A Language for Particle Interactions in Rule 54 and Other Cellular Automata

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This is a study of localized structures in one-dimensional cellular automata, with the elementary cellular automaton rule 54 as a guiding example.

A formalism for particles on a periodic background is derived, applicable to all one-dimensional cellular automata. We can compute which particles collide and in how many ways. We can also compute the fate of a particle after an unlimited number of collisions—whether they only produce other particles, or the result is a growing structure that destroys the background pattern.

For rule 54, formulas for the four most common particles are given and all two-particle collisions are found. We show that no other particles arise, which particles are stable and which can be created, provided that only two particles interact at a time. More complex behavior of rule 54 requires therefore multi-particle collisions.

1. Introduction

This paper is part of a project to develop a higher-level language for the dynamical behavior of cellular automata. In the current investigation, we search for an intermediate-level description of the elementary cellular automaton rule 54, in order to learn how to handle periodic background structures and simple particle interactions. The investigation leads to further streamlining and an extension of the existing formalism. This article started as an extension of [1], but has now grown considerably and is completely rewritten.

The formalism is called *Flexible Time*. It was introduced in [2] and further developed in [3]. Flexible Time makes it possible to "calculate" with the localized structures in a cellular automaton and to determine their development over time. The structures in Flexible Time resemble the way in which a human observer views an evolution diagram of a cellular automaton (like Figure 1): by grouping the states of cells from different times and places to a single pattern in spacetime.

Rule 54 is an elementary cellular automaton that was first investigated in detail by Boccara et al. [4]. When evolving from random ini-

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tial configurations, it develops a simple background pattern with a small number of interacting particles that move on this background. While it has not been shown to be computationally universal, it can at least evaluate Boolean expressions [5]. So it is a rather simple system (but not too simple) and therefore an ideal test object for a formalism that is still under development.

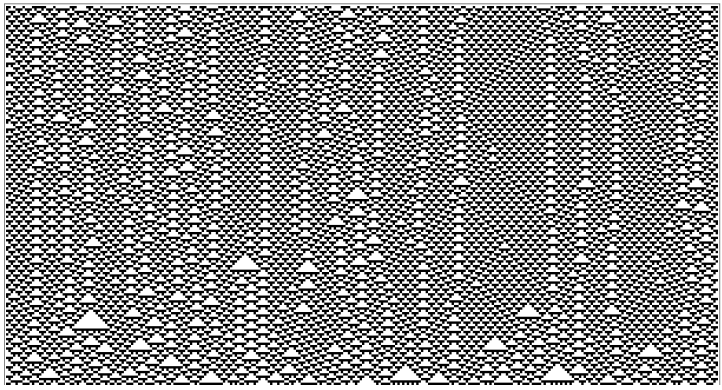

Figure 1. Development of a random initial configuration under rule 54. Time runs from bottom to top.

The right methods to handle large complex structures must still be found. In this paper new questions are asked about the behavior of rule 54, and Flexible Time must "learn" how to handle them. As a result, there are differences and extensions of the formalism in this paper that were not present in [3]. They will be pointed out and reviewed at the end.

1.1 Context

Researchers on cellular automata have developed a number of concepts to describe the localized structures that arise in a cellular automaton.

The oldest named structures must be the particles (also called gliders or signals) and their collisions. This goes back at least to Zuse [6], whose cellular automaton simulates idealized physical particles. Particle-based research has continued since then, with Cook's construction of a universal computer in rule 110 as its most spectacular result [7].

This rule and rule 54 belonged also to those rules in which a stable periodic background pattern occurred; it was called *ether* by Cook.

For rule 54, the starting point was the work by Boccara et al. [4]; they identified the most common particles that arise from random ini-

tial configurations, described their interactions and gave them the names that are still used. This research was later continued by the group around McIntosh [5, 8, 9], who found more complex particles and interactions.

The descriptions of these particles were mostly given by pictures and by a simple symbolism that showed which particle collides with which. But, especially to find general theorems about cellular automata, more abstract representations were developed too.

There is a more detailed investigation of particles and what they can achieve [10, 11]. For rule 110 there is an approach for the systematic specification of initial configurations with interacting gliders [9], and to express the behavior of the cellular automaton through a block substitution system [12].

There are also the approaches by Hordijk et al. [13] and by Martin [14], who use properties of the background and draw conclusions about the particles and particle interactions that are possible. More generally, the cellular automaton is subdivided into "regular" regions and the boundaries between them [15–19]; the boundaries move, often in an almost random fashion, and are thus a generalization of the more straight-moving particles.

Other approaches view the evolution of the cellular automaton as two dimensional, with one space and one time dimension. The cellular spacetime is then subdivided into finite patches that represent, for example, a piece of the background or a collision between particles. The theory of cellular automata then becomes a special tiling problem. We can do this in a more informal way, like McIntosh and Martínez [20], or develop a complex formal theory around it, as Ollinger and Richard [21, 22] do. (This approach is closest to the work described here.)

1.2 Overview

After an introductory section about cellular automata and rule 54, Section 3 recapitulates the work in [3], as far as it is relevant for the present work. At its end, a representation of rule 54 as a *reaction system* (defined below) is shown, the same that was derived in [3]. In Section 4, we then find a way to compress this and similar systems, and we use the compressed reaction system to understand the local behavior of rule 54 better. Section 5 then turns to larger patterns and describes the triangular structures in rule 54 and the stable background pattern that is formed by them. Then, in Section 6, the four kinds of particles found by Boccara et al. [4] are represented in Flexible Time, together with the collision between the particles. A summary follows in Section 7.

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2. Cellular Automata and Rule 54

2.1 Elementary Cellular Automata

Rule 54 is a one-dimensional cellular automaton, more specifically an elementary cellular automaton. This kind of cellular automaton was made popular by Stephen Wolfram [23], who also introduced the system of code numbers from which rule 54 got its name.

One-dimensional cellular automata are dynamical systems with discrete time. The state of such an automaton is called a *configuration*. It consists of an infinite sequence of simpler objects, the *cells*. The state of each cell is an element of a finite set Σ; the configuration at time t is therefore a function $c_t : \mathbb{Z} \to \Sigma$. We write $\Sigma^{\mathbb{Z}}$ for the set of configurations; $c_t(x)$ is then the state of the cell at position x at time t.

The *evolution* of the automaton is then a sequence $(c_0, c_1, c_2, \ldots)$ of configurations that follow a common rule that is described below in equation (2). While the sequence here starts at time 0, we will also accept other starting times.

An elementary cellular automaton is a one-dimensional cellular automaton with two states and a three-cell neighborhood. The set of states is $\Sigma = \{0, 1\}$, and its behavior is given by its *local transition rule*:

$$\varphi : \Sigma^3 \to \Sigma. \tag{1}$$

This is the function with which the configuration c_{t+1} is computed from its predecessor c_t. To do this, we apply φ to every three-cell neighborhood of c_t, and the result is the next state of the middle cell:

$$c_{t+1}(x) = \varphi\big(c_t(x-1),\, c_t(x),\, c_t(x+1)\big) \quad \text{for all } t,\, x \in \mathbb{Z}. \tag{2}$$

The function φ defines then a *global transition rule* φ_{global}: it is the function that maps the configuration c_t to its successor c_{t+1} according to equation (2).

