

Chaotic Properties of the Elementary Cellular Automaton Rule 40 in Wolfram's Class I

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This paper examines the chaotic properties of the elementary cellular automaton rule 40.

Rule 40 has been classified into Wolfram's class I and also into class C_1 by G. Braga *et al.* These classifications mean that the time-space patterns generated by this cellular automaton die out in a finite time and so are not interesting. As such, we may hardly realize that rule 40 has chaotic properties.

In this paper we show that the dynamical system defined by rule 40 is Devaney chaos on a class of configurations of some particular type and has every periodic point except prime period one, four, or six. In the process of the proof, it is noticed that the dynamical properties of rule 40 can be related to some interval dynamical systems. These propositions are shown in Theorems 2 and 4.

1. Introduction and preliminaries

Cellular automata were first introduced by J. von Neuman [1] as a mathematical model for biological self-replication phenomena, and have played a basic role for understanding and explaining various complex physical, social, chemical, and biological phenomena. S. Wolfram [2–4], using extensive computer simulation, classified cellular automata into four classes according to the time-space patterns generated by them. This classification has been logically verified by G. Braga *et al.* [5], focusing on the class of the quiescent cellular automata and 0-finite configurations. They also give a powerful tool for classifying the quiescent elementary cellular automata. Following the work in [5], more detailed verifications of the classes C_1 , C_2 , and C_3 have been made in [6–8].

The cellular automata of Wolfram's class IV, especially, generate time-space patterns called "edge-of-chaos" by Langton [9, 10], who started the research area of artificial life. The cellular automaton rule 184, which belongs to Wolfram's class II, is one of the simplest but can be used as a basic model of traffic flow, see B. Chopard, *et al.* [11]. Rule 184

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also relates to Nagel’s work [12] for constructing a wide-range traffic flow simulator. On the other hand, the time-space patterns generated by the cellular automata of class I have been said to die out in a finite time, based on computer simulations, and not to have any significant meaning. And so we may hardly realize that the class I cellular automata could have chaotic properties.

Rule 40 belongs to Wolfram’s class I, and the time-space pattern starting a 0-finite configuration of Braga *et al.* [5] dies out in a finite time, which is easily observed during computer simulation. The rule number 40 is the decimal number transformed from the series 00101000 in Table 1 as a binary number.

In this paper we show that the dynamical system defined by rule 40 is Devaney chaos on a class of configurations of some particular type and has every periodic point except prime period one, four, or six. This property for the variety of periodic points differs from Sarkovskii’s theorem for a continuous interval dynamical system which asserts that every kind of periodic point exists if a period-three periodic point exists.

In the process of proving the given propositions for rule 40, its dynamical properties are noticed to relate to some figures of interval dynamical systems. We find that the class I cellular automata have interesting hidden characteristics.

Focusing on rule 40, we examine the trajectory of each configuration and show the statements (1), (2), and (3) later but first we present some basic terminology and settings.

An elementary cellular automaton (ECA) is a tuple $(\{0, 1\}, g)$, where g is a mapping from $\{0, 1\}^3$ to $\{0, 1\}$ and is called a *local transition function*. An ECA is determined by g and is simply called an ECA g .

An ECA g defines a mapping \mathbf{g} from $\mathcal{A} \equiv \{0, 1\}^{\mathbb{Z}}$ to \mathcal{A} , which is called a *global transition function*, as

$$\mathbf{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{A}, \quad (\mathbf{g}(\mathbf{x}))_i = g(x_{i-1}, x_i, x_{i+1}), \quad i \in \mathbb{Z}.$$

Defining a metric d on \mathcal{A} as

$$\mathbf{x}, \mathbf{y} \in \mathcal{A}, \quad d(\mathbf{x}, \mathbf{y}) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}},$$

we have a topological dynamical system $(\mathcal{A}, \mathbf{g})$. An element of \mathcal{A} is called a *configuration* and the trajectory of an initial configuration $\mathbf{x} \in \mathcal{A}$ is a series $\{\mathbf{g}^t(\mathbf{x})\}_{t=1}^{\infty}$ defined as

$$\mathbf{g}^0(\mathbf{x}) = \mathbf{x}, \quad \mathbf{g}^{t+1}(\mathbf{x}) = \mathbf{g}(\mathbf{g}^t(\mathbf{x})), \quad t > 0.$$

A topological dynamical system $(\mathcal{A}, \mathbf{g})$ is called *Devaney chaos* when it is transitive and regular [13].

We denote the local and global transition functions of the ECA rule 40 as g_{40} and \mathbf{g}_{40} , respectively. $g_{40} : \{0, 1\}^3 \rightarrow \{0, 1\}$ is given by Table 1.

(a, b, c)	$(1, 1, 1)$	$(1, 1, 0)$	$(1, 0, 1)$	$(1, 0, 0)$
$g_{40}(a, b, c)$	0	0	1	0
(a, b, c)	$(0, 1, 1)$	$(0, 1, 0)$	$(0, 0, 1)$	$(0, 0, 0)$
$g_{40}(a, b, c)$	1	0	0	0

Table 1. Local transition function of rule 40.

In this paper we show the following properties of the discrete-time dynamical system (\mathcal{A}, g_{40}) .

