Coexistence of Dynamics for Two-Dimensional Cellular Automata

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This paper is concerned with the study of six rules from the family of square Boolean cellular automata (CAs) having a neighborhood consisting of four peripheral neighbors and with periodic boundary conditions. Based on intensive computations, we are able to conclude, with statistical support, that these rules have a common feature: they all show coexistence of dynamics, in the sense that as the size n of the side of the square increases with fixed parity, the relative size of the basin of attraction of the homogeneous final state tends to a constant value that is neither zero nor one. It is also statistically shown that the values of the constant levels—one for n odd and the other for n even—can be considered as equal for five of the rules, while for the sixth rule these values are one-half of the others. Some results obtained for the one-dimensional CAs with four peripheral neighbors are also reported, to support our claim that with periodic boundary conditions, the coexistence of dynamics can only appear for automata with dimension higher than one.

1. Introduction

Elementary cellular automata (ECAs)—that is, binary, one-dimensional cellular automata (CAs) in which the state of each cell is updated according to its own state and the states of its two immediate neighbors—were extensively studied by Wolfram in the 1980s [1–5]. Based on the analysis of the behavior of the patterns generated by the time evolution of ECAs, Wolfram [2] proposed a classification of these automata into four classes. This phenomenological type of classi-

https://doi.org/10.25088/ComplexSystems.25.1.1

fication, later refined by Li and Packard [6] and Li [7], was also used for more general CAs [8]. In these classification schemes, a first class (class 1) is reserved for those automata that have very simple dynamics: evolution to a homogeneous state for almost all initial configurations.

In the case of finite ECAs with periodic boundary conditions, there is a clear dichotomy in what concerns the proportion of initial configurations that lead to a homogeneous final state as the size of the automaton increases: it either tends to one, for class 1 automata, or to zero, for all other automata. This means that either "almost all" or "almost no" initial configurations evolve to a homogeneous final state. As we will show, the situation is, however, totally different when dealing with two-dimensional CAs: the family of binary square automata with a neighborhood consisting of four peripheral neighbors and with periodic boundary conditions contains six rules, for which the proportion of initial configurations leading to a homogeneous final state stabilizes at two values (one for automata with even side and the other for automata with odd side) that are neither zero nor one. This means that for these rules, we can have "homogeneous dynamics" coexisting with other dynamics. We should mention that the peculiar behavior of these rules has already been observed—although not fully studied—by Freitas and Severino [9].

The main purpose of this paper is to study in a thorough way these six exceptional rules. Based on a large number of computations, we are able to statistically determine the values of the constant levels for the proportions of initial configurations leading to a homogeneous final state and to show that for five of the rules these values can be considered as equal, while for the sixth rule these values are one-half of the others. We also describe results obtained for the family of one-dimensional CAs with four peripheral neighbors that support our conviction that with periodic boundary conditions, the coexistence of dynamics can only appear for automata with dimension higher than one.

The paper proceeds as follows: Section 2 presents the main definitions and notations; Section 3 contains the first computational results obtained for each of the six rules that enable us to conclude, with statistical support, that all these rules show coexistence of dynamics; Section 4 describes the study conducted to determine the values of the constant levels for the proportions of initial configurations leading to a homogeneous final state; in Section 5 we perform some comparisons on the basins of attraction of the homogeneous final state for the six rules; Section 6 discusses some results to support our belief that coexistence does not exist for one-dimensional CAs with periodic boundary conditions; finally, Section 7 concludes.

https://doi.org/10.25088/ComplexSystems.25.1.1

2. Definitions and Notations

We consider finite square $n \times n$ synchronous Boolean CAs with a four-neighbor peripheral neighborhood, that is, a neighborhood of von Neumann type with the center cell not considered, and with periodic boundary conditions. If we denote by

$$\mathsf{C}(t) = \left(c_{i,j}(t)\right); \quad i, j = 1, \ldots, n,$$

the system state configuration at time t, then the state of the cell $c_{i,j}$ at time $t+1$ is given by the value that a Boolean function ϕ of four variables—the *local update rule*—takes on the 4-tuple consisting of the states of the up, left, right, and down neighbors of that cell at the previous time t:

$$\begin{aligned} &c_{i,j}\left(t+1\right) = \phi\left(c_{i-1,j}(t), c_{i,j-1}(t), c_{i,j+1}(t), c_{i+1,j}(t)\right); \\ &i, j = 1, \ldots, n, \end{aligned} \tag{1}$$

where, for $k = 1, \ldots, n$, we take

$$\begin{aligned} &c_{k,0}(t) = c_{k,n}(t), \;\; c_{k,n+1}(t) = c_{k,1}(t), \\ &c_{0,k}(t) = c_{n,k}(t), \;\; c_{n+1,k}(t) = c_{1,k}(t). \end{aligned} \tag{2}$$

Each configuration is, in this case, an $n \times n$ binary matrix. If we denote by C the set of all such configurations, equation (1), together with the prescribed boundary conditions in equation (2), defines the so-called *global transition function* $\Phi : C \to C$.

A state configuration in which all the cells have the same value is called a *homogeneous configuration*. The all-0 configuration will be denoted by C_0 and the all-1 configuration will be denoted by C_1.

Since we are considering automata with periodic boundary conditions, all the cells in a homogeneous configuration have their neighbors in the same state and, consequently, all the cells are updated by the automaton rule in the same manner. This means that a homogeneous configuration can only be transformed into a homogeneous configuration, and so there exist only three possible situations for the dynamics of the homogeneous configurations: (*i*) C_0 and C_1 form a 2-cycle; (*ii*) both C_0 and C_1 are fixed points; and (*iii*) one of the two homogeneous configurations is a fixed point and the other is mapped into it by applying the automaton rule once.

It happens that for the six rules analyzed in this paper, only situations (*i*) and (*ii*) occur.