The transition rule in equation (2) is also called a rule of radius 1, because only the $c_t(y)$ with $|x - y| \le 1$ contribute to $c_{t+1}(x)$. Rules with other radii are defined similarly.

2.2 Rule 54

Rule 54 has a left-right symmetric transition rule:

$$\varphi(s) = \begin{cases} 1 & \text{for } s \in \{(0,0,1), (1,0,0), (0,1,0), (1,0,1)\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The rule is easier to remember in the form of the following slogan [3],

> "$\varphi(s) = 1$ if s contains at least one 1, except if the cells in state 1 touch."

Here we say that two cells "touch" if they are direct neighbors. Thus the two cells in state 1 touch in the neighborhood $(1, 1, 0)$ but not in the neighborhood $(1, 0, 1)$.

Figure 2 shows how the neighborhoods influence the next state of the central cell. White squares are in state 0, black squares are in state 1, and the time runs upward. This is also our convention in the other diagrams, even if white and black may also become dark and bright gray in the parts of the diagram that are less emphasized.

Figure 2. Rule icon for rule 54.

3. Flexible Time

3.1 Situations

We need a means to describe and understand the interactions of gliders and other patterns under rule 54. Flexible Time was developed in [3] for this task. The motivation was that it is easier to find patterns in the evolution of cellular automata if we work with structures that involve the states of cells at different times. These structures are called here *situations*.

They generalize the finite sequences of cells that are part of the configurations c_t described earlier. In order to express, for example, that $c_t(0) = c_t(1) = 0$ and $c_t(2) = 1$, we would often write that the subsequence of c_t that begins at cell position 0 is 001. Situations generalize this notation. They may extend not only over space but also over time. To write them, we use additional symbols that express a jump in spacetime.

Under rule 54, situations are written as sequences of the symbols 0, 1, $\ominus_i$ and $\oplus_i$, for $i \in \{1, 2\}$. The intended interpretation can most easily be described in terms of instructions to write symbols on squares in a grid. The squares are labeled by pairs $(t, x) \in Z^2$; x is the position of a cell and t a time step in its evolution. The writing rules are:

- Start reading at the first symbol. For writing, place the cursor at square $(0, 0)$ of the grid.
- If the cursor is at (t, x) and the current symbol is:
  an element of Σ, write it down and move the cursor one square to the right, to $(t, x + 1)$

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$\ominus_i$, move the cursor to $(t-1, x-i)$
$\oplus_i$, move the cursor to $(t+1, x-i)$
Then continue with the next symbol.

- No overwriting: different symbols cannot be written on the same square.

To get an example for such a writing process, let us set for a moment $\Sigma = \{0, 1, 2, 3\}$ and look at the situation $01 \oplus_1 23$. First, the cell states 0 and 1 are written to the squares $(0, 0)$ and $(0, 1)$. The cursor is then at square $(0, 2)$. Now the symbol $\oplus_1$ moves the cursor to $(1, 1)$. The following symbols 2 and 3 are then written to the squares $(1, 1)$ and $(1, 2)$, leaving the cursor at $(1, 3)$. The result is then the following grid:

$t=1$				2	3	—	—
$t=0$	—	—	0	1			
$x=$	-2	-1	0	1	2	3	4

The horizontal rules mark the beginning and end of the symbol sequence, or, more exactly, the squares left of the starting point and right of the end point of the state sequence. Similar lines will later appear in the illustrations.

Now we need to express this construction in a mathematical form. We will use two-dimensional coordinates and call a coordinate pair $(t, x) \in \mathbb{Z}^2$ a *spacetime point*. A pair $(p, \sigma) \in \mathbb{Z}^2 \times \Sigma$ is a *cellular event*. The event $((t, x), \sigma)$ provides the information "at time t, the cell at position x is in state σ." We will usually write them $[t, x]\sigma$ or $[p]\sigma$ for better readability. A situation is then a sequence of cellular events, together with the final cursor position:

$s = (([p_0]\sigma_0, \ldots, [p_{n-1}]\sigma_{n-1}), p_n)$.

For the final cursor position of s, we write $\delta(s)$, the *size* of s. This means that we have in our example $\delta(s) = p_n$.

In a situation, the sequence of the cellular events is significant, and the size too, since they make algebraic operations possible. In many cases, however, we want to ignore this information: then we will use the *cellular process* that belongs to a situation; it is simply the set of its cellular events. The cellular process of a situation s is written $\mathrm{pr}(s)$. In our example, with $s = 01 \oplus_1 23$, we have therefore

$$s = (([0, 0]0, [0, 1]1, [1, 1]2, [1, 2]3), (1, 3)).$$

This long expression means that $\delta(s) = (1, 3)$ and $\mathrm{pr}(s) = \{[0, 0]0, [0, 1]1, [1, 1]2, [1, 2]3\}$.

Usually we will not need this explicit form, since situations are meant to make this unnecessary. However, it helps us to explain the "no overwriting" rule above. This rule concerns expressions like $01 \oplus_1 2 \ominus_1 3$, where the cursor reaches the same point twice. If it were a situation, its cellular process would be $\{[0, 0]0, [0, 1]1, [1, 1]2, [0, 1]3\}$. This would provide contradictory information about the spacetime point $(0, 1)$: At time 0, is the cell at position 1 in state 0 or 3? The overwriting rule prevents this problem.

The most important algebraic property of situations is that they can be multiplied. The product of s_1 and s_2 is found by first writing s_1 and then, with the cursor at $\delta(s_1)$, writing s_2. The resulting product is written $s_1 s_2$, but due to the overwrite rule, it may not always exist.

More complex terms of situations are defined in the usual way: s^2 is the result of writing s twice, and so on. The *Kleene closure* of a situation s is the set

$$s^* = \{s^k : k \geq 0\}. \tag{4}$$

The Kleene closure always contains the *empty situation*, which is written $[0]$.

In Flexible Time, situations are used to express the evolution of a cellular automaton. But in order to understand how this is done, we first have to look at the way in which the evolution of a cellular automaton is expressed by cellular processes.

3.2 Evolution Expressed with Cellular Processes

In a similar way to that in which a configuration $c_0 \in \Sigma^{\mathbb{Z}}$ can be the starting point of an evolution $(c_0, c_1, c_2, \ldots)$, a cellular process π can be extended to a larger process $\mathrm{cl}\,\pi$, its *closure*. Figure 3 shows how this is meant for the initial configuration $\pi = \mathrm{pr}(10^{13}\,1)$. The cellular events of the original process π are displayed in black and white; together with the events in gray they form the process $\mathrm{cl}\,\pi$. Each horizontal row in the diagram contains the events that belong to a specific time step. We see that the diagram becomes smaller at the top; this means that as time progresses, fewer cell states can be deduced from the information given by the initial process π.

To motivate the exact definition of the closure, we first express the configurations of the cellular automaton and their evolution in terms of cellular processes. This will then allow us to generalize the definition of evolution to processes that do not correspond to configurations.

Now let c be the configuration of a cellular automaton. We define the *embedding* of c at time t to be the process

$$\eta_t(c) = \{[t, x]c(x) : x \in \mathbb{Z}\}. \tag{5}$$

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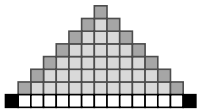

Figure 3. A process and its closure.

