots0}_{3m-r}\;\;\underbrace{1\,0\,0\ldots0}_{3m-r}\;\;\ldots\;\;\underbrace{1\,0\,0\ldots0}_{3m-r}\;\;\ldots \tag{4}$$

we distinguish the following two cases.

Case 1. $\exists\,\lambda \in \mathbb{N}$ such that $\alpha + k = 2r + \lambda(3m - r)$.

Let us denote $\mathrm{contr1}(r) = \{i/x^r_{\alpha+k-i} = 1 \text{ and } 1 \leq i \leq k\}$. From equation (4), one can easily prove that $\mathrm{contr1}(r) = \{6m - 2r, 3m - r\}$.

Case 2. $\exists\,\lambda, \beta \in \mathbb{N}, 0 < \beta < 3m - r$ such that $\alpha + k = 2r + \lambda(3m - r) + \beta$.

Let us denote $\mathrm{contr2}(r,\beta) = \{i/x^r_{\alpha+k-i} = 1 \text{ and } 1 \leq i \leq k\}$. From equation (4), one easily shows that:

1. $1 \leq \beta \leq 2r$
   $\mathrm{contr2}(r,\beta) = \{6m - 2r + \beta, 3m - r + \beta, \beta\}$
2. $2r + 1 \leq \beta \leq 3m - r - 1$
   $\mathrm{contr2}(r,\beta) = \{3m - r + \beta, \beta\}$.

Let the predicate $Q(m, r1, r2, r3)$ be defined as follows:

$$Q(m, r1, r2, r3) : \begin{cases} r1 \neq r3, \;\; r2 \neq r3 \\ \exists\,\beta 1, 1 \leq \beta 1 \leq 3m - r3 - 1 \text{ and } \exists\, e, \;fe \neq f \\ e \in \mathrm{contr2}(r3,\beta 1), \; f \in \mathrm{contr2}(r3,\beta 1) \\ e \in \mathrm{contr1}(r1) \text{ and } f \in \mathrm{contr1}(r2). \end{cases} \tag{5}$$

Note that when $r1 = r2$, the predicate $Q(m, r1, r2, r3)$ becomes:

$$Q(m, r1, r2, r3) : \begin{cases} r1 \neq r3 \\ \exists\,\beta 1, 1 \leq \beta 1 \leq 3m - r3 - 1 \text{ and } \exists\, e, \;fe \neq f \\ e \in \mathrm{contr2}(r3,\beta 1) \text{ and } f \in \mathrm{contr2}(r3,\beta 1) \\ \{e, \; f\} = \mathrm{contr1}(r1). \end{cases} \tag{6}$$

Let us denote $H = \{w_1, w_2, \ldots, w_\alpha\}$ as any subset of indices r of sequences $\{(x^r_n); n \geq 0\}$ (i.e., of $\{1, 2, \ldots, m-1\}$) such that $\forall\, r1, r2, r3 \in H$, $r1 \neq r3$ and $r2 \neq r3 \Rightarrow \neg Q(r1, r2, r3)$ (i.e., $Q(m, r1, r2, r3)$ is false).

Let us define $G = \cup_{w_j \in H} \mathrm{contr1}(w_j)$. Consider the following neuronal recurrence equation of memory length k:

$$y_n = \mathbf{1}\left[\sum_{i=1}^{k} b_i y_{n-i} - \theta_1\right] \tag{7}$$

where:

$$\begin{cases} b_i = a_i + 2k(\theta + m) & \text{if } 1 \leq i \leq k \text{ and } i \notin G \\ b_i = a_i + 6k(\theta + m) & \text{if } 1 \leq i \leq k \text{ and } i \in G \\ \theta_1 = \theta + 12k(\theta + m). \end{cases} \tag{8}$$

The parameters a_i, θ, k, and m intervening in equation (8) are those defined in equation (2). The weighting coefficients $(b_i)_{1\le i\le k}$ are all positive. For all $w_j \in H$, we are considering the sequence $\{(y_n^{w_j}); n \ge 0\}$ generated by neuronal recurrence equation (7) with the first k terms being equal to the first k terms of the sequence $\{(x_n^{w_j}); n \ge 0\}$ defined in equation (3). In other words:

$$y_0^{w_j}, y_1^{w_j}, \ldots, y_{k-1}^{w_j} = \underbrace{0\,0\ldots0}_{2w_j}\ \underbrace{1\,0\,0\ldots0}_{3m-w_j}\ \underbrace{1\,0\,0\ldots0}_{3m-w_j}. \qquad (9)$$

Lemma 1 characterizes the evolution of the sequence $\{(y_n^{w_j}); n \ge 0\}$ by showing in particular that it is the same as those of the sequence $\{(x_n^{w_j}); n \ge 0\}$.