We will denote by $\mathcal{B}_h$ the set of all configurations that evolve to a homogeneous configuration. In case (*i*), $\mathcal{B}_h$ is simply the basin of attraction of the 2-cycle attractor, while in case (*ii*), $\mathcal{B}_h$ is the union of the basins of attraction of C_0 and C_1, which we will denote by $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively. With a slight abuse of notation, even in this last situation we will refer to $\mathcal{B}_h$ as the *basin of attraction of the homogeneous final state*.

When $\mathcal{B}_h$ is not the union of two basins—that is, when C_0 and C_1 form a 2-cycle—we will still be interested in partitioning it into two sets that will play a role identical to the basins of attraction of C_0 and C_1 in case (*ii*): the set consisting of the configurations for which the first homogeneous configuration reached by the automaton is C_0 and the set of configurations for which the first homogeneous configuration attained is C_1. Although in this case these sets are not basins of attraction of any attractor, we will still denote them by $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively.

The symbol r_h will be reserved for the proportion of configurations that belong to $\mathcal{B}_h$, that is, for the number given by

$$r_h = \frac{|\mathcal{B}_h|}{2^{n \times n}},$$

where we use the notation $|S|$ for the number of elements in a set S. We will frequently refer to r_h as *the relative size of* $\mathcal{B}_h$. Similar notations, r_0 and r_1, will be used for the proportion of elements in $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively. Note that since $\mathcal{B}_0$ and $\mathcal{B}_1$ form a partition of $\mathcal{B}_h$, we always have $r_h = r_0 + r_1$. (Naturally, $\mathcal{B}_h$, r_h, etc. are functions of n, but in general we will not explicitly state this dependency unless it is strictly necessary.)

For convenience, we will follow the usual procedure of associating a code number N_ϕ with each automaton ϕ. The numbering scheme considered here is the one used in [9] and is defined as follows.

To each neighborhood state $(\mathsf{u}, \mathsf{l}, \mathsf{r}, \mathsf{d})$; $\mathsf{u}, \mathsf{l}, \mathsf{r}, \mathsf{d} \in \{0, 1\}$, we associate the number whose binary representation is $(\mathsf{u}, \mathsf{l}, \mathsf{r}, \mathsf{d})_2$, and the 16 different neighborhood states n_k; $k = 0, \ldots, 15$ are ordered according to these numbers; that is, $\mathsf{n}_0 = (0, 0, 0, 0)$, $\mathsf{n}_1 = (0, 0, 0, 1)$, … , $\mathsf{n}_{15} = (1, 1, 1, 1)$. The integer code N_ϕ corresponding to the rule ϕ is then given by the formula

$$N_\phi = \sum_{k=0}^{15} \phi(\mathsf{n}_k) 2^k. \tag{3}$$

In what follows, we will indiscriminately refer to a cellular automaton (CA) by the associated Boolean function ϕ, the global function Φ, or the integer code N_ϕ.

As already mentioned, a detailed study of the dynamics of the two-dimensional-binary four-neighbor CAs, in a manner similar to what was done by Wolfram for the case of ECAs, was initiated in [9]. In particular, the authors have identified all the CAs for which homogeneous final states play a significant role and, among these, called attention to the special behavior of the following six rules: 383, 575, 831, 43240, 59624, and 60072, which we now investigate more deeply.

3. Coexistence of Dynamics

In this section, we report the first computational results obtained for the six rules.

We should note that with respect to the dynamics of the homogeneous configurations, the situation is the following: for the odd-numbered rules, that is, for rules 383, 575, and 831, the configurations C_0 and C_1 form a 2-cycle, while the even-numbered rules, that is, rules 43240, 59624, and 60072, have C_0 and C_1 as fixed points.

The first automaton considered was rule 383. To compute approximations $\tilde{r}_h(n)$ to $r_h(n)$, we randomly generated 10 000 initial configurations and counted the number of those configurations that evolved to the 2-cycle homogeneous final state. The results for the values of the size n of the automaton $n = 20(4)500$ and $n = 21(4)501$ are displayed in Figure 1. This figure, obtained by allowing values of n up to 501, reinforces a claim made in [9] that was based on a study with n taking the maximum value of 140: *the relative size of the basin of attraction of the homogeneous final state tends to a constant, as n increases with fixed parity*. Before we give statistical arguments to mathematically support this statement, we present graphics illustrating some aspects of the behavior of this rule.

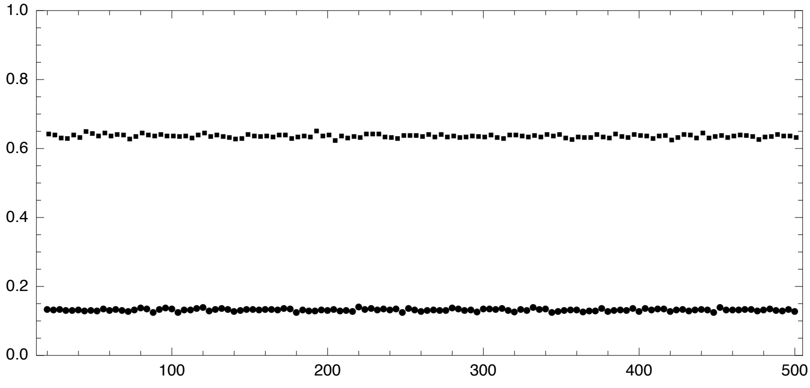
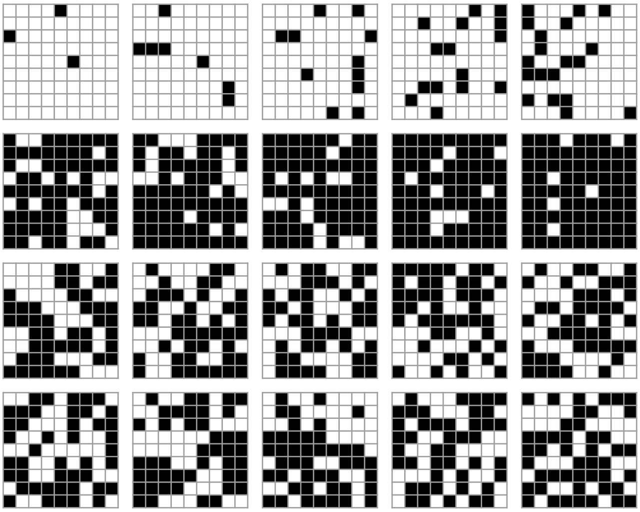
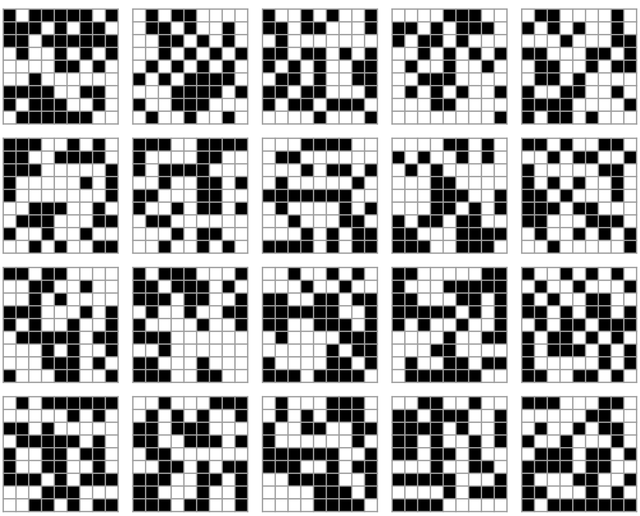