A kind of inverse of the function η_t is the concept of *time slices*. The time slice at time t of a process π is the process

$$\pi^{(t)} = \{[t, x]\sigma : x \in \mathbb{Z}\}. \tag{6}$$

The time slice is a process and not a configuration because $\pi^{(t)}$ must exist for all processes, not just for embeddings of configurations.

With these concepts, the cellular process that belongs to the evolution sequence $(c_o, c_1, c_2 \ldots)$ is $\gamma = \bigcup_{t\geq 0} \eta_t(c_t)$. It has the time slices $\gamma^{(t)} = \eta_t(c_t)$, which represent the configurations c_t. The process γ must then be the closure of $\eta_0(c_0)$.

A time slice $\pi^{(t)}$ of an arbitrary process is then understood as partial knowledge about the state of a cellular automaton at time t. In order to determine the state of the automaton at time $t+1$, we take all configurations that are compatible with this knowledge, evolve them for one time step and accept only the states of those cells about which all configurations agree. The result is the cellular process

$$\Delta_t(\pi) = \bigcap \left\{\eta_{t+1}\left(\varphi_{\text{global}}(c)\right) : \eta_t(c) \supseteq \pi^{(t)}\right\} \tag{7}$$

of those events that are determined by $\pi(t)$. The cellular events of which it consists all belong to time $t+1$.

We can now easily check that the process γ has the property that $\gamma^{(t)} = \Delta_t(\gamma)$ for all $t > 0$. Every time slice, except the first, can be computed from the previous one. Only $\gamma^{(0)}$, which represents the initial configuration, must still be handled separately.

This inconvenience is resolved in the full definition of the closure. In it, the initial process no longer needs to be the embedding of a configuration. This is possible because it is now split into time slices and then added piecewise to the partial results of the computation.

Definition 1. (Closure [3, Def. 3.10]) Let π be a cellular process for which there is a time $t_0 \in \mathbb{Z}$ such that $\pi^{(t)} = \emptyset$ for all $t < t_0$.

If there is a process γ with the property that

$$\gamma^{(t)} = \begin{cases} \Delta_t(\gamma) \bigcup \pi^{(t)} & \text{for } t \geq t_0, \\ \emptyset & \text{for } t < t_0, \end{cases} \tag{8}$$

then we write $\gamma = \operatorname{cl} \pi$ and say that it is the closure of π.

It is easy to see that the choice of t_0 has no influence on $\operatorname{cl}\pi$.

We can now see that the set γ that was defined above satisfies equation (8) if we set $t_0 = 0$ and $\pi = \eta_0(c_0)$: then we have $\gamma^{(t)} = \emptyset$ for $t < 0$, $\gamma^{(0)} = \eta_0(c_0)$, and $\gamma^{(t)} = \Delta_t(\gamma)$ for $t > 0$, and indeed $\gamma = \operatorname{cl}\eta_0(c_0)$. Definition 1 is thus a generalization of the transition rule in equation (2) to cellular processes.

Not to all cellular processes, however. One of the requirements of Definition 1 is that γ must be a process, and this can easily be broken. All we need is conflicting information in $\Delta_t(\gamma)$ and $\pi^{(t)}$: if there is a time step t at which there is an event $[t, x]\sigma \in \Delta_t(\gamma)$ and another event $[t, x]\tau \in \gamma^{(t)}$ with $\sigma \neq \tau$, then $\gamma^{(t)}$ is no cellular process, and neither is γ.

However, in the next subsection we introduce a class of situations whose cellular processes all have a closure: they will then be used to describe the evolution of cellular automata in an economical way.

3.3 Reactions

The evolution of a cellular automaton is represented in Flexible Time by reactions. We will say that there is a reaction between two situations a and b if the situation b consists only of events that are determined by the events of a. They belong to the future of a, so to speak.

Figure 4 shows a reaction. On the left side we see the process of the situation $a = 10^{13}\,1$, together with its closure. As in Figure 3, the events of $\operatorname{pr}(a)$ are highlighted, while the remaining cells of the closure are displayed in gray. On the right side we see the same closure, but with different events highlighted. This time they belong to the situation $b = \big(10 \oplus\big)^7 1\big(\ominus 01\big)^7$. With these diagrams, we therefore see that the events of the process b are determined by the process a.

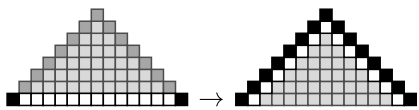

Figure 4. A reaction under rule 54.

The formal definition of reactions is then:

Definition 2. (Reactions [3, Def. 4.8]) Let a and b be two situations with

$$\operatorname{cl}\operatorname{pr}(a) \supseteq \operatorname{pr}\big(b\big) \quad \text{and} \quad \delta(a) = \delta\big(b\big). \tag{9}$$

Then the pair (a, b) is the reaction from a to b. It is usually written $a \to b$.

We will use the expression $a \to b$ also as a proposition, meaning that there is a reaction from a to b. The symbol "$\to$" then specifies a relation, and as it is normal for relations, we can also write longer chains of reactions, like $a \to b \to c$. We can verify that if such a chain exists, then there is also a reaction $a \to c$.

Reactions are useful because they can be *applied* to situations. It can be shown [3, Th. 4.11] that if there are situations x, y and a for which $\mathrm{cl}\,\mathrm{pr}(xay)$ exists and if there is a reaction $a \to b$, then there is also a reaction $xay \to xby$. This reaction is then called the *application* of $a \to b$ to xay.

Now it is possible that there is also a reaction that can be applied to xby. We would then have a reaction $b' \to c$ and two processes x' and y' such that $xby = x'b'y' \to x'cy'$ and therefore, by transitivity, also a reaction $xay \to x'cy'$. This way application allows us to specify a large set of reactions by a small set of "generator reactions," provided only that there is a large enough set of situations to which they can be applied.

The result is a *reaction system*. It is the foundation of all calculations in Flexible Time.

Definition 3. (Reaction System [3, Def. 4.13]) Let D be a set of situations and R a set of reactions between them. We say that R is a reaction system with domain D if the following is true:

1. If $a \in D$, then $a \to a$ is in R.
2. If $a \to b$ and $b \to c$ are in R, then $a \to c$ is in R.
3. R is closed under application of reactions to the situations in D.

We will now define a reaction system by specifying D and a set $G \subseteq R$ of generators; it is then extended by repeated application and concatenation of reactions, as described above. The system describes rule 54; its derivation is described in detail in Chapters 6 and 7 of [3].

The reaction system is summarized in Table 1. The top of the table, entitled "Generating Slopes," specifies the domain D of Φ. More specifically, it lists the neighborhoods that a $\ominus$ or $\oplus$ operator may have if it is part of a situation $s \in D$. The first entry, $\ominus_1 00$, specifies that a $\ominus_1$ may occur in s at the left of the term 00; the second entry, $1 \ominus_1 01$, specifies that it may occur between a 1 (at its left) and a 01 (at its right). No other possibilities exist, since the remaining entries refer to other operators. It can be proved [3, Th. 6.10] that all situations in D have a closure.