Lemma 1. $\forall\, w_j \in H$, the sequence $\{(y_n^{w_j}); n \ge 0\}$ is of period $3m - w_j$.

Proof. It suffices to show by strong recurrence that $\forall\, n \in \mathbb{N}$, $y_n^{w_j} = x_n^{w_j}$.

Basic case: $n = 0, 1, 2, \ldots, k-1$. From equations (9) and (3) we deduce the result.

Recurrence hypothesis: We suppose that the property is true for the rank $t-k, t-k+1, \ldots, t-1$.

Rank t:

$$\begin{aligned} y_t^{w_j} &= \mathbf{1}\left[\sum_{i=1}^{k} b_i y_{t-i}^{w_j} - \theta_1\right] \\ &= \mathbf{1}\left[\sum_{i\in G} b_i x_{t-i}^{w_j} + \sum_{i\notin G} b_i x_{t-i}^{w_j} - \theta - 12k(\theta+m)\right]. \end{aligned}$$

On the basis of the fact that $\forall\, r1, r2, r3 \in H$, $r1 \ne r3$ and $r2 \ne r3 \Rightarrow \neg Q(m, r1, r2, r3)$, we distinguish the following two cases.

Case 1. $\exists w_z \in H$ such that $\{i/x_{t-i}^{w_j} = 1, \text{ and } 1 \le i \le k\} = \mathrm{contr1}(w_z)$.

We have:

$$\begin{aligned} y_t^{w_j} &= \mathbf{1}\left[\sum_{i\in \mathrm{contr1}(w_z)} b_i x_{t-i}^{w_j} + \sum_{i\notin \mathrm{contr1}(w_z)} b_i x_{t-i}^{w_j} - \theta - 12k(\theta+m)\right] \\ &= \mathbf{1}\left[\sum_{i\in \mathrm{contr1}(w_z)} a_i x_{t-i}^{w_j} + 12k(\theta+m) - \theta - 12k(\theta+m)\right] \\ &= \mathbf{1}\left[\sum_{i\in \mathrm{contr1}(w_z)} a_i x_{t-i}^{w_j} - \theta\right] \end{aligned}$$

$$= \mathbf{1}\left[\sum_{i=1}^{k} a_i x_{t-i}^{w_j} - \theta\right], \text{ because } i \notin \text{contr1}(w_z) \Rightarrow x_{t-i}^{w_j} = 0$$
$$= x_t^{w_j} \text{ by definition.}$$

From line one to line two, we use the fact that card contr1(w_z) = 2 and leaned on equation (8).

Case 2. $\nexists\, w_z \in H$ such that $\{i/x_{t-i}^{w_j} = 1, \text{ and } 1 \le i \le k\} = \text{contr1}(w_z)$.

Case 2 implies that $\exists\ \lambda, \gamma \in \mathbb{N}$, $0 < \gamma < 3m - w_j$ such that $t = 2r + \lambda(3m - w_j) + \gamma$. Consequently, we have $\{i/x_{t-i}^{w_j} = 1, \text{ and } 1 \le i \le k\} = \text{contr2}(w_j, \gamma)$ and $x_t^{w_j} = 0$. From the definitions of H, G, and equation (5); and on the fact that $x_t^{w_j} = 0$, we deduce that card$\{i/x_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \in G\} \le 1$. By the recurrence hypothesis, we deduce that card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k$, $i \in G\}$ = card$\{i/x_{t-i}^{w_j} = 1$ and $1 \le i \le k$, $i \in G\}$ and card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k\}$ = card$\{i/x_{t-i}^{w_j} = 1$ and $1 \le i \le k\}$. From equation (4), we have: $2 \le$ card$\{i/x_{t-i}^{w_j} = 1$ and $1 \le i \le k\} \le 3$. We deduce that:

$$\text{card}\{i/y_{t-i}^{w_j} = 1, 1 \le i \le k,\ i \in G\} \le 1 \tag{10}$$
$$\text{card}\{i/y_{t-i}^{w_j} = 1, 1 \le i \le k,\ i \notin G\} \le 2 \tag{11}$$
$$2 \le \text{card}\{i/y_{t-i}^{w_j} = 1, 1 \le i \le k\} \le 3. \tag{12}$$

From equations (10) through (12), we deduce the following different cases.