Figure 1. Rule 383–Proportion of initial configurations that lead to a homogeneous final state, as a function of n: ■ – n odd, • – n even; number of randomly chosen initial configurations: 10 000.

Figure 2 contains a sample of 20 configurations belonging to the basin of attraction of the homogeneous final state (Figure 2(a)) and a sample of 20 configurations that are not in that basin (Figure 2(b)), considering a square of size 9×9.



(a)



(b)

Figure 2. Rule 383–Examples of configurations that (a) lead to the 2-cycle homogeneous final state and (b) do not lead to the homogeneous final state.

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The first two lines of Figure 2(a) show configurations with a very unbalanced number of cells in states 0 and 1, for which a homogeneous final state would be easily anticipated. If we compare the configurations in the last two lines of Figure 2(a) with the ones in Figure 2(b), we see that when the number of cells in states 0 and 1 is similar, it is impossible to distinguish a priori whether or not the configuration is in the basin of attraction of the homogeneous final state.

Figures 3–5 show the evolution of the automaton starting from four different well-balanced configurations: the last configuration in Figure 2(a) (leading to a homogeneous state) and the first three configurations in Figure 2(b) (leading to a non-homogeneous final state). When the final state is a non-homogeneous cycle, to help to identify the cycle, we highlight in red the first repeating configurations.

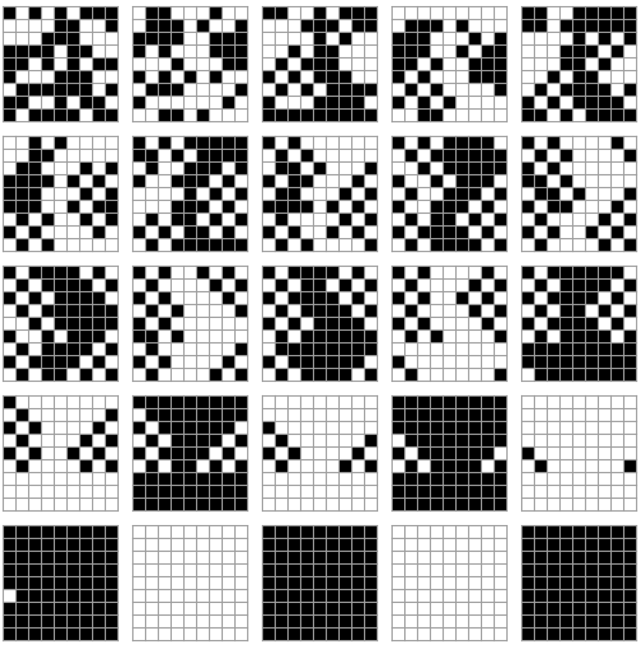

Figure 3. Rule 383–Evolution (left-to-right, top-to-bottom) to the 2-cycle homogeneous final state of a given initial configuration.

Figure 4 shows the evolution to a cycle of very short length, which is a typical behavior of Wolfram's class 2 automata.

Also typical of Wolfram's class 2 automata, Figure 5 shows the evolution to a cycle related to shifts of a simple pattern.

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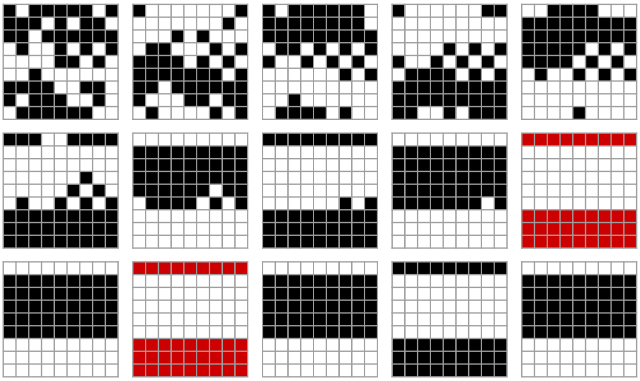

Figure 4. Rule 383–Evolution (left-to-right, top-to-bottom) to a non-homogeneous 2-cycle of a given initial configuration.