The bottom of Table 1 contains the generating reactions of Φ. Its upper part (i.e., the middle of the whole table) contains the reactions

that involve a single $\ominus$ or $\oplus$ operator. If we had only them, no reaction could have an element of Σ^* at its left side: therefore we have at the bottom left of the table a set of reactions that create a $\ominus$ and a $\oplus$ operator from an element of Σ^*. Their converses are listed at the bottom right: reactions that destroy a $\ominus$ and a $\oplus$ operator. All reactions of Φ are the results of repeated applications of these four types of generators.

The arrangement of the reactions in Table 1 also has another purpose. It allows us to read off two important subsystems of Φ.

GeneratingSlopes :		
$\ominus_1 00,\ 1 \ominus_1 01,\ 1 \ominus_2 10,\ 00 \ominus_2 11,$		$00 \oplus_1,\ 01 \oplus_2 1,\ 10 \oplus_1 1,\ 11 \oplus_2 00.$
Reactions:	$\ominus_1 000 \to 0 \ominus_1 00$	$000 \oplus_1 \to 00 \oplus_1 0$
	$\ominus_1 001 \to 1 \ominus_1 01$	$100 \oplus_1 \to 10 \oplus_1 1$
	$1 \ominus_1 010 \to 111 \ominus_2 10$	$010 \oplus_1 1 \to 01 \oplus_2 111$
	$1 \ominus_1 011 \to 100 \ominus_2 11$	$110 \oplus_1 1 \to 11 \oplus_2 001$
	$1 \ominus_2 100 \to 1 \ominus_1 00$	$001 \oplus_2 1 \to 00 \oplus_1 1$
	$1 \ominus_2 101 \to 1 \ominus_1 01$	$101 \oplus_2 1 \to 10 \oplus_1 1$
	$00 \ominus_2 110 \to 001 \ominus_2 10$	$011 \oplus_2 00 \to 01 \oplus_2 100$
	$00 \ominus_2 111 \to 000 \ominus_2 11$	$111 \oplus_2 00 \to 11 \oplus_2 000$
	$00 \to 00 \oplus_1 \ominus_1 00$	$\ominus_1 00 \oplus_1 \to [0]$
	$01 \to 01 \oplus_2 1 \ominus_1 01$	$1 \ominus_1 01 \oplus_2 1 \to 1$
	$10 \to 10 \oplus_1 1 \ominus_2 10$	$1 \ominus_2 10 \oplus_1 1 \to 1$
	$11 \to 11 \oplus_2 00 \ominus_2 11$	$00 \ominus_2 11 \oplus_2 00 \to 00$

Table 1. The local reaction system for rule 54, long form.

Definition 4. (Slopes) Let R be a reaction system with domain D.

The system R_+ (with domain D_+) of positive slopes consists of the situations of D that only contain $\oplus$ operators and the reactions between these situations.

The system R_- (with domain D_-) of negative slopes consists of the situations of D that only contain $\ominus$ operators and the reactions between these situations.

In the case of rule 54, we can find the generators of Φ_- if we take only the generating slopes at the right and the generator reactions at the top right of the middle section in Table 1. Similarly, Φ_+ is represented by the slopes and reactions at the top right of the table.

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3.4 Details of the Reaction System

We will now have a closer look at the way in which the reaction system Φ represents rule 54.

We begin with the slopes. Figure 5 displays the generating slopes for Φ, first the negative slopes and then their mirror images, the positive slopes.

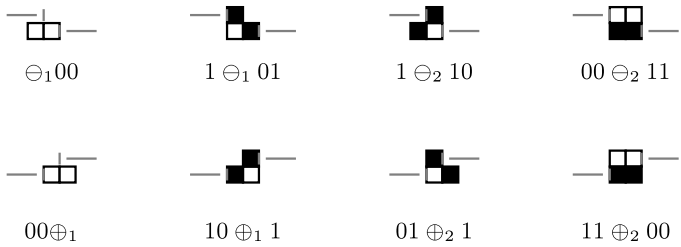

Figure 5. Generating slopes.

In this and in later diagrams, the endpoints of the situations are marked by horizontal lines. They represent the places where the surrounding events would be expected if the slopes were parts of larger situations. Or, in the interpretation of Section 3.1, the square at which the left horizontal line ends is always one point left of the coordinate origin, while the right horizontal line always begins at $\delta(s)$. The beginning of the situation is also marked by the small vertical bar, which is located at the left boundary of the square at the coordinate origin.

An important property of the generating slopes is that they trace the boundaries of the closure. We can see in Figure 6 what this means. It shows a situation, 110 101 000, together with two generations of its closure. We see at its left the slope $00 \oplus_2 11$ (the mirror image of $11 \ominus_2 00$ in Figure 5), and at its right, the term $\ominus_1$, both in bolder colors. Note that the situation $\ominus_1 00$ reaches over two time steps and has its starting point directly at the right end of the second time step of the closure. This is the way the slope terms trace the boundary of a closure.

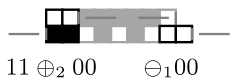

Figure 6. Generating slopes as boundaries of the closure.

The generator reactions of Φ_- are designed with the goal that the reaction result consists of events near the right boundary of the closure of the initial situation. (For Φ_+ it is similar, with left and right

exchanged.) How this is done is shown in Figures 7 and 8. They contain reactions of the form $a \to b$ and display pr(a) and pr(b) in relation to the closure of pr(a). Figure 7 shows the generator reactions of Φ_-. In it, we see that the process of b is always located more to the right than pr(a) and that it touches the right boundary of cl pr(a). The reactions involve only two time steps, and one of the $\ominus$ operators must always be present. To get the system started from situations in Σ^*, we need the reactions at the right side of Figure 8. Here we see reactions in which pr(b) completely fills the closure of pr(a), and b is a situation with both a $\oplus$ and a $\ominus$ operator.

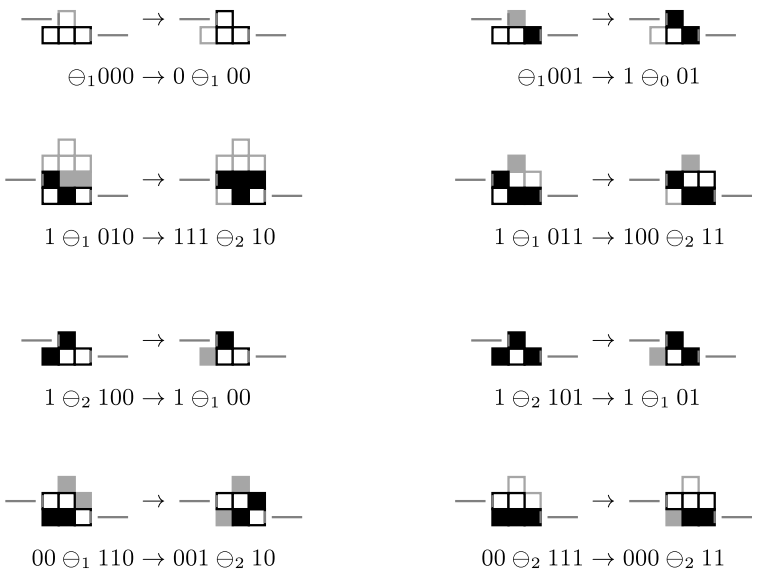

Figure 7. Reactions of Φ_- as motion toward the boundaries of the closure.