- card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \in G\} = 0$ and card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \notin G\} = 2$.

  We obtain:

$$\sum_{i=1}^{k} b_i y_{t-i}^{w_j} \le \sum_{i=1}^{k} a_i y_{t-i}^{w_j} + 4k(\theta + m).$$

- card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \in G\} = 0$ and card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \notin G\} = 3$.

  We obtain:

$$\sum_{i=1}^{k} b_i y_{t-i}^{w_j} \le \sum_{i=1}^{k} a_i y_{t-i}^{w_j} + 6k(\theta + m).$$

- card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \in G\} = 1$ and card$\{i/y_{t-i}^{w_j} = 1$ and $1 \le i \le k, i \notin G\} = 1$.

We obtain:

$$\sum_{i=1}^{k} b_i y_{t-i}^{w_j} \leq \sum_{i=1}^{k} a_i y_{t-i}^{w_j} + 6k(\theta + m) + 2k(\theta + m).$$

- $\operatorname{card}\{i / y_{t-i}^{w_j} = 1 \text{ and } 1 \leq i \leq k, i \in G\} = 1$ and $\operatorname{card}\{i / y_{t-i}^{w_j} = 1 \text{ and } 1 \leq i \leq k, i \notin G\} = 2$.

  We obtain:

$$\sum_{i=1}^{k} b_i y_{t-i}^{w_j} \leq \sum_{i=1}^{k} a_i y_{t-i}^{w_j} + 6k(\theta + m) + 4k(\theta + m).$$

By basing ourselves in the different cases, we deduce that

$$\sum_{i=1}^{k} b_i y_{t-i}^{w_j} \leq \sum_{i=1}^{k} a_i y_{t-i}^{w_j} + 11k(\theta + m).$$

From the recurrence hypothesis we have $x_{t-i}^{w_j} = y_{t-i}^{w_j}$, it follows that:

$$\sum_{i=1}^{k} b_i y_{t-i}^{w_j} \leq \sum_{i=1}^{k} a_i x_{t-i}^{w_j} + 11k(\theta + m).$$

From [11] we know that $\sum_{i=1}^{k} a_i x_{t-i}^{w_j} \leq \theta$. This implies:

$$y_t^{w_j} = 0 = x_t^{w_j}. \blacksquare$$

Lemma 2. $\forall\, m \in \mathbb{N}, m \geq 3$, the subset $H1 = \{1, 2\}$ verify:

$$\forall\, r1, r3 \in H1 r1 \neq r3 \Rightarrow \neg Q(m, r1, r1, r3)$$

and $\forall\, m \in \mathbb{N}, m \geq 5$, the subset $H2 = \{1, 2, 4\}$ verify:

$$\forall\, r1, r2, r3 \in H2 r1 \neq r3 \text{ and } r2 \neq r3 \Rightarrow \neg Q(m, r1, r2, r3).$$

Proof. We are providing the proof for only $H1$. The proof for $H2$ is similar. We have

$$\begin{aligned}
&\operatorname{contr1}(1) = \{6m-2, 3m-1\}, \operatorname{contr1}(2) = \{6m-4, 3m-2\}\\
&\operatorname{contr2}(1,\beta) = \{6m-2+\beta, 3m-1+\beta, \beta\} \text{ for } \beta \text{ such that } \beta \in \{1, 2\}\\
&\operatorname{contr2}(1,\beta) = \{3m-1+\beta, \beta\} \text{ for } \beta \text{ such that } 3 \leq \beta \leq 3m-2\\
&\operatorname{contr2}(2,\beta) = \{6m-4+\beta, 3m-2+\beta, \beta\} \text{ for } \beta \text{ such that } \beta \in \{1, 2, 3, 4\}\\
&\operatorname{contr2}(2,\beta) = \{3m-2+\beta, \beta\} \text{ for } \beta \text{ such that } 5 \leq \beta \leq 3m-3
\end{aligned}$$

One can easily check that $6m - 2 \in \text{contr2}(2, \gamma) \Rightarrow \gamma = 2$, and so $\text{contr2}(2, 2) = \{6m - 2, 3m, 2\}$, it follows that $3m - 1 \notin \text{contr2}(2, 2)$.

$3m - 1 \in \text{contr2}(2, \gamma) \Rightarrow \gamma = 1$, and so $\text{contr2}(2, 1) = \{6m - 3, 3m - 1, 1\}$, it follows that $6m - 2 \notin \text{contr2}(2, 1)$.