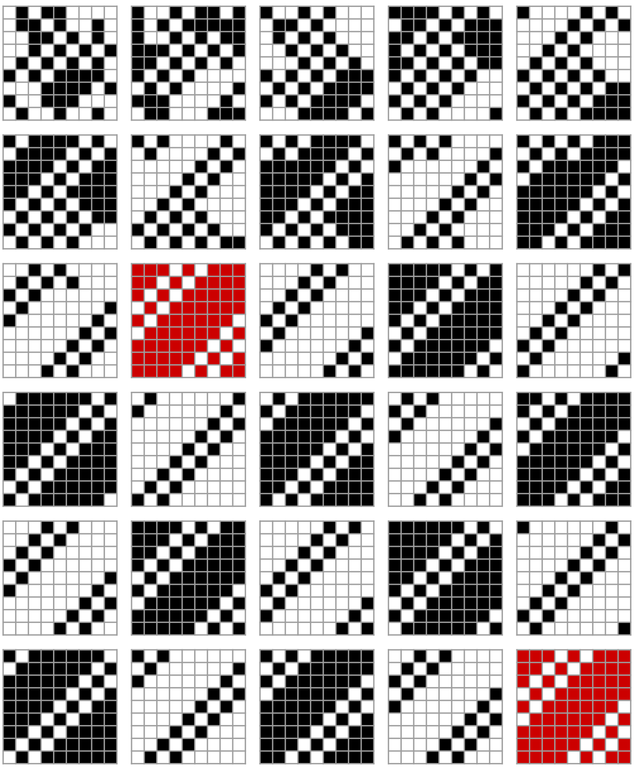

Figure 5. Rule 383–Evolution (left-to-right, top-to-bottom) to a non-homogeneous 18-cycle of a given initial configuration.

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Figure 6 shows a more curious kind of behavior: we still identify a cycle related to shifts, but in this case, the pattern involved is not so simple.

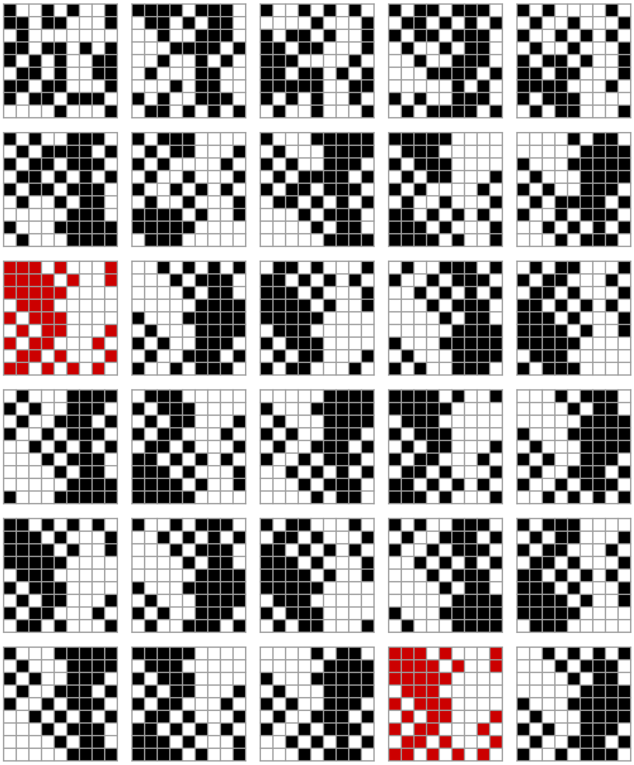

Figure 6. Rule 383–Evolution (left-to-right, top-to-bottom) to a non-homogeneous 18-cycle of a given initial configuration.

Returning to the computational results obtained for the values of $\tilde{r}_h(n)$ as approximations to $r_h(n)$, we now describe the statistical study conducted to support the claim of a constant limit for $r_h(n)$ as n increases with fixed parity. We begin by fitting a classical simple linear regression model to the observed values $\tilde{r}_h$ as a function of n with fixed parity. We then test the null hypothesis of zero slope, and if this hypothesis is not rejected, then our claim is sustained. We report the p-value of the test, which leads to no rejection at the usual 5% significance level if it exceeds 0.05. The classical assumptions of normal errors with a constant variance (homoscedasticity) for this regression model are also checked by performing quantile–quantile plots and residuals plots, respectively.

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For n odd, there is no evidence to reject zero slope (p-value 0.1813), and thus it is statistically reasonable to assume a constant limit value for the relative size of $\mathcal{B}_h$. The same conclusion holds for n even (p-value 0.5419). An analogous behavior was observed for all the other rules. The smallest p-value observed when testing the hypothesis of zero slope was obtained for rule 59624, n odd, and was equal to 0.1573. Quantile–quantile plots show no deviations from normality, and residuals plots indicate no heteroscedasticity.

For each of the rules, there are always two values for the constant limits for r_h, depending on whether we are considering automata with even or odd size length. For simplicity of notation, we will use the same symbol ℓ_h for these limiting values. In what follows, when comparing values of ℓ_h for different rules it should always be understood that the comparison is made for automata of the same kind of parity.

4. Values of l_h for the Six Rules

In order to find more precise estimates for the values of ℓ_h, for each rule we computed the proportion of initial configurations that evolved to a homogeneous final state using a much larger number, 20 000 000, of random initial configurations. The computational effort of dealing with such a large number of initial configurations, however, imposed a severe limitation on the size of the automata considered: in this case, the maximum value of n used was $n = 72$, except for rule 43240, for which we used n up to 88.

4.1 Rules 383, 831, 59624, and 60072

We now analyze the results obtained with the procedure described above for a first set of four of the rules: rules 383, 831, 59624, and 60072. The respective values for $\tilde{r}_h$ are given in Tables A.1, A.2, A.3, and A.4 in Appendix A.

By observing these tables, it is natural to conjecture that the values of ℓ_h are the same for the four rules. Here we apply the classical analysis of variance (ANOVA) technique to test for equality of those values. The classical framework assumes that the four samples are normally distributed with common unknown variance. The ANOVA applied to the data for n odd allowed us to conclude that there is no evidence to reject the null hypothesis of equality of the four ℓ_h values (p-value 0.211) and led to a common estimated value of $\ell_h = 0.6322657 \pm 0.0000292$, corresponding to a 95% confidence value for the true common value of ℓ_h. Notice that the four datasets were consistent with normal distributions (all p-values above 0.524)

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with equal variances (p-value 0.298 for the Bartlett test), thus supporting the classical ANOVA assumptions.