The converses of the reactions at the left side of Figure 8 are shown at its right side: reactions $a \to b$ in which a contains one $\ominus$ and one $\oplus$, while b contains none. We can use them for cleanup, since they remove pairs of neighboring $\ominus$ and $\oplus$ operators. The same maneuver is also possible in all other cases where a $\ominus$ is left of a $\oplus$, and we get a result that for every situation a there is a reaction $a \to b_+ b_-$ with $b_+ \in D_+$ and $b_- \in D_-$. If we start from a and continue to apply the generator reactions as long as possible, we can even enforce that b_+ and b_- trace the boundaries of cl pr(a).

This was a summary of the content of [3] as far as it concerns rule 54.

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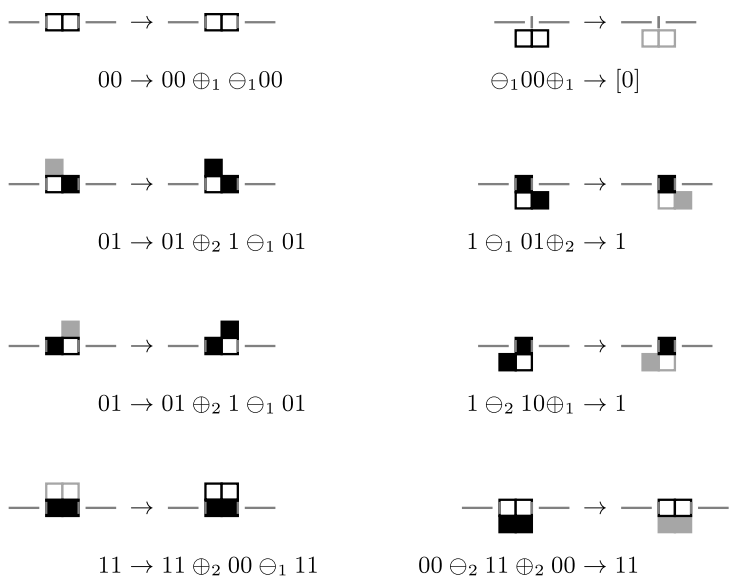

Figure 8. Reactions that generate and destroy slopes. The generator reactions are shown at the left, the destructors at the right.

4. Understanding the Reaction System

Up to now, the representation of rule 54 in Table 1 looks complex and does not provide much insight. This makes it difficult to do calculations about rule 54 without always looking at the table. We will therefore develop a more compact representation of the reaction system. The goal is to find "slogans" for it that are easy to remember, analogous to the slogan for φ in Section 2.2.

4.1 A Simpler Rule Table

As a first simplification, we omit the indices from the $\oplus$ and $\ominus$ operators. This is possible because the indices of the operators are always determined by the environment. We can see from the list of generating slopes in Table 1 that if $\ominus_i$ is followed by a 0, then always $i = 1$, and if it is followed by a 1, then $i = 2$. A similar law is valid for $\oplus_i$, and we can recover the indices of $\ominus$ and $\oplus$ from the equations

$$\begin{aligned} \ominus 0 &= \ominus_1 0, \quad \ominus 1 = \ominus_2 1, \\ 0 \oplus &= 0 \oplus_1, \quad 1 \oplus = 1 \oplus_2 . \end{aligned} \tag{10}$$

This kind of abbreviation is possible in every reaction system, because in a generating slope $u \ominus_i v$, the term $u \ominus_i$ is completely determined by v.

For the same reason, we can shorten the generator reactions by removing common factors from their left and right sides. The generator reactions of Φ_- all have the form $u \ominus v\sigma \to ux \ominus v'$, with a generating slope $u \ominus v$. When such a reaction is applied to a situation s, there must always be a factor u to the left of $\ominus v$ in s. Therefore we can shorten these generator reactions to the form $\ominus v\sigma \to x \ominus v'$ and do not get new reactions when the shortened reactions are applied.

We then get four pairs of reactions as generators for Φ_-:

$$\begin{aligned} &\ominus 000 \to 0 \ominus 00, \qquad \ominus 010 \to 11 \ominus 10, \\ &\ominus 001 \to 0 \ominus 01, \qquad \ominus 011 \to 00 \ominus 11, \end{aligned} \tag{11a}$$

$$\begin{aligned} &\ominus 100 \to \ominus 00, \qquad \ominus 110 \to 1 \ominus 00, \\ &\ominus 101 \to \ominus 01, \qquad \ominus 111 \to 0 \ominus 01\,. \end{aligned} \tag{11b}$$

They can be compressed further with the help of a new notation. For a cell state $\sigma \in \Sigma$, we will write $\overline{\sigma}$ for the *complementary state*, such that $\overline{0} = 1$ and $\overline{1} = 0$. Then we can write the following reactions, valid for all σ (the bottom-left reaction has been shortened even more; it should have been $\ominus 10\sigma \to \ominus 0\sigma$),

$$\ominus 00\sigma \to \sigma \ominus 0\sigma, \qquad \ominus 01\sigma \to \overline{\sigma}\overline{\sigma} \ominus 1\sigma, \tag{12a}$$

$$\ominus 10 \to \ominus 0, \qquad \ominus 11\sigma \to \overline{\sigma} \ominus 1\sigma\,. \tag{12b}$$

Written in this form, we will analyze the reaction system and show what the generator reactions actually mean. But before we can do this, we must see how to simplify the rest of Table 1.

The reactions at the bottom of the table can be brought easily to a common form, when we define the set of negative generating slopes $G_- = \{\ominus 00,\ 1 \ominus 01,\ 1 \ominus 10,\ 00 \ominus 11\}$. With this name at hand, we can see that the bottom reactions have the common form

$$v \to v \oplus u \ominus v \quad u \ominus v \oplus u \to u \tag{13}$$

whenever $u, v \in \Sigma^*$ and $u \ominus v \in G_-$. This then completes the compression of Table 1. The result is Table 2.

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Generating Slopes
$G_- = \{\ominus 00,\ 1 \ominus 01,\ 1 \ominus 10,\ 00 \ominus 11\}$
$G_+ = \{00 \oplus,\ 01 \oplus 1,\ 01 \oplus 1,\ 11 \oplus 00\}$
Reactions
$\ominus 00\sigma \to \sigma \ominus 0\sigma$ $\quad$ $\sigma 00 \oplus \to \sigma 0 \oplus$
$\ominus 10 \to \ominus 0$ $\quad$ $01 \oplus \to 0 \oplus$
$\ominus 01\sigma \to \overline{\sigma}\overline{\sigma} \ominus 1\sigma$ $\quad$ $\sigma 10 \oplus \to \sigma 1 \oplus \overline{\sigma}\overline{\sigma}$
$\ominus 11\sigma \to \overline{\sigma} \ominus 1\sigma$ $\quad$ $\sigma 11 \oplus \to \sigma 1 \oplus \overline{\sigma}$
$u \ominus v \oplus u \to u$
$v \to v \oplus u \ominus v$ $\quad$ for $u \ominus v \in G_-$
Abbreviations
$\ominus 0 = \ominus_1 0$ $\quad$ $0 \oplus = 0 \oplus_1$
$\ominus 1 = \ominus_2 1$ $\quad$ $1 \oplus = 1 \oplus_2$

Table 2. The local reaction system for rule 54, short form.