Likewise, $6m - 4 \in \text{contr2}(1, \gamma) \Rightarrow \gamma = 3m - 3$, and so $\text{contr2}(1, 3m - 3) = \{6m - 4, 3m - 3\}$, it follows that $3m - 2 \notin \text{contr2}(1, 3m - 3)$.

$3m - 2 \in \text{contr2}(1, \gamma) \Rightarrow \gamma = 3m - 2$, and so $\text{contr2}(1, 3m - 2) = \{6m - 3, 3m - 2\}$, it follows that $6m - 4 \notin \text{contr2}(1, 3m - 2)$.

We have $\max H1 = 2$, and by definition of H or $H1$, we have $\max H1 \le m - 1$, it follows that $m \ge 3$. ■

Proposition 1. If there exists a neuronal recurrence equation of memory length h

$$x_n = \mathbf{1}\left[\sum_{i=1}^{h} a_i x_{n-i} - \theta\right]$$

which describes a transient of length T and a cycle of length P such that

$$\forall\, i \in \mathbb{N},\ 1 \le i \le h; \qquad a_i \ge 0$$
$$\exists\, j \in \mathbb{N},\ j \ge h; \qquad x_j = 0$$

then there exists a neuronal recurrence equation of memory length h

$$y_n = \mathbf{1}\left[\sum_{i=1}^{h} b_i y_{n-i} - \theta\right]$$

which describes a transient of length T and a cycle of length P such that

$$\forall\, i \in \mathbb{N},\ 1 \le i \le h; \qquad b_i > 0.$$

Proof. Set

$$\lambda = \max\left\{\sum_{i=1}^{h} a_i x_{n-i} - \frac{\theta}{n} \ge h \text{ and } \sum_{i=1}^{h} a_i x_{n-i} < \theta\right\}.$$

λ is well defined because there exists $j \in \mathbb{N}$, $j \ge h$ such that $x_j = 0$. Let us denote $\beta = \lambda/2h$. We define $b_i = a_i - \beta$, for $i = 1, 2, \ldots, h$. From the definition of β and from the fact that $a_i \ge 0$, we deduce $b_i > 0$. Let us initialize the first h terms of the sequence $\{(y_n); n \ge 0\}$ as follows:

$$y_i = x_i, \qquad 0 \le i \le h - 1. \tag{13}$$

We want to show by strong recurrence that $\forall\, n \in \mathbb{N}$, $y_n = x_n$.

Basic case: Rank $0, 1, 2, \ldots, h - 1$.
From equation (13), we deduce the result.

Recurrence Hypothesis: Rank $n-h, n-h+1, \ldots, n-2, n-1$.
We suppose that: $y_i = x_i$, $n-h \le i \le n-1$.

Rank n:

$$y_n = \mathbf{1}\left[\sum_{i=1}^{h} b_i y_{n-i} - \theta\right] \tag{14}$$

$$= \mathbf{1}\left[\sum_{i=1}^{h} b_i x_{n-i} - \theta\right] \tag{15}$$

$$= \mathbf{1}\left[-\theta - \beta\sum_{i=1}^{h} x_{n-i} + \sum_{i=1}^{h} a_i x_{n-i}\right]. \tag{16}$$

From the definition of β, we deduce

$$0 \le -\beta\sum_{i=1}^{h} x_{n-i} \le -\beta h. \tag{17}$$

From equations (14) and (15), we use the recurrence hypothesis and from equations (15) and (16), we use the definition of b_i. By basing ourselves on values of $\sum_{i=1}^{h} a_i x_{n-i}$, the following two cases have to be distinguished.

Case 1. $\sum_{i=1}^{h} a_i x_{n-i} < \theta$.

From hypothesis, we deduce that $x_n = 0$. By definition of λ and hypothesis, we can deduce that $\sum_{i=1}^{h} a_i x_{n-i} < \lambda$. From equation (17), we deduce $-\theta + \sum_{i=1}^{h} a_i x_{n-i} - \beta\sum_{i=1}^{h} < \lambda - \beta h < 0$. It follows that $y_n = 0 = x_n$.

Case 2. $\sum_{i=1}^{h} a_i x_{n-i} \ge \theta$.