- (1) For each configuration $x \in \mathcal{A}$, we make the dynamical property of the trajectory $\{g_{40}^t(x)\}_{t=0}^\infty$ clear. We set

$$S = \left\{ (0, \mathbf{1}_{m_i})_{i=-\infty}^\infty \mid m_i = 1 \text{ or } 2, i \in \mathbf{Z} \right\},$$

where $\mathbf{1}_{m_i} = \underbrace{(1, \dots, 1)}_{m_i}$ and S is the set of all the two-sided infinite sequences composed of blocks 01 or 011. In other words, S is the language generated by the words 01 and 011. It is shown that

$$g_{40}(S) \subseteq S \quad \text{and} \quad \forall x \in S, \quad g_{40}(x) = \sigma_L(x),$$

which means that (S, g_{40}) is a left-shift dynamical subsystem of (\mathcal{A}, g_{40}) , so g_{40} shifts each configuration of S left by one site, and is Devaney chaos. σ_L is the left-shift transformation on \mathcal{A} .

- (2) For each configuration $x \in \mathcal{A} \setminus S$,

$$\lim_{t \rightarrow \infty} g^t(x) = \mathbf{0} \quad \text{or} \quad \exists t \geq 0, \quad g^t(x) = \mathbf{0},$$

which means that the trajectories of the configurations of $\mathcal{A} \setminus S$ have the unique attractor $\{\mathbf{0}\}$. In section 3 $\mathcal{A} \setminus S$ is essentially divided into five mutually exclusive classes and dynamical properties of the trajectories in each class are precisely examined.

- (3) The dynamical system (g_{40}, S) has every periodic point except prime period one, four, or six.

Notations

In this paper we use the following notations for rigorous arguments.

- $\mathbf{N}_+ \equiv \{1, 2, 3, \dots\}$ is the set of positive integers and the set of negative integers is denoted by $\mathbf{N}_- \equiv \{\dots, -3, -2, -1\}$. \mathbf{Z} is the set of all integer numbers.
- σ_L is the left-shift transformation on \mathcal{A} , that is to say, $(\sigma_L(x))_i = x_{i+1}$ for every $x \in \mathcal{A}$. σ_L shifts each configuration to the left by one site.

- 3. For $x \in \mathcal{A}$, $x_{i,j} \equiv (x_i, \dots, x_j)$ ($i \leq j$), $x_{\leftarrow i} \equiv (\dots, x_{i-1}, x_i)$, and $x_{j,\rightarrow} \equiv (x_j, x_{j+1}, \dots)$.

$x_{i,j}$ is the block of elements from x with coordinate numbers between i and j .

- 4. For $\alpha_i \in \{0, 1\}^{n_i}$, $\beta_i \in \{0, 1\}^{m_i}$, $n_i \geq 1$, $m_i \geq 1$, $i \in \mathbf{Z}$, we define

$$(\alpha_i, \beta_i)_{i=-\infty}^{+\infty} = (\dots, \alpha_1^{-1}, \dots, \alpha_{n-1}^{-1}, \beta_1^{-1}, \dots, \beta_{m-1}^{-1}, \alpha_0^0, \dots, \alpha_{n_0}^0, \beta_1^0, \dots, \beta_{m_0}^0, \alpha_1^1, \dots, \alpha_{n_1}^1, \beta_1^1, \dots, \beta_{m_1}^1, \dots),$$

where $\alpha_i = (\alpha_1^i, \dots, \alpha_{n_i}^i)$, $\beta_i = (\beta_1^i, \dots, \beta_{m_i}^i)$, $i \in \mathbf{Z}$.

When coordinate numbers are necessary, we write them over an element of the configuration and then the coordinate number of every element is easily obtained since the lengths of blocks α_i and β_i ($i \in \mathbf{Z}$) are given.

- 5. $\mathbf{0}$ means one of the three types $(\dots, 0, 0, 0, \dots)$, $(\dots, 0, 0, 0)$, or $(0, 0, \dots)$. It is clear from the context which type $\mathbf{0}$ means. We also use the terminology $\mathbf{0}_n = (0, \dots, 0)$, $n \in \mathbf{N}_+$. $\mathbf{1}$ and $\mathbf{1}_n$ are interpreted similarly to $\mathbf{0}$ and $\mathbf{0}_n$, respectively.

- 6. For a $x \in \mathcal{A}$ and a block $\alpha \in \{0, 1\}^n$ with finite length n , which is denoted as $|\alpha| = n$, $\alpha \in x$ means that $\alpha = (x_i, \dots, x_{i+n-1})$ for some $i \in \mathbf{Z}$.

- 7. For $x \in \mathcal{A}$ and a block c with finite length, we define

$$N(c|x_{i,j}) \equiv \sharp\{k | (x_k, \dots, x_{k+|c|-1}) = c, i \leq k \leq j - |c| + 1\}$$

as the number of blocks c contained in $x_{i,j}$.

- 8. \oplus means the exclusive or and for x and y of \mathcal{A} ,

$$x \oplus y = (\dots, x_{-1} \oplus y_{-1}, x_0 \oplus y_0, x_1 \oplus y_1, \dots).$$

2. Trajectories and time-space patterns of rule 40

In this section we precisely examine the trajectory $\{g_{40}^t(x)\}_{t=0}^\infty$ for each $x \in \mathcal{A}$ and clarify the dynamical properties of (\mathcal{A}, g_{40}) . We simply write g_{40} and \mathbf{g}_{40} as g and \mathbf{g} , respectively.