A similar test applied to the values corresponding to n even also enabled us to conclude that it is statistically reasonable to assume equality of the ℓ_h for the four rules (p-value 0.744), with a common estimated value of $\ell_h = 0.11311135 \pm 0.0000143$ (all p-values were above 0.192 for normality tests, and the p-value for the Bartlett test for equality of variances under normal populations was equal to 0.794).

We now give a result concerning the relative sizes of $\mathcal{B}_0$ and $\mathcal{B}_1$, valid for the four rules considered in this section. The significance of this result will be clarified later on.

Proposition 1. For rules 383, 831, 59624, and 60072, we have $r_0 = r_1$.

Proof. From the binary representation of equation (3) of each of the rules and by a simple inspection, we can conclude that all the rules satisfy the following relation

$$\phi\left(\overline{\mathsf{u}}, \overline{\mathsf{l}}, \overline{\mathsf{r}}, \overline{\mathsf{d}}\right) = \overline{\phi(\mathsf{u}, \mathsf{l}, \mathsf{r}, \mathsf{d})}; \quad \mathsf{u}, \mathsf{l}, \mathsf{r}, \mathsf{d} \in \{0, 1\}.$$

It thus follows that given any configuration $\mathsf{C} \in C$, we have $\Phi(\overline{\mathsf{C}}) = \overline{\Phi(\mathsf{C})}$, and hence, that $\Phi^k(\overline{\mathsf{C}}) = \overline{\Phi^k(\mathsf{C})}$ for any $k \in \mathbb{N}$. So we can conclude that

$$\forall\, k \in \mathbb{N}, \quad \Phi^k(\mathsf{C}) = \mathsf{C}_0 \Longleftrightarrow \Phi^k(\overline{\mathsf{C}}) = \mathsf{C}_1,$$

from which the assertion of the proposition immediately follows. □

In view of Proposition 1, it makes sense to consider the limiting values for r_0 and r_1 which, with an obvious adaptation of notation, we denote by ℓ_0 and ℓ_1, respectively, and also to conclude that, for these four rules, we have $\ell_h = \ell_0 + \ell_1 = 2\ell_0$.

4.2 Rule 43240

The results of the computational experiments conducted for rule 43240 are given in Table A.5. At a first glance, a striking difference from the rules studied so far is immediately apparent: although the numbers still indicate the existence of two plateaus, one for n even and the other for n odd, the relative size of $\mathcal{B}_h$ seems to stabilize at values that are about one-half of the corresponding values for the other four rules.

Actually, the hypothesis that for rule 43240, ℓ_h is one-half of the common value for the other four rules was not rejected for either n odd or even (p-values 0.711 and 0.873, resp.). In both cases, the data

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was consistent with two normal populations, which is commonly assumed for the test about the true means.

It should be noted that the result of Proposition 1 is no longer true for rule 43240. In this case, there exists exactly one pair of conjugate neighborhood states whose images by ϕ are not conjugate numbers: we have $\phi(1, 1, 1, 0) = \phi(0, 0, 0, 1) = 0$. From this asymmetry in favor of a zero output, it is clear that we now must have $r_1 < r_0$. For very small values of n, namely for $n = 4$ and $n = 5$, it is possible to explicitly analyze the behavior of the automaton for all the initial configurations and determine with exactitude the values of $|\mathcal{B}_1|$. The results for $n = 4$ are $|\mathcal{B}_1| = 25$, and for $n = 5$, $|\mathcal{B}_1| = 11$. These numbers show that the proportion of configurations that lead to C_1 is extremely small. We have $r_1 \approx 0.38\times 10^{-3}$ for $n = 4$ and $r_1 \approx 0.33\times 10^{-6}$ for $n = 5$. For larger values of n, it is not feasible to do a complete scrutiny of the evolution of the automaton for all possible initial configurations, and as before, we have to rely on a statistical approach. For $n = 6$, the number of initial configurations is $2^{36} > 10^{10}$. Using 20 000 000 randomly chosen initial configurations, we did not obtain a single configuration leading to C_1, and so it seems reasonable to conjecture that the limiting value of r_1 is equal to zero, and hence, that in this case, we have $\ell_h = \ell_0$.

4.3 Rule 575

The last rule studied in detail was rule 575. The corresponding computational results obtained to determine ℓ_h are given in Table A.6.

Here, as with rule 43240, there exists exactly one pair of neighborhood states that are conjugate with each other but have zero as common output: we have $\phi(1, 0, 0, 0) = \phi(0, 1, 1, 1) = 0$ in this case. A similar study to the one conducted for rule 43240 has shown that, here also, it is reasonable to assume that r_1 tends in a very fast way to a limiting value of zero, and so we can say that $\ell_h = \ell_0$. However, this rule behaves in a different manner from rule 43240: as the numbers in Table A.6 show, in this case the values for ℓ_h are no longer one-half of the ones obtained for the four initial rules but seem to be equal to those values. This equality was statistically supported based on a test for equality of means (p-values 0.711 and 0.873, respectively, for n odd and n even), under the normality assumptions described before, which were once again consistent with the data.

4.4 Overview for the Six Rules

Table 1 summarizes the results for ℓ_h and its relation to ℓ_0 for the six rules studied. Here the estimated ℓ_h values are based on 95% confidence intervals under the usual normality assumptions that were actually consistent with the data.

	ℓ_h		
Rules	n odd	n even	
383, 831, 59624, 60072	0.6322657 ± 0.0000292	0.1311135 ± 0.0000143	$\ell_h = 2\ell_0(\ell_0 = \ell_1)$
43240	0.3161242 ± 0.0000450	0.0655550 ± 0.0000207	$\ell_h = \ell_0(\ell_1 = 0)$
575	0.6322371 ± 0.0000501	0.1311203 ± 0.0000267	$\ell_h = \ell_0(\ell_1 = 0)$

Table 1. Estimated values of ℓ_h and relation to ℓ_0 for the six rules.