4.1.1 Relation to the Transition Rule

In order to understand this new form of the reaction system and to see how it is related to the transition rule φ, we write the reactions of Φ_- in the following manner:

	τ_0	τ_1	τ_2
$\ominus_1 00\sigma \to \tau_0 \ominus_1 0\sigma$	$\varphi(0, 0, \sigma) = \sigma$	$\varphi(0, \sigma, \cdot)\uparrow$	
$\ominus_2 10 \to \ominus_1 0$	$\varphi(0, \sigma, \cdot)\uparrow$		
$\ominus_1 01\sigma \to \tau_0\tau_1 \ominus_2 1\sigma$	$\varphi(0, 1, \sigma) = \overline{\sigma}$	$\varphi(1, \sigma, \cdot) = \overline{\sigma}$	$\varphi(\sigma, \cdot, \cdot)\uparrow$
$\ominus_2 11\sigma \to \tau_0 \ominus_2 1\sigma$	$\varphi(1, \sigma, \cdot) = \overline{\sigma}$	$\varphi(\sigma, \cdot, \cdot)\uparrow$	

In the reactions in the leftmost column of the table, each variable τ_i stands for the state of the cell at position $(0, i)$. The other columns then show for each τ_i the computation that determines its value—or, if it cannot be computed, which application of φ fails to have a determined value.

We can see, for example, in the first row that the state of the cell at $(0, 0)$ can be computed from the information presented in the initial situation $\ominus_1 00\sigma$. The cellular process of this situation consists of the events $[-1, -1]0$, $[-1, 0]0$ and $[-1, 1]\sigma$, and therefore the state τ_0 of the cell at $(0, 0)$ must be $\varphi(0, 0, \sigma)$.

In the same way, we can see that in the third row, τ_0 is $\varphi(0, 1, \sigma)$. However, the diagram also contains entries for which not all arguments of φ are known. The missing arguments are marked by a dot. When the value of φ is independent of the missing argument, it is entered in the table; otherwise, the entry is marked with an arrow.

We can see that the values of the τ_i only depend on three equations,

$$\varphi(0, 0, \sigma) = \sigma, \quad \varphi(0, 1, \sigma) = \overline{\sigma}, \quad \varphi(1, \sigma, \cdot) = \overline{\sigma}. \tag{14}$$

They all can be derived from the rule that a pair of touching ones cause a φ value of 0, while one or more isolated ones make the value equal to 1. In the case of $\varphi(0, 0, \sigma)$, a pair of touching ones cannot occur, therefore the value of φ is one if and only if $\sigma = 1$. In the other two cases, $\sigma = 1$ creates a touching pair and $\sigma = 0$ inhibits it, therefore the function value is $\overline{\sigma}$. In a similar way, we can see that in the remaining entries of the table, the value of φ is undefined. This is how φ influences the reactions in Φ.

In the table, the $\ominus$ have been written once again with indices—not just to ease the translation from situations to cellular processes, but also because with them we can see how many new events are generated in the reactions. We can thus see that in the first reaction one new event is generated because $\delta(\ominus_1 00\sigma)$ must be equal to $\delta(\tau_0 \ominus_1 0\sigma)$, and so on. If the left side of a reaction has a $\ominus_i$ operator and the right side a $\ominus_j$, then $j - i$ new cell states must be generated in the reaction.

4.1.2 Slogans

These considerations may help to understand the reactions of the system Φ a bit better. To help with memorizing them, we introduce two slogans. Both refer to the left side of the reactions of $\Phi\_$. This side can always be written as $\ominus\alpha\beta\sigma$, with $\alpha, \beta, \sigma \in \Sigma$. The first slogan tells in which cases the value of $\alpha\beta$ makes the reaction product longer or shorter than the initial term:

"01 causes growth, 10 shrinking, everything else no change."

The second slogan describes the influence of $\beta\sigma$ on the newly generated cell states. They can either be a copy (σ) or the inversion ($\overline{\sigma}$) of the variable σ, and the rule is:

"0σ copies and 1σ inverts."

5. Triangles and Ether

In the rest of this paper, we will describe the behavior of larger systems of cells under rule 54. We want to describe the interaction of particles that move on a periodic background, the so-called ether. So we will now introduce, as a first step, reactions for the ether. Since it has

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been done already to some extent in [3, Ch. 8], we will do it here in a shorter form and from a higher point of view.

The first tool that we will use is *reaction families*, which allow us to represent many similar reactions in a single formula. Reaction families appeared already in [3], but here we use a more streamlined notation.

Definition 5. (Reaction Families) If there is a reaction $a_k \to b_k$ for every $k \geq 0$, we will write this as

$$\left(a_k \to b_k\right)_k. \tag{15}$$

The notation will be extended in the usual way to expressions like $\left(a_k \to b_k\right)_{k \geq N}$ or $\left(a_{j,k} \to b_{j,k}\right)_{j,k}$. We will also speak of $(a_k)_k$ as a situation family.

5.1 Triangle Reactions

We will first find general formulas for reactions that represent triangular structures like that in Figure 4.

There are two general laws that we will use here. The first one makes it possible to iterate a reaction of a special form. This can be done in two ways,

$$\text{if} \quad ax \to ya, \quad \text{then} \quad \left(ax^k \to y^k b\right)_k, \tag{16a}$$

$$\text{if} \quad xa \to ay, \quad \text{then} \quad \left(x^k a \to b y^k\right)_k. \tag{16b}$$

The second law iterates a specific reaction family; in it, n is a constant:

$$\text{if} \quad (a_{k+n} \to x a_k y)_k, \quad \text{then} \quad \left(a_{kn+i} \to x^k a_i y^k\right)_{i,k}. \tag{17}$$

Both laws can easily be proved by induction [3, Ch. 8.1].

We now search for cases in which the first law can be applied and in which the left side is a generator reaction. There are two candidates, $\ominus 000 \to 0 \ominus 00$ and $\ominus 111 \to 0 \ominus 11$. The first one has $a = \ominus 00$ and $x = y = 0$ and leads to

$$\left(\ominus 0^{k+2} \to 0^k \ominus 00\right)_k, \tag{18a}$$

while the second reaction has $a = \ominus 11$, $x = 1$ and $y = 0$ and leads to

$$\left(\ominus 1^{k+2} \to 0^k \ominus 11\right)_k. \tag{18b}$$

The family in equation (18a) is the more interesting one. It becomes the core of another reaction family,

$$\left(10^{k+2}\,1 \to 10 \oplus 10^k\,1\right)_k, \tag{19}$$

whose derivation is shown here in detail, as an example for calculation with reactions:

$$\begin{aligned}\underline{10}00^k\,1 &\to 10 \oplus 1\underline{\ominus 1000^k}\,1\\ &\to 10 \oplus 1\underline{\ominus 000^k}1\\ &\to 10 \oplus 10^k\underline{\ominus 001} \to 10 \oplus 10^k\,1 \ominus 01\,.\end{aligned}$$

Parts of the situations are underlined; they are the places that change in the next reaction step. We will use this notation in later calculations without special notice.