Part 1 of Lemma 1 shows that when a configuration contains a block 00, it extends to the left by one site at each step and to the right by at least one site at each two steps. Part 2 of Lemma 1 tells us that the block 00 in the configuration plays a wall-like role and the development of the time-space patterns on the right and left sides of the block 00 are mutually independent.

Lemma 1. For $x = (\overset{0}{\beta}, \overset{1}{0}, 0, \alpha)$, $\alpha \in \{0, 1\}^{\mathbf{N}^+}$, $\beta \in \{0, 1\}^{\mathbf{N}^-}$ we have the following.

$$(1) \forall t \geq 0, \quad (g^t(x))_{-t,1} = \underbrace{(0, \dots, 0)}_{t+2}, \quad (g^{2t}(x))_{1,1+t} = \underbrace{(0, \dots, 0)}_{t+1}.$$

$$(2) \forall t \geq 0, \quad g^t(x) = g^t(\beta, 0) \oplus g^t(0, \alpha).$$

Proof. The first halves of (1) and (2) are obvious since

$$g(0, 0, 0) = g(0, 0, 1) = g(1, 0, 0) = 0.$$

Proof of the second half of (1). It is sufficient to examine $g^2(x)$ for the following three cases:

$$\begin{aligned} \mathbf{x} &= (\dots, \overset{0}{*}, \overset{1}{0}, \overset{2}{0}, \overset{3}{*}, \dots) \\ &(\dots, \overset{0}{*}, \overset{1}{0}, \overset{2}{0}, \overset{3}{1}, \overset{4}{0}, \overset{5}{*}, \dots) \\ &(\dots, \overset{0}{*}, \overset{1}{0}, \overset{2}{0}, \overset{3}{1}, \overset{4}{1}, \overset{5}{*}, \dots). \end{aligned}$$

For every configuration of this type, $g(x)$ is one of the following configurations:

$$\begin{aligned} \mathbf{g}(x) &= (\dots, \overset{-1}{*}, \overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0}, \overset{4}{0} *, \dots) \\ &(\dots, \overset{-1}{*}, \overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0}, \overset{4}{1} *, \dots) \\ &(\dots, \overset{-1}{*}, \overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{1}, \overset{4}{0} *, \dots). \end{aligned}$$

Since $g(*, *, 0) = 0$ and $g(1, 1, 1) = 0$, we have

$$g^2(x) = (\dots, \overset{-2}{*}, \overset{-1}{0}, \overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0} *, \dots),$$

and the proof is terminated by using mathematical induction. ■

Theorem 1. $g^t(x)$ does not converge to $\mathbf{0}$ when $t \rightarrow \infty$, if and only if, $(0, 0) \notin x$ and $(1, 1, 1) \notin x$, or equivalently,

$$x = (0, \mathbf{1}_{m_i})_{i=-\infty}^{\infty} = (\dots, 0, \mathbf{1}_{m_{-1}}, 0, \mathbf{1}_{m_0}, 0, \mathbf{1}_{m_1}, \dots), \quad \forall i, m_i = 1 \text{ or } 2.$$

A configuration x of this type is shifted left forward by g , that is,

$$\forall t \geq 0, \quad g^t(x) = \sigma_L^t(x).$$

Proof of the only if part. If x has a block $(0, 0)$, that is, $(0, 0) \in x$, then we have from Lemma 1

$$\lim_{t \rightarrow \infty} g^t(x) = \mathbf{0}.$$

For a configuration x having a block $(1, 1, 1)$ and $x_i = x_{i+1} = x_{i+2} = 1$ for some i , the local transition function of rule 40 satisfies $g(1, 1, 1) =$

$g(1, 1, *) = 0$, and we have $(g(x))_{i+1} = (g(x))_{i+2} = 0$, which means that $g(x)$ has a block $(0, 0)$, as such $\lim_{t \rightarrow \infty} g^t(x) = \mathbf{0}$ also holds from Lemma 1.

Proof of the if part. Since $g(0, 1, 0) = g(1, 1, 0) = 0$ and $g(0, 1, 1) = g(1, 0, 1) = 1$, $g^t(x) = \sigma_L^t(x)$ ($t \geq 0$) holds for x of the type given, which means that $g^t(x)$ does not converge to $\mathbf{0}$. ■

Theorems 2 and 3 show that the two discrete dynamical subsystems (S, g) and $(\mathcal{A} \setminus S, g)$ differ greatly from each other: the former is Devaney chaos, and the latter has the unique attractor $\{\mathbf{0}\}$ and $\lim_{t \rightarrow \infty} g^t(x) = \mathbf{0}$ for every $x \in \mathcal{A} \setminus S$. A computer simulation usually shows the dynamics of $(\mathcal{A} \setminus S, g)$ and not those for (S, g) .

Theorem 2. (S, g) is Devaney chaos.

Proof. From Theorem 1 $g(S) \subseteq S$ holds and then (S, g) is a discrete time left-shift dynamical system. It is sufficient to prove that (S, g) has a dense trajectory and that the set of all the periodic configurations is dense in S . For terminology, see [13].

Note that each configuration $x \in S$ consists of two types of blocks $(0, 1)$ and $(0, 1, 1)$, where we denote each block as a and b , respectively, every configuration $x \in S$ is a bi-infinite configuration consisting of a or b .