Finally, we conclude that it is reasonable to assume that the values ℓ_h are the same for all the rules, with the exception of rule 43240, and are given by:

$$\ell_h = \begin{cases} 0.6322600 \pm 0.0000253, & \text{for } n \text{ odd,} \\ 0.1311149 \pm 0.0000125, & \text{for } n \text{ even.} \end{cases}$$

For rule 43240, the values of ℓ_h can be considered as equal to one-half of the values of the other five rules. On the other hand, we can also say that all the rules, with the exception of rule 575, have the same values of ℓ_0, while for this rule, these values are twice the values of the others.

5. Basins of Attraction of the Homogeneous Final State

The purpose of this section is to clarify the relation between the basins of attraction of the homogeneous final state for the various rules.

We first considered CAs of size 20×20 and randomly selected 400 initial configurations. For each of these configurations and for each of the six rules, we determined whether or not there was evolution to the homogeneous final state. The results are shown in Figure 7, where the convergence to the homogeneous final state is indicated with a bullet. In the graphic we only show the cases where at least one of the rules reached a homogeneous final state.

Figure 7 shows some unexpected regularity. In fact, the configurations that evolve to a homogeneous final state under rules 383 and 60072 are exactly the same, and this happens also for rules 575, 831, and 59624; furthermore, the set of configurations that lead to a homogeneous final state under rule 43240 is a subset, with about one-half of the elements, of the corresponding set for rules 575, 831, and 59624. The experiments were repeated for a much larger number of initial configurations (2 000 000), and the conclusions were the same as above. Furthermore, for rules 43240 and 59624 (both of which have C_0 and C_1 as fixed points), we observed that the configurations that evolved to the fixed point C_0 are exactly the same. We did a similar study for CAs with odd size 21×21 and obtained the same type of conclusions.

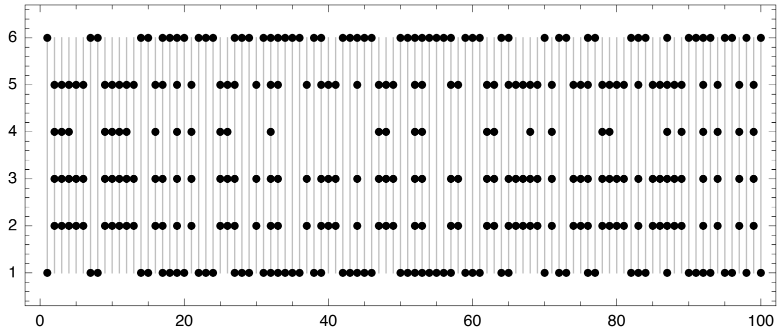


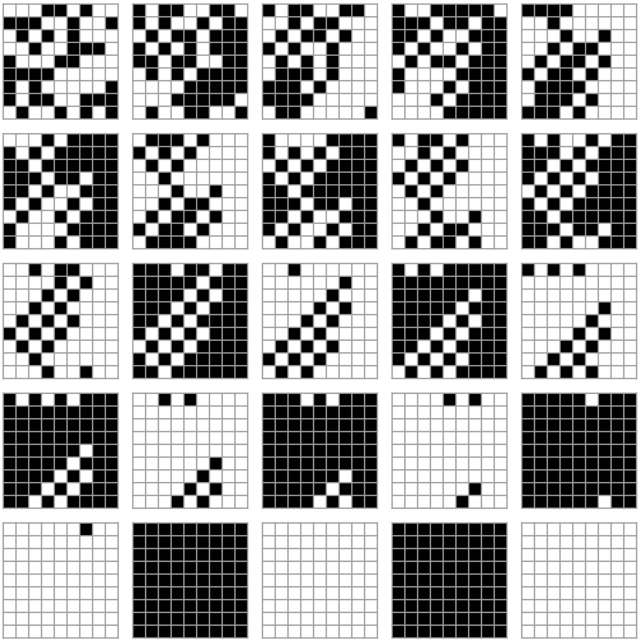
Figure 7. On the x axis: different initial configurations (from a set of 400 randomly chosen initial configurations) for which at least one of the six rules evolved to the homogeneous final state; y axis: different rules. Convergence for the homogeneous final state is indicated with a bullet.

For automata of very small sizes 4×4 and 5×5 it was possible to do a complete scrutiny of the behavior of the automata for all the different initial configurations. The conclusions confirm what we had observed considering 2 000 000 initial configurations and lead us to the following conjecture.

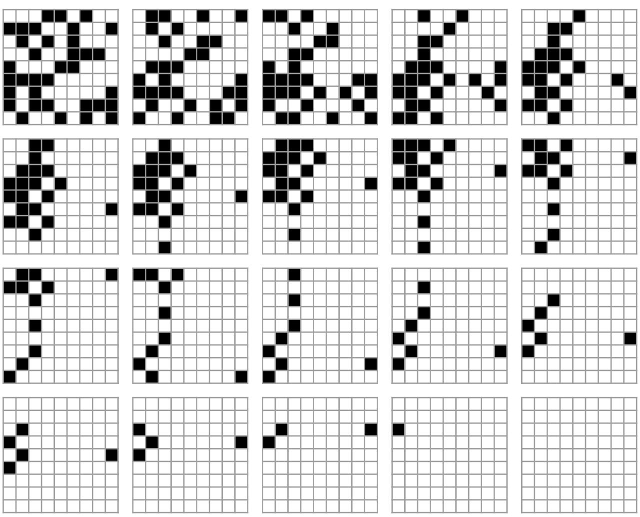
Conjecture 1.

1. Rules 383 and 60072 have the same basin of attraction $\mathcal{B}_h$ of the homogeneous final state.
2. Rules 575, 831, and 59624 have the same basin of attraction $\mathcal{B}_h$ of the homogeneous final state.
3. Rules 43240 and 59624 have the same basin of attraction $\mathcal{B}_0$ of the fixed point C_0.