From hypothesis, it follows that $x_n = 1$. From equation (17) and hypothesis, we conclude that $0 \le -\theta + \sum_{i=1}^{h} a_i x_{n-i} - \beta\sum_{i=1}^{h} x_{n-i}$. It follows that $y_n = 1 = x_n$. ■

Proposition 2. Let the subset $H = \{w_1, w_2, \ldots, w_\alpha\}$ of $\{1, 2, \ldots, m-1\}$ such that $\forall\, r1, r2, r3 \in H, r1 \ne r3$ and $r2 \ne r3 \Rightarrow \neg Q(m, r1, r2, r3)$. There exists a neuronal recurrence equation of memory length αk such that all the weighting coefficients are strictly positive and which generates a sequence of period $\alpha * \mathrm{lcm}(3m - w_1, 3m - w_2, \ldots, 3m - w_\alpha)$.

Proof. Let us denote $G = \cup_{w_j \in H}\mathrm{contr1}(w_j)$. By applying Lemma 1, we deduce that $\forall\, w_j \in H$, the sequence $\{(y_n^{w_j}); n \ge 0\}$ whose first k terms are given in equation (9) and whose terms are generated by neuronal recurrence equation (7) of memory length k, is of period $3m - w_j$. By application of the fundamental lemma of composition of memory automata [5], according to which if there is a neuronal recurrence equation of memory length k that generates r sequences of periods $p_0, p_1, \ldots, p_{r-1}$, then there exists a neuronal recurrence equation of memory length rk that generates a sequence of period $r * \text{lcm}(p_0, p_1, \ldots, p_{r-1})$. We deduce that there exists a neuronal recurrence equation of memory length αk that generates a sequence of period $\alpha * \text{lcm}(3m - w_1, 3m - w_2, \ldots, 3m - w_\alpha)$. The weighting coefficients defined in equation (8) are all positive; and, on the basis of the construction of the neuronal recurrence equation of size αk [5], we deduce that all the weighting coefficients of the neuronal equation of memory length αk are positive. By applying Proposition 1, we deduce the result. ∎

Corollary 1. $\forall\, m \in \mathbb{N}$, $m \geq 3$, there is a neuronal recurrence equation of memory length $2k$ with all the weighting coefficients being strictly positive that generates a sequence of period $2(3m - 1)(3m - 2)$.

Proof. It suffices to take $H = \{1, 2\}$ and $G = \text{contr1}(1) \cup \text{contr1}(2) = \{3m-2, 3m-1, 6m-4, 6m-2\}$. By applying Lemma 2 and Proposition 2, we deduce the result. ∎

Corollary 2. $\forall\, m \in \mathbb{N}$, $m \geq 5$, and m odd, there is a neuronal recurrence equation of memory length $3k$ with all weighting coefficients strictly positive that generates a sequence of period $3(3m - 1)(3m - 2)(3m - 4)$.

Proof. It suffices to take $H = \{1, 2, 4\}$ and $G = \text{contr1}(1) \cup \text{contr1}(2) \cup \text{contr1}(4) = \{3m - 4, 3m - 2, 3m - 1, 6m - 8, 6m - 4, 6m - 2\}$. By applying Lemma 2 and Propositon 2, we deduce that there exists a neuronal recurrence equation of memory length $3k$ where the weighting coefficients are all strictly positive that generates a sequence of period $3 * \text{lcm}(3m - 1, 3m - 2, 3m - 4)$. m is odd, this implies that $3m - 2 \not\equiv 2 \pmod 3$ and $3m - 2 \not\equiv 0 \pmod 2$. On the basis of Lemma 4 from [5], we deduce that $3m - 1$, $3m - 2$, and $3m - 4$ are relatively prime. It follows that $\text{lcm}(3m - 1, 3m - 2, 3m - 4) = (3m - 1)(3m - 2)(3m - 4)$, and so the result holds. ∎

3. Conclusion

For any integer m, $m \geq 3$, we have exhibited the neuronal recurrence equation of memory length $2k$ ($k = 6m$) where the weighting coefficients are all strictly positive that generates a sequence of period $O(k^2)$. For odd integer m, $m \geq 5$, we have exhibited the neuronal recurrence equation of memory length $3k$ where the weighting coefficients are all strictly positive that generates a sequence of period $O(k^3)$. The technique used to pass from a neuronal recurrence equation where the weighting coefficients are positive or negative to the neuronal recurrence equation where the weighting coefficients are all strictly positive is inscribed in the frame of structural construction. Structural construction methods are the general and more powerful tools used in the study of sequences generated by neuronal recurrence equations [5, 8, 10, 11, 15]. By exhibiting another set H where any distinct triplet of elements do not verify the predicate Q, we can hope to exhibit the cycles $O(k^s)$, where s is any element of $\mathbb{N}$. The open question remains: Is there a neuronal recurrence equation of memory length k where the weighting coefficients are all positive that generates a sequence of period exponential respective to k?

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