(1) *Transitive property.* The trajectory $\{g^t(x_0)\}_{t=0}^\infty = \{\sigma_L^t(x_0)\}_{t=0}^\infty$ of the following configuration x_0 is dense in S :

$$x_0 \equiv (\dots, aa, ab, ba, bb, b, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \dots).$$

(2) *Regular property.* For $x \in S$ and $\epsilon > 0$, l is supposed to be the least positive integer number which satisfies $\sum_{k=l}^\infty 1/2^k < \epsilon/2$.

Since $x \in S$, there exist m_1 and m_2 satisfying

$$l \leq m_1, \quad l \leq m_2, \quad x_{-m_1} = 0, \quad x_{m_2} = 1.$$

A configuration defined by

$$(x_{-m_1, m_2})^{\mathbb{Z}} \equiv (\dots, x_{-m_1, m_2}, x_{-m_1, m_2}, x_{-m_1, m_2}, \dots) \in S$$

is periodic and

$$d(x, (x_{-m_1, m_2})^{\mathbb{Z}}) < \epsilon,$$

and then (S, g) becomes regular. ■

Since $g(1, 1, 1) = 0$, every non-0-configuration x or $g(x)$ has one of these four types:

$$x = \begin{cases} (\dots, \mathbf{1}_{m_2}, \mathbf{0}_{n_2}, \mathbf{1}_{m_1}, \mathbf{0}_{n_1}, \mathbf{1}_{m_0}, \mathbf{0}_{n_0}, \mathbf{1}_{m_1}, \mathbf{0}_{n_1}, \mathbf{1}_{m_2}, \mathbf{0}_{n_2}, \dots), \\ \underbrace{(\dots, 0, 0, 0, \mathbf{1}_{m_0}, \mathbf{0}_{n_0}, \mathbf{1}_{m_1}, \mathbf{0}_{n_1}, \mathbf{1}_{m_2}, \mathbf{0}_{n_2}, \dots)}_{\text{only } 0}, \\ (\dots, \mathbf{1}_{m_2}, \mathbf{0}_{n_2}, \mathbf{1}_{m_1}, \mathbf{0}_{n_1}, \mathbf{1}_{m_0}, \mathbf{0}_{n_0}, \underbrace{0, 0, 0, \dots}_{\text{only } 0}), \\ (\dots, 0, 0, 0, \mathbf{1}_{m_0}, \mathbf{0}_{n_0}, \mathbf{1}_{m_1}, \mathbf{0}_{n_1}, \dots, \mathbf{1}_{m_k}, \mathbf{0}_{n_k}, \underbrace{0, 0, 0, \dots}_{\text{only } 0}), \end{cases}$$

where $\forall i, n_i \geq 1, m_i \geq 1$. Following the mutually exclusive five classes of configurations defined in Table 2, we have

$$\mathcal{AS} = (C_1 \cup C_{1'} \cup C_2 \cup C_3 \cup C_4) \cup (\mathcal{AS} \setminus C),$$

$$g(\mathcal{AS} \setminus C) = C_1 \cup C_{1'} \cup C_2 \cup C_3 \cup C_4 \cup \{0\},$$

where $C = C_1 \cup C_{1'} \cup C_2 \cup C_3 \cup C_4$, and then by the dynamical properties of g summarized in Table 2 (proved in Theorem 3), we have

$$\forall x \in \mathcal{AS}, \lim_{t \rightarrow \infty} g^t(x) = 0.$$

We need another lemma to prove the dynamical properties of rule 40 presented in Table 2.

Class of configurations	Conditions defining the class	Dynamics
C_1	$\exists i, n_i = 1$ $\forall j, m_j \geq 3$	$g(C_1) \subseteq C_2$
$C_{1'}$	$\forall i, n_i \geq 2$ $\forall j, m_j \geq 3$	$g(C_{1'}) \subseteq C_3$
C_2	$\forall i, n_i \geq 2$ $\forall j, 1 \leq m_j \leq 2,$ $\exists j, m_j = 2$	$g(C_2) \subseteq C_3$
C_3	$\forall i, n_i \geq 2$ $\forall j, m_j = 1$	$g(C_3) = \{0\}$
C_4	$\exists i, n_i = 1,$ $\exists i, n_i \geq 2$ $\forall j, 1 \leq m_j \leq 2$	$\exists t \geq 1, g^t(x) = 0,$ or $\forall t \geq 1, g^t \neq 0,$ $\lim_{t \rightarrow \infty} g^t(x) = 0$

Table 2. Dynamical properties of rule 40.

Lemma 2. For $\alpha \in \{01, 011\}^{\mathbb{N}^+} \cup \cup_{n \in \mathbb{N}_+} \{01, 011\}^n \times \{0\}$, $\beta \in \{0, 1\}^{\mathbb{N}^-}$, $x = (\beta, \overset{0}{0}, \alpha)$, we define

$$T \equiv \sup\{t \mid \{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = t\} \neq \emptyset\},$$

$$n_t \equiv \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = t\}, \quad 0 \leq t \leq T.$$

Then we have these three relationships:

- (1) $N(011|x_{1,n}) \leq N(01|x_{1,n})$.
- (2) $t \leq T, x_{n_t} = 1, n_t = 2N(01|x_{1,n_t}) + N(011|x_{1,n_t})$.
- (3) $1 \leq t < T, (g^t(x))_{0, \rightarrow} = (0, \mathbf{0}_{N(01|x_{1,n_t}), x_{n_t+1, \rightarrow}}), t \geq T,$
 $(g^t(x))_{0, \rightarrow} = \mathbf{0}$.