We should note that the above equality of basins of attraction must be understood simply as an equality of sets and not as an equality of graphs; in fact, the kind of dynamics that the rules show when they start from the same initial configuration leading to a homogeneous state can be different. Figure 8 illustrates this fact for the case of rules 383 and 60072.



(a)



(b)

Figure 8. Evolution to the homogeneous final state of rules (a) 383 and (b) 60072, starting from the same initial configuration.

6. Two Dimensions versus One Dimension

To our knowledge, the type of behavior of the six rules considered here—what we have called coexistence of dynamics—was never referred to for any one-dimensional CA. It is our conviction that when periodic conditions are used, this phenomenon can only occur if the automata have dimension higher than one.

We considered the family of one-dimensional Boolean four-neighbor peripheral automata with periodic boundary conditions. It is a simple matter to show that there are 16 704 dynamically nonequivalent such automata. A detailed study conducted for these rules enabled us to conclude that in what concerns the proportion of initial configurations that evolve to a homogeneous final state, all of these automata still have a behavior similar to ECAs: this proportion either tends to one or to zero. This reinforces our belief that coexistence does not appear in one-dimensional automata. To illustrate this in a different way, we considered again the two-dimensional rule 59624 and computed approximations to r_h for rectangular systems of size $n \times 15$, with increasing values of n. The results, given in Figure 9, support our conviction: as the rectangles become thinner, that is, as the two-dimensional systems get closer to a line, the relative size of $\mathcal{B}_h$ tends to zero.

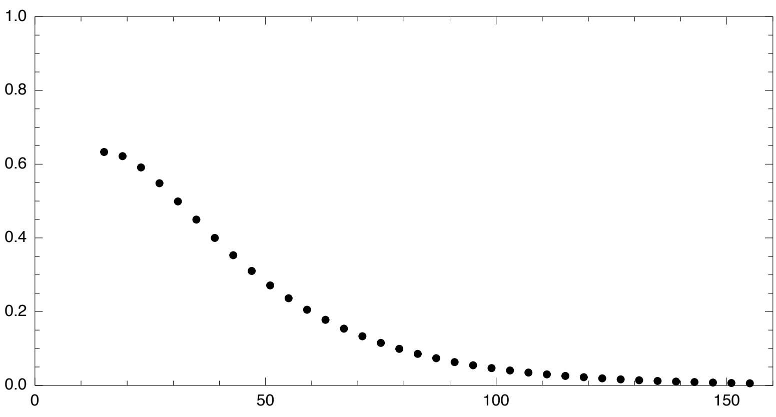


Figure 9. Rule 59624–Proportion of initial configurations that evolved to a homogeneous final state, as a function of n, for systems of size $n \times 15$; number of initial configurations used: 20 000.

7. Conclusion

In this paper we studied in detail six rules from the family of Boolean square cellular automata (CAs) with a four-neighbor peripheral neighborhood and periodic boundary conditions: rules 383, 575, 831, 43240, 59624, and 60072. We concluded, with statistical support, that all these rules have a common feature: if we consider CAs whose side length n is of fixed parity (always odd or always even), then as n increases, the proportion of configurations that evolve to a final homogeneous state tends to a constant value that is neither one nor zero. Moreover, this constant value ℓ_h is the same for all the rules, with the exception of rule 43240, for which it can be considered as equal to one-half of the value for the others.

Some comparisons made on the basins of attraction of the homogeneous final state of the six rules led us to conjecture the equality of these sets for some of the rules.

We also described briefly results obtained for a family of one-dimensional CAs with a neighborhood with a radius larger than elementary cellular automata (ECAs), more precisely, with a four-neighbor peripheral neighborhood. These results support our conjecture that for automata with periodic conditions, the phenomenon exhibited by the rules here studied—what we called coexistence of dynamics—cannot occur for one-dimensional automata.

Finally, we would like to mention that some work with the family of two-dimensional CAs with a full five-cell von Neumann neighborhood has already been conducted; the preliminary results obtained seem to indicate that in this wider family of automata, there exist rules showing coexistence of dynamics with values of ℓ_h different from the ones obtained in this paper.

Acknowledgments

This work was partially supported by FEDER funds through Programa Operacional Factores de Competitividade – COMPETE and by national funds through FCT – Fundação para a Ciência e Tecnologia within the remits of projects PEst-OE/MAT/UI0013/2014 and FCOMP-01-0124-FEDER-037268 (PEst-C/EGE/UI3182/2013).

We would like to thank the anonymous referee for many valuable suggestions.

Appendix

A. Tables

In this appendix we present the tables with approximate values for the relative size of $\mathcal{B}_h$ for each of the rules studied. Each table contains the proportion $\tilde{r}_h$ of initial configurations that, from a total of 20 000 000 initial configurations, led to a homogeneous final state, for different values of n.

n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$
21	0.63250405	47	0.63198800	22	0.13112230	48	0.13108145
23	0.63246340	49	0.63219220	24	0.13118795	50	0.13104350
25	0.63257230	51	0.63238375	26	0.13117710	52	0.13120515
27	0.63243530	53	0.63223945	28	0.13112705	54	0.13111365
29	0.63238665	55	0.63219555	30	0.13107245	56	0.13105605
31	0.63244675	57	0.63210865	32	0.13107825	58	0.13111595
33	0.63214195	59	0.63226600	34	0.13110725	60	0.13102290
35	0.63227580	61	0.63222105	36	0.13119645	62	0.13101170
37	0.63234710	63	0.63212150	38	0.13110935	64	0.13107045
39	0.63230295	65	0.63202320	40	0.13096805	66	0.13101770
41	0.63226475	67	0.63224535	42	0.13106870	68	0.13121885
43	0.63235485	69	0.63204095	44	0.13116430	70	0.13099020
45	0.63235005	71	0.63226570	46	0.13108720	72	0.13127985