The reaction family in equation (19) can now be iterated by equation (17), with $a_k = 10^k\,1$ and $n = 2$. The result is

$$\left(10^{2k+i}\,1 \to \left(10 \oplus\right)^k 10^i\,1\left(\ominus 01\right)^k\right)_{i,\,k}. \tag{20}$$

In families like these, the cases with $i < 2$ are the most important ones, since the reactions in equation (19) have been applied in them for the largest number of times. For $i = 1$, we can add one more step, since we have $\underline{10}1 \to 10 \oplus 1\underline{\ominus 10}1 \to 10 \oplus 1 \ominus 01$. Therefore, equation (20) can be written as two families,

$$\left(10^{2k}\,1 \to \left(10 \oplus\right)^k 11\left(\ominus 01\right)^k\right)_k, \tag{21a}$$

$$\left(10^{2k+1}\,1 \to \left(10 \oplus\right)^{k+1} 1\left(\ominus 01\right)^{k+1}\right)_k\,. \tag{21b}$$

They, and all reactions of the form $\left(a_{k+n} \to x^k a_n y^k\right)_k$, are called *triangle reactions*.

Diagrams for the reactions with $k = 3$ are shown in Figure 9.

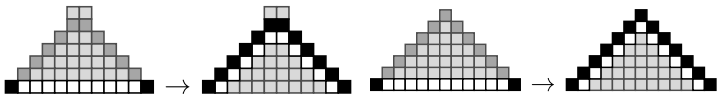

Figure 9. Triangle reactions for $k = 3$.

If we try the same maneuver with the other reaction family, equation (18b), we get $\left(01^{k+2}\,0 \to 01 \oplus 10^{k+2}\,1 \ominus 10\right)_k$. This is a family to which equation (17) cannot be applied. Therefore, we will now use the reaction families in equation (21) as our base for the description of the ether.

5.2 The Ether

We will now represent the ether of rule 54 by reactions. The reactions for rule 54 will turn out to be a special case of a generic scheme that

applies to periodic patterns in any one-dimensional cellular automaton.

In rule 54 [4], the ether is a periodic structure whose configurations consist of alternating the two patterns …100 010 001… and …011 101 110…. When one of them occurs again, it is shifted horizontally by two cells, so that the true time period is four.

Our starting point for representing them by reactions must be the configuration …100010001… , since to it we can apply one reaction of the type in equation (21b),

$$10\,001 \to \big(10 \oplus\big)^2 1\big(\ominus 01\big)^2. \tag{22}$$

It would therefore be advantageous to decompose the initial configuration into components of the form 10 001. With a small extension of our notation, this is actually possible.

Definition 6. (Overlapping Situations) Let ax be a situation. The $a\langle x\rangle$ is also a situation, and $\langle x\rangle$ is the overlapping part. We have

$$\mathrm{pr}(a\langle x\rangle) = \mathrm{pr}(ax) \quad \text{and} \quad \delta(a\langle x\rangle) = \delta(a)\,. \tag{23}$$

A product of situations with overlap, like $a\langle x\rangle b\langle y\rangle$, is only allowed if the situation by begins with x; then $a\langle x\rangle b\langle y\rangle = ab\langle y\rangle$.

A reaction that begins with $a\langle x\rangle$ must have the form

$$a\langle x\rangle \to a'\langle x\rangle; \tag{24}$$

it exists if $ax \to a'x$ is a reaction.

If we remind ourselves that the transitions of a cellular automaton are defined in terms of overlapping cell neighborhoods, then the new extension looks quite natural.

We can now write a term like $\big(1000\big)^k 1$ as a product $\big(1000\langle 1\rangle\big)^k 1$ and apply the ether reactions in parallel to each factor, except for the final 1. In this style, the reaction of equation (22) is best written in the form $1000\big\langle 1\big\rangle \to \big(10 \oplus \langle 1\rangle\big)^2\big(1 \ominus 0\langle 1\rangle\big)^2$.

But now we should better introduce abbreviations. We will write

$$\varepsilon_+ = 10 \oplus \big\langle 1\big\rangle \quad \text{and} \quad \varepsilon_- = 1 \ominus 0\big\langle 1\big\rangle, \tag{25}$$

such that equation (22) becomes

$$1000\big\langle 1\big\rangle \to \varepsilon_+^2 \varepsilon_-^2. \tag{26}$$

The terms ε_+ and ε_- are the simplest of the higher-level structures in rule 54 that we will identify.

There is also a complementary reaction to equation (26),

$$\varepsilon_-^2 \varepsilon_+^2 \to 1000\big\langle 1\big\rangle. \tag{27}$$

In contrast to equation (26), this reaction does not belong to a known family, and we will derive it by hand (see below). Together the two reactions form a type that naturally represents the periodic patterns of one-dimensional cellular automata. Before a formal definition is given, we introduce the abbreviations

$$e_- = \varepsilon_-^2, \quad e_+ = \varepsilon_+^2, \quad b = 1000\langle 1\rangle. \tag{28}$$

Then we see that equations (26) and (27) are examples of the following general pattern:

Definition 7. (Background Pairs) Two situations, e_-, e_+, form a background pair if there is a reaction

$$e_- e_+ \to e_+ e_-. \tag{29a}$$

If there is also a situation $b \in \Sigma^*$ with

$$e_- e_+ \to b \to e_+ e_-, \tag{29b}$$

then b is the *baseline* of the background pair.

A background pair represents the elementary region of a tiling of the two-dimensional spacetime (Figure 10). If a background pair is present, we automatically get the reaction families

$$\left(b^k \to e_+^k e_-^k\right)_k, \tag{30a}$$

$$\left(e_-^k e_+^\ell \to e_+^\ell e_-^k\right)_{k,\,\ell}, \tag{30b}$$

which represent larger patches of the background. As we can see in Figure 10, the reactions of equation (30a) represent the generation of a larger piece of ether from an initial configuration, while equation (30b) represents the development of a background fragment at a later time.

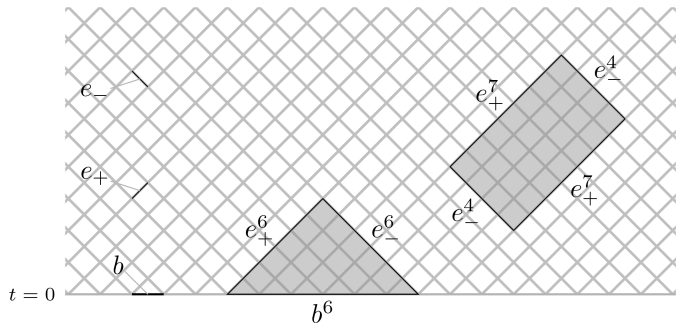

Figure 10. An ether, represented by a background pair e_-, e_+ with baseline b.

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5.2.1 Derivation of the Remaining Ether Reaction

We have not yet proved equation (27), the reaction $e\_e_+ \to 1000\langle 1\rangle$. This will be done now.