Proof. (1) and (2) are obvious. (3) is proved by using mathematical induction on t . ■

Lemma 2 may be figured out by Example 1.

Example 1. For the following x , we show some specific values of $n_t, N(01|x_{1,n_t})$ and $N(011|x_{1,n_t}), t = 1, 2, 3, 4, 5, 6$.

$$(x)_{0, \rightarrow} = (0, 0, \overset{0}{0}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{0}{0}, \overset{1}{1}, \overset{1}{1}, \dots)$$

$$n_1 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 1\} = 1,$$

$$N(01|x_{1,n_1}) = 1, \quad N(011|x_{1,n_1}) = 0,$$

$$n_2 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 2\} = 4,$$

$$N(01|x_{1,n_2}) = 2, \quad N(011|x_{1,n_2}) = 0,$$

$$n_3 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 3\} = 6,$$

$$N(01|x_{1,n_3}) = 3, \quad N(011|x_{1,n_3}) = 0,$$

$$n_4 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 4\} = 7,$$

$$N(01|x_{1,n_4}) = 3, \quad N(011|x_{1,n_4}) = 1,$$

$$n_5 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 5\} = 9,$$

$$N(01|x_{1,n_5}) = 4, \quad N(011|x_{1,n_5}) = 1,$$

$$n_6 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 6\} = 10,$$

$$N(01|x_{1,n_6}) = 4, \quad N(011|x_{1,n_6}) = 2,$$

$$n_7 = \min\{n \mid N(01|x_{1,n}) + N(011|x_{1,n}) = 7\} = 12,$$

$$N(01|x_{1,n_7}) = 5, \quad N(011|x_{1,n_6}) = 2,$$

$$\begin{aligned}
 (\mathbf{x})_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{1}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{1}}, \overset{4}{\underset{\sim}{0}}, \overset{5}{\underset{\sim}{1}}, \overset{6}{\underset{\sim}{1}}, \overset{7}{\underset{\sim}{0}}, \overset{8}{\underset{\sim}{1}}, \overset{9}{\underset{\sim}{1}}, \overset{10}{\underset{\sim}{0}}, \overset{11}{\underset{\sim}{1}}, \overset{12}{\underset{\sim}{0}}, \overset{13}{\underset{\sim}{1}}, \overset{14}{\underset{\sim}{1}}, \overset{15}{\underset{\sim}{0}}, \overset{16}{\underset{\sim}{1}}, \overset{17}{\underset{\sim}{0}}, \dots) \\
 (\mathbf{g}(\mathbf{x}))_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{0}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{1}}, \overset{4}{\underset{\sim}{0}}, \overset{5}{\underset{\sim}{1}}, \overset{6}{\underset{\sim}{1}}, \overset{7}{\underset{\sim}{0}}, \overset{8}{\underset{\sim}{1}}, \overset{9}{\underset{\sim}{1}}, \overset{10}{\underset{\sim}{0}}, \overset{11}{\underset{\sim}{1}}, \overset{12}{\underset{\sim}{0}}, \overset{13}{\underset{\sim}{1}}, \overset{14}{\underset{\sim}{1}}, \overset{15}{\underset{\sim}{0}}, \dots) \\
 (\mathbf{g}^2(\mathbf{x}))_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{0}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{0}}, \overset{4}{\underset{\sim}{1}}, \overset{5}{\underset{\sim}{0}}, \overset{6}{\underset{\sim}{1}}, \overset{7}{\underset{\sim}{1}}, \overset{8}{\underset{\sim}{0}}, \overset{9}{\underset{\sim}{1}}, \overset{10}{\underset{\sim}{0}}, \overset{11}{\underset{\sim}{1}}, \overset{12}{\underset{\sim}{0}}, \overset{13}{\underset{\sim}{1}}, \overset{14}{\underset{\sim}{1}}, \dots) \\
 (\mathbf{g}^3(\mathbf{x}))_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{0}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{0}}, \overset{4}{\underset{\sim}{0}}, \overset{5}{\underset{\sim}{1}}, \overset{6}{\underset{\sim}{0}}, \overset{7}{\underset{\sim}{1}}, \overset{8}{\underset{\sim}{1}}, \overset{9}{\underset{\sim}{0}}, \overset{10}{\underset{\sim}{1}}, \overset{11}{\underset{\sim}{0}}, \overset{12}{\underset{\sim}{1}}, \overset{13}{\underset{\sim}{1}}, \dots) \\
 (\mathbf{g}^4(\mathbf{x}))_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{0}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{0}}, \overset{4}{\underset{\sim}{0}}, \overset{5}{\underset{\sim}{0}}, \overset{6}{\underset{\sim}{1}}, \overset{7}{\underset{\sim}{0}}, \overset{8}{\underset{\sim}{1}}, \overset{9}{\underset{\sim}{1}}, \overset{10}{\underset{\sim}{0}}, \overset{11}{\underset{\sim}{1}}, \overset{12}{\underset{\sim}{0}}, \dots) \\
 (\mathbf{g}^5(\mathbf{x}))_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{0}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{0}}, \overset{4}{\underset{\sim}{0}}, \overset{5}{\underset{\sim}{0}}, \overset{6}{\underset{\sim}{0}}, \overset{7}{\underset{\sim}{1}}, \overset{8}{\underset{\sim}{0}}, \overset{9}{\underset{\sim}{1}}, \overset{10}{\underset{\sim}{1}}, \overset{11}{\underset{\sim}{0}}, \dots) \\
 (\mathbf{g}^6(\mathbf{x}))_{0,\rightarrow} &= (0, \overset{0}{\underset{\sim}{0}}, \overset{1}{\underset{\sim}{0}}, \overset{2}{\underset{\sim}{0}}, \overset{3}{\underset{\sim}{0}}, \overset{4}{\underset{\sim}{0}}, \overset{5}{\underset{\sim}{0}}, \overset{6}{\underset{\sim}{0}}, \overset{7}{\underset{\sim}{0}}, \overset{8}{\underset{\sim}{1}}, \overset{9}{\underset{\sim}{0}}, \overset{10}{\underset{\sim}{1}}, \dots)
 \end{aligned}$$