Table A.1 Rule 383.

n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$
21	0.63255560	47	0.63204770	22	0.13122340	48	0.13112200
23	0.63264620	49	0.63226580	24	0.13121510	50	0.13108150
25	0.63249625	51	0.63229275	26	0.13117635	52	0.13113135
27	0.63240225	53	0.63241935	28	0.13118605	54	0.13112360
29	0.63254050	55	0.63218600	30	0.13096990	56	0.13111380
31	0.63249610	57	0.63203740	32	0.13108475	58	0.13119885
33	0.63241480	59	0.63234845	34	0.13120600	60	0.13110360
35	0.63225965	61	0.63197705	36	0.13105795	62	0.13122660
37	0.63223845	63	0.63221705	38	0.13113740	64	0.13103100
39	0.63223220	65	0.63210390	40	0.13120780	66	0.13104330
41	0.63247520	67	0.63212325	42	0.13107560	68	0.13108625
43	0.63229085	69	0.63227155	44	0.13115865	70	0.13106995
45	0.63215455	71	0.63216015	46	0.13107365	72	0.13105850

Table A.2 Rule 831.

n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$
21	0.63250695	47	0.63234350	22	0.13115760	48	0.13113155
23	0.63248935	49	0.63227795	24	0.13112455	50	0.13100860
25	0.63238374	51	0.63228580	26	0.13119370	52	0.13112765
27	0.63229035	53	0.63208580	28	0.13118475	54	0.13115020
29	0.63247490	55	0.63223650	30	0.13123490	56	0.13107415
31	0.63231077	57	0.63218545	32	0.13122870	58	0.13105255
33	0.63222110	59	0.63220850	34	0.13113710	60	0.13115345
35	0.63251685	61	0.63221720	36	0.13114210	62	0.13101470
37	0.63224725	63	0.63218720	38	0.13104565	64	0.13102720
39	0.63244120	65	0.63197130	40	0.13109645	66	0.13107345
41	0.63225780	67	0.63202405	42	0.13120980	68	0.13106574
43	0.63233500	69	0.63205270	44	0.13103325	70	0.13108375
45	0.63242970	71	0.63233240	46	0.13103140	72	0.13101260

Table A.3 Rule 59624.

n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$
21	0.63217445	47	0.63227700	22	0.13135765	48	0.13113545
23	0.63237285	49	0.63225515	24	0.13122415	50	0.13105940
25	0.63233800	51	0.63222485	26	0.13123890	52	0.13107930
27	0.63220575	53	0.63217250	28	0.13110165	54	0.13105940
29	0.63230225	55	0.63214215	30	0.13119760	56	0.13114325
31	0.63225480	57	0.63229200	32	0.13101015	58	0.13113860
33	0.63225355	59	0.63210050	34	0.13107535	60	0.13117050
35	0.63213885	61	0.63193885	36	0.13111145	62	0.13102565
37	0.63228220	63	0.63206735	38	0.13114490	64	0.13112420
39	0.63232575	65	0.63218455	40	0.13117940	66	0.13103985
41	0.63203060	67	0.63221050	42	0.13116205	68	0.13110645
43	0.63252560	69	0.63218965	44	0.13095880	70	0.13113585
45	0.63209800	71	0.63217625	46	0.13105190	72	0.13112365

Table A.4 Rule 60072.

n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$
17	0.31641345	53	0.31591780	18	0.06554450	54	0.06549985
19	0.31626700	55	0.31604220	20	0.06548615	56	0.06553715
21	0.31628450	57	0.31609890	22	0.06565500	58	0.06551935
23	0.31638440	59	0.31615095	24	0.06561730	60	0.06560750
25	0.31611410	61	0.31598130	26	0.06562320	62	0.06559645
27	0.31616185	63	0.31627940	28	0.06566555	64	0.06553250
29	0.31638995	65	0.31593965	30	0.06565455	66	0.06560865
31	0.31602650	67	0.31619210	32	0.06554155	68	0.06551425
33	0.31613925	69	0.31608420	34	0.06554945	70	0.06553820
35	0.31613905	71	0.31584750	36	0.06557370	72	0.06553110
37	0.31614435	73	0.31618065	38	0.06558840	74	0.06551845
39	0.31607500	75	0.31603735	40	0.06551125	76	0.06546320
41	0.31603275	77	0.31619285	42	0.06556440	78	0.06549560
43	0.31610125	79	0.31609550	44	0.06554360	80	0.06561705
45	0.31615545	81	0.31624085	46	0.06565425	82	0.06554765
47	0.31621440	83	0.31596995	48	0.06545670	84	0.06543905
49	0.31603735	85	0.31600945	50	0.06553815	86	0.06553950
51	0.31595295	87	0.31617575	52	0.06564255	88	0.06546545

Table A.5 Rule 43240.

n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$	n	$\tilde{r}_h$
21	0.63246910	47	0.63219795	22	0.13115020	48	0.13111355
23	0.63246590	49	0.63234765	24	0.13104800	50	0.13117210
25	0.63236015	51	0.63210440	26	0.13119405	52	0.13112515
27	0.63227290	53	0.63209875	28	0.13102370	54	0.13111180
29	0.63234905	55	0.63219755	30	0.13117040	56	0.13122840
31	0.63232840	57	0.63220200	32	0.13115590	58	0.13115140
33	0.63234180	59	0.63207400	34	0.13122000	60	0.13118720
35	0.63228955	61	0.63219545	36	0.13104350	62	0.13103665
37	0.63215325	63	0.63209600	38	0.13104515	64	0.13110000
39	0.63227005	65	0.63205750	40	0.13101875	66	0.13112140
41	0.63234055	67	0.63200390	42	0.13113825	68	0.13098895
43	0.63224780	69	0.63234930	44	0.13108920	70	0.13118970
45	0.63217235	71	0.63218040	46	0.13115630	72	0.13114740

Table A.6 Rule 575.

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