Theorem 3. For the classes given in Table 2, we have the following relationships.

- (1) $\mathbf{g}(C_1) \subseteq C_2, \mathbf{g}(C_{1'}) \subseteq C_3, \mathbf{g}(C_2) \subseteq C_3, \mathbf{g}(C_3) = \{\mathbf{0}\}.$
- (2) For any $\mathbf{x} \in C_4$, we have

$$\exists t \geq 1, \mathbf{g}^t(\mathbf{x}) = \{\mathbf{0}\}$$

or

$$\forall t \geq 1, \mathbf{g}^t(\mathbf{x}) \neq \mathbf{0} \text{ and } \lim_{t \rightarrow \infty} \mathbf{g}^t(\mathbf{x}) = \mathbf{0}.$$

Statement (2) is made more precise in the following proof by using the distinguishable patterns of the configurations in C_4 .

Proof of (1). Noticing that

$$\mathbf{g}(1, 0, 1) = 1, \mathbf{g}(1, 1, *) = 0, \mathbf{g}(0, 1, 1) = 1, \text{ and } \mathbf{g}(0, 1, 0) = 0,$$

we have the following four statements.

- (i) For $\mathbf{x} = (\dots, \mathbf{0}_{n-1}, \mathbf{1}_{m-1}, \mathbf{0}_{n_0}, \mathbf{1}_{m_0}, \mathbf{0}_{n_1}, \mathbf{1}_{m_1}, \dots) \in C_1$, since $n_j \geq 1$ for all j , $n_i = 1$ for some i and $m_j \geq 3$ for all j , then $\mathbf{g}(\mathbf{x})$ is of this type:

$$\mathbf{g}(\mathbf{x}) = (\dots, \mathbf{1}, \mathbf{1}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1}, \mathbf{1}, \underbrace{\mathbf{0}, \dots, \mathbf{0}}_{\geq 2}, \mathbf{1}, \mathbf{1}, \dots) \in C_2.$$

- (ii) For $\mathbf{x} = (\dots, \mathbf{0}_{n-1}, \mathbf{1}_{m-1}, \mathbf{0}_{n_0}, \mathbf{1}_{m_0}, \mathbf{0}_{n_1}, \mathbf{1}_{m_1}, \dots) \in C_{1'}$, since $n_j \geq 2$ for all i and $m_j \geq 2$ for all j , then $\mathbf{g}(\mathbf{x})$ is of this type:

$$\mathbf{g}(\mathbf{x}) = (\dots, \mathbf{1}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1}, \dots) \in C_3.$$

(iii) For $x = (\dots, \mathbf{0}_{n_{-1}}, \mathbf{1}_{m_{-1}}, \mathbf{0}_{n_0}, \mathbf{1}_{m_0}, \mathbf{0}_{n_1}, \mathbf{1}_{m_1}, \dots) \in C_2$, since $n_i \geq 2$ for all i , $1 \leq m_j \leq 2$ for all j and $m_j = 2$ for some j , then $g(x)$ is of this type:

$$g(x) = (\dots, \underbrace{1, 0, \dots, 0}_{\geq 2}, \underbrace{1, 0, \dots, 0}_{\geq 2}, \dots) \in C_3.$$

(iv) For $x = (\dots, \mathbf{0}_{n_{-1}}, \mathbf{1}_{m_{-1}}, \mathbf{0}_{n_0}, \mathbf{1}_{m_0}, \mathbf{0}_{n_1}, \mathbf{1}_{m_1}, \dots) \in C_3$, since $n_i \geq 2$ for all i and $m_j = 1$ for all j , noticing that $g(0, 1, 0) = g(0, 0, 1) = g(1, 0, 0) = 0$, we have $g(x) = \mathbf{0}$.

Proof of (2). $x \in C_4$ may be expressed as

$$\begin{aligned} x &= (\dots, \mathbf{0}_{n_{-1}}, \alpha_{-1}, \mathbf{0}_{n_0}, \alpha_0, \mathbf{0}_{n_1}, \alpha_1, \dots), \\ &\quad \forall i, n_i \geq 2, \quad \alpha_i \in \cup_{n \in \mathbb{N}} \{01, 011\}^n, \\ \text{or } x &= (\alpha, \mathbf{0}_{n_0}, \alpha_0, \mathbf{0}_{n_1}, \alpha_1, \dots), \\ &\quad \forall i, n_i \geq 2, \quad \alpha_i \in \cup_{n \in \mathbb{N}} \{01, 011\}^n, \quad \alpha \in \{01, 011\}^{\mathbb{N}^-}, \\ \text{or } x &= (\dots, \mathbf{0}_{n_{-1}}, \alpha_{-1}, \mathbf{0}_{n_0}, \alpha), \\ &\quad \forall i, n_i \geq 2, \quad \alpha_i \in \cup_{n \in \mathbb{N}} \{01, 011\}^n, \quad \alpha \in \{01, 011\}^{\mathbb{N}^+}. \end{aligned}$$

Since 00 plays a wall-like role from Lemma 1(2), it is sufficient to examine the following three cases:

- (i) $g^t(\mathbf{0}, \alpha, \mathbf{0})$, $\alpha \in \cup_{n \in \mathbb{N}} \{01, 011\}^n$,
- (ii) $g^t(\mathbf{0}, \alpha)$, $\alpha \in \{01, 011\}^{\mathbb{N}^+}$,
- (iii) $g^t(\alpha, \mathbf{0})$, $\alpha \in \{01, 011\}^{\mathbb{N}^-}$.

Applying Lemma 2(3) to case (i) we have

$$g^t(\mathbf{0}, \alpha, \mathbf{0}) = \mathbf{0}, \quad t \geq N(1|\alpha).$$

Applying Lemma 2(3) to case (ii) and Lemma 1 to case (iii) we have

$$\begin{aligned} \forall t \geq 0, \quad g^t(\mathbf{0}, \alpha) &\neq \mathbf{0}, \quad \lim_{t \rightarrow \infty} g^t(\mathbf{0}, \alpha) = \mathbf{0}, \\ \forall t \geq 0, \quad g^t(\alpha, \mathbf{0}) &\neq \mathbf{0}, \quad \lim_{t \rightarrow \infty} g^t(\alpha, \mathbf{0}) = \mathbf{0}. \blacksquare \end{aligned}$$

The periodic configurations of g exist only in S and Theorem 4 shows us the variety of periods.

First note that for $a = 2n + 3m$, where n and m are positive integers,

$$\begin{aligned} &(\dots, \overbrace{01, 01, \dots, 01, 01}^n, \overbrace{011, 011, \dots, 011, 011}^m, \dots) \\ &\quad \underbrace{\hspace{10em}}_{\text{one set}} \\ &(\dots, \overbrace{01, 01, \dots, 01, 01}^n, \overbrace{011, 011, \dots, 011, 011}^m, \dots) \in S \\ &\quad \underbrace{\hspace{10em}}_{\text{one set}} \end{aligned}$$

is a periodic configuration with the prime period a . Then it is sufficient to examine how many kinds of type $2n + 3m$ integers exist. Since

$$\begin{aligned} 2p &= 2(p - 3) + 3 \cdot 2, & p \geq 4, \\ 2p + 1 &= 2(p - 1) + 3, & p \geq 2, \end{aligned}$$

there exist clearly periodic configurations with odd prime period more than four and even prime period more than seven. And also there apparently exist periodic configurations with prime period two and three. On the other hand, we can easily verify that no periodic points exist with prime period one, four, or six.

Theorem 4. The dynamical system (g_{40}, \mathcal{S}) has every periodic point except those of prime period one, four, or six.

3. Concluding remarks

The elementary cellular automaton (ECA) rule 40 belongs to Wolfram's class I. It has been said that the time-space pattern generated by rule 40 dies out in a finite time. This observation of rule 40 was made during a computer simulation for a randomly given initial configuration, in other words, the initial configurations were determined according to a Bernoulli measure.

In this paper we directly handled bi-infinite configurations and showed the chaotic properties of rule 40 on the class \mathcal{S} , which is not noted by Wolfram, because the Bernoulli measure of \mathcal{S} is 0 and so every element of \mathcal{S} cannot be chosen as an initial configuration in the case of randomly given initial configurations.

About the proof of the chaotic properties of rule 40, corresponding 01 to 0 and 011 to 1, we could easily have an idea that (\mathcal{S}, g_{40}) is homeomorphic to $(\{0, 1\}^{\mathbb{Z}}, \sigma_L)$. This homeomorphism does not hold, because the left-shift by one site on $\{0, 1\}^{\mathbb{Z}}$ corresponds to a left-shift by two or three sites on \mathcal{S} .

We have also shown that (g_{40}, \mathcal{S}) has every periodic point except prime period one, four, or six. This differs from Sharkovsky's theorem which states that when a continuous interval dynamics has a prime period three point, then it has necessarily every periodic point.

The left-shift dynamical system (g_{40}, \mathcal{S}) may be easily related to an interval dynamical system. Because the proof of the chaotic properties in Theorem 2 is similar to that of the chaotic left-shift dynamical system (σ_L, \mathcal{A}) , which is homeomorphic to a logistic map or triangular transformation. Precise examination of the relationship between (g_{40}, \mathcal{S}) and an interval dynamical system will be presented in a forthcoming paper.

The results obtained in this paper suggest that the ECA of Wolfram's class I, like rule 40, may not be tedious. For example, as shown in Table 2 of [5], the quiescent ECAs of Wolfram class I are rules 0, 8, 32,

40, 64, 96, 128, 136, 160, 168, 192, 224, 234, 238, 248, 250, 252, and 254. When the local transition function g satisfies $g(0, 0, 0) = 0$, the ECA is called *quiescent*. The dynamics of these quiescent rules are roughly summarized as follows.

promising $a = (\dots, 1, 0, 1, 0, 1, 0, 1, 0, \dots)$,

rules 0, 8, 64, 128, 136, 192

$$\forall x \in \mathcal{A}, \lim_{t \rightarrow \infty} g^t(x) = \mathbf{0}$$

rules 32, 160

$$g(a) = \sigma_L(a)$$

$$\forall x \in \mathcal{A} (x \neq a), \lim_{t \rightarrow \infty} g^t(x) = \mathbf{0}$$

rule 40 examined in this paper

rule 96 $\forall x \in \mathcal{S}, g_{96}^t(x) = \sigma_R^t(x)$

$$\forall x \in \mathcal{A} \setminus \mathcal{S}, \lim_{t \rightarrow \infty} g_{96}^t(x) = \mathbf{0}$$

rule 168

if x of which every 0-state site is isolated includes no right-infinite 1-block $(1, 1, \dots)$,

$$g_{168}(x) = \sigma_L(x),$$

if x of which every 0-state site is isolated includes a right-infinite 1-block,

$$\lim_{t \rightarrow \infty} g_{168}^t(x) = \mathbf{1},$$

if x includes a right-infinite 1-block and a 0-block $(0, 0)$,

$$\lim_{t \rightarrow \infty} g_{168}^t(x) = (\mathbf{0}, \mathbf{1}),$$

and for any other x ,

$$\lim_{t \rightarrow \infty} g_{168}^t(x) = \mathbf{0}$$

rule 224

if x of which every 0-state site is isolated includes no left-infinite 1-block $(\dots, 1, 1)$,

$$g_{224}(x) = \sigma_R(x),$$

if x of which every 0-state site is isolated includes a left-infinite 1-block,

$$\lim_{t \rightarrow \infty} g_{224}^t(x) = \mathbf{1},$$

if x includes a left-infinite 1-block and a 0-block $(0, 0)$,

$$\lim_{t \rightarrow \infty} g_{224}^t(x) = (\mathbf{1}, \mathbf{0}),$$

and for any other x ,

$$\lim_{t \rightarrow \infty} g_{224}^t(x) = \mathbf{0}$$

rule 234

if x of which every 1-state site is isolated includes no right-infinite 0-block $(0, 0, \dots)$,

$$g_{234}(x) = \sigma_L(x),$$

if x of which every 1-state site is isolated includes a right-infinite 0-block,

$$\lim_{t \rightarrow \infty} g_{234}^t(x) = \mathbf{0},$$

if x includes a right-infinite 0-block and a 1-block $(1, 1)$,

$$\lim_{t \rightarrow \infty} g_{234}^t(x) = (\mathbf{1}, \mathbf{0}),$$

and for any other $x (\neq \mathbf{0})$,

$$\lim_{t \rightarrow \infty} g_{234}^t(x) = \mathbf{1}$$

- rules 238, 254 $\forall \mathbf{x} \in \mathcal{A}(\mathbf{x} \neq \mathbf{0}), \lim_{t \rightarrow \infty} \mathbf{g}^t(\mathbf{x}) = \mathbf{1}$
- rule 248 if \mathbf{x} of which every 1-state site is isolated includes no left-infinite 0-block $(\dots, 0, 0)$,
 $\mathbf{g}_{248}^t(\mathbf{x}) = \sigma_R^t(\mathbf{x})$,
 if \mathbf{x} of which every 1-state site is isolated includes a left-infinite 0-block,
 $\lim_{t \rightarrow \infty} \mathbf{g}_{248}^t(\mathbf{x}) = \mathbf{0}$,
 if \mathbf{x} includes a left-infinite 0-block and a 1-block $(1, 1)$,
 $\lim_{t \rightarrow \infty} \mathbf{g}_{248}^t(\mathbf{x}) = (\mathbf{0}, \mathbf{1})$,
 and for any other $\mathbf{x}(\neq \mathbf{0})$,
 $\lim_{t \rightarrow \infty} \mathbf{g}_{248}^t(\mathbf{x}) = \mathbf{1}$
- rule 250 for $\mathbf{x} = (\mathbf{0}, 1, 0, 1, 0, 1, 0, 1, 0, \dots)$,
 $\mathbf{g}_{250}^t(\mathbf{x}) = \sigma_L^t(\mathbf{x})$, $\lim_{t \rightarrow \infty} \mathbf{g}_{250}^t(\mathbf{x}) = \mathbf{a}$
 for any other $\mathbf{x}(\neq \mathbf{0})$,
 $\lim_{t \rightarrow \infty} \mathbf{g}_{250}^t(\mathbf{x}) = \mathbf{1}$
- rule 252 for $\mathbf{x} = (\dots, 0, 1, 0, 1, 0, 1, 0, \dots)$,
 $\mathbf{g}_{252}^t(\mathbf{x}) = \sigma_R^t(\mathbf{x})$, $\lim_{t \rightarrow \infty} \mathbf{g}_{252}^t(\mathbf{x}) = \mathbf{a}$
 for any other $\mathbf{x}(\neq \mathbf{0})$,
 $\lim_{t \rightarrow \infty} \mathbf{g}_{252}^t(\mathbf{x}) = \mathbf{1}$

These properties are easily verified and rules 168, 224, 234, and 248 are especially interesting in the context of their relationship with interval dynamics. Exact calculation of the Lyapunov exponents and the spreading rate of ECAs in Wolfram’s class I, extending Lemma 2, is also attractive. Precise examination of the dynamic properties of ECAs in Wolfram’s class I remain an open problem.

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