The computation is an example for a larger calculation with Flexible Time. We will prove equation (27) via the two reactions

$$\varepsilon\_\varepsilon_+ \to 1^2\langle 1\rangle, \tag{31a}$$

$$\varepsilon_-^2\varepsilon_+^2 \to 1000\langle 1\rangle \tag{31b}$$

and the auxiliary step

$$01^3\,0 \to 01 \oplus 10^3\,1 \ominus 10. \tag{31c}$$

The last reaction is an element of the reaction family

$$\left(01^{k+2}\,0 \to 01 \oplus 10^{k+2}\,1 \ominus 10\right)_k. \tag{32}$$

Its derivation uses the reaction family of equation (18b) and is done in the following way:

$$\begin{aligned}
\underline{01}11^k\,0 &\to 01 \oplus 1\underline{\ominus\, 0111}^k\,0 \\
&\to 01 \oplus 1\underline{\ominus\, 111}^k 0 \\
&\to 01 \oplus 1000^k\underline{\ominus\, 110} \to 01 \oplus 10^{k+2}\,1 \ominus 10.
\end{aligned}$$

Now we can derive the other two reactions of equation (31):

$$\begin{aligned}
\varepsilon_\varepsilon_+ &= 1 \ominus 0\langle 1\rangle 10 \oplus \langle 1\rangle \\
&= 1\underline{\ominus\, 010} \oplus \langle 1\rangle \\
&\to 111\underline{\ominus\, 10 \oplus \langle 1\rangle} \to 11\langle 1\rangle, \\
\varepsilon_\underline{\varepsilon_\varepsilon_+}\varepsilon_+ &\to \varepsilon_ 11\langle 1\rangle\varepsilon_+ \\
&= 1 \ominus 0\langle 1\rangle 11\langle 1\rangle 10 \oplus \langle 1\rangle \\
&= 1 \ominus \underline{01^3\quad 10} \oplus \langle 1\rangle \\
&\to \underline{1 \ominus 01 \oplus 1}0^3\underline{1 \ominus 01 \oplus \langle 1\rangle} \to 10^3\langle 1\rangle.
\end{aligned}$$

In the second computation, we have used equations (31a) and (31c).

6. Particles

In the ether, particles move. Boccara et al. [4] have found four of them and called them $\overleftarrow{w}$, $\overrightarrow{w}$, g_o and g_e (Figure 11). We will refer to the moving particles $\overleftarrow{w}$ and $\overrightarrow{w}$ sometimes as gliders, in contrast to the static particles g_o and g_e.

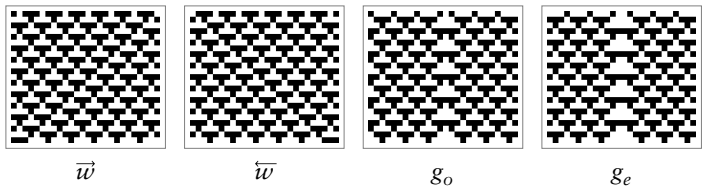

Figure 11. Particles under rule 54. The diagrams show the four types of particles on an ether background.

Now we will represent these particles by situations and reactions. The characterization of particles is a natural generalization of that of a background:

Definition 8. (Particles) Let (b_-, b_+) be a background pair. A particle that moves in this background is a situation p for which there is a reaction

$$b_-^m p b_+^n \to b_+^n p b_-^m. \tag{33}$$

The pair (m, n) is the type of the particle.

The type of p represents its speed relative to the background. To convert it to a more conventional form, we notice that in the initial situation of the reaction in equation (33), the left side of p is located at the spacetime point $m\delta(b_-)$, while in its final situation, it is at $n\delta(b_+)$. The *period vector* $(\Delta t, \Delta x) = n\delta(b_+) - m\delta(b_-)$ is therefore the displacement that p undergoes during one cycle of its existence. After Δt time steps, the particle is in the same state, and it has Δx positions to the right. The speed of p is then $\Delta x / \Delta t$ (Figure 12).

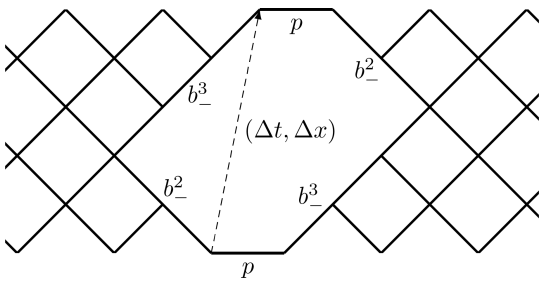

Figure 12. A particle of type (2, 3) as part of a periodic background. Its relative speed is 1 / 5.

Often it is simpler to work with speeds relative to the background. For this we use the vectors $T = \delta(b_+) - \delta(b_-)$ and $X = \delta(b_+) + \delta(b_-)$ as

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our base, the first one pointing to the future and the second one to the right. A particle of type (m, n) has then a period vector of

$$\frac{n+m}{2}T + \frac{n-m}{2}X$$

and we can say that its relative speed is $(n-m)/(n+m)$.

6.1 The Particles of Rule 54

For rule 54 we use the following definitions:

$$\begin{aligned} \overleftarrow{w} &= \varepsilon_- 1^2 \langle 1 \rangle, & g_o &= \varepsilon_+ \varepsilon_-, \\ \overrightarrow{w} &= 1^2 \varepsilon_+, & g_e &= \varepsilon_+ 1 \varepsilon_-. \end{aligned} \tag{34}$$

They have this specific form because we can then use a simple subset of our reaction system to represent their behavior. This subset consists of two reaction families and one extra reaction,

$$\left(\varepsilon_- 1^{2k} \varepsilon_+ \to \varepsilon_+^{k+1} \varepsilon_-^{k+1}\right)_{k \geq 1}, \quad \varepsilon_- \varepsilon_+ \to 1^2 \langle 1 \rangle, \tag{35a}$$

$$\left(\varepsilon_- 1^{2k+1} \varepsilon_+ \to \varepsilon_+^{k+1} 1 \varepsilon_-^{k+1}\right)_k, \tag{35b}$$

which transform situations that consist only of ε_-, ε_+ and 1 into each other. They can easily be derived from the reaction families in equations (32) and (21). With the reactions of equation (35a), the ether reaction $\varepsilon_- \varepsilon_+$ can be proved, as we have seen in Section 5.

With these reactions, we can now verify that the terms in equation (34) are indeed particles:

$$\overrightarrow{w} e_+ = \varepsilon_- 1^2 \varepsilon_+^2 \to \varepsilon_+^2 \varepsilon_-^2 \varepsilon_+ \to \varepsilon_+^2 \varepsilon_- 1^2 \langle 1 \rangle = e_+ \overrightarrow{w}, \tag{36a}$$

$$e_- g_o e_+ = \varepsilon_-^2 \varepsilon_+ \varepsilon_- \varepsilon_+^2 \to \varepsilon_- 1^2\, 1^2 \varepsilon_+ \to \varepsilon_+^3 \varepsilon_-^3 = e_+ g_o e_-, \tag{36b}$$

$$e_- g_e e_+ = \varepsilon_-^2 \varepsilon_+ 1 \varepsilon_- \varepsilon_+^2 \to \varepsilon_- 1^2\, 11^2 \varepsilon_+ \to \varepsilon_+^3 1 \varepsilon_-^3 = e_+ g_e e_-. \tag{36c}$$

The reaction $e_- \overleftarrow{w} \to \overleftarrow{w} e_-$ has been omitted since the reactions in equation (34) are left-right symmetric. We see from these reactions that the types of $\overrightarrow{w}$ and $\overleftarrow{w}$ are $(0, 1)$ and $(1, 0)$, while g_o and g_e both have type $(1, 1)$. Figure 13 contains diagrams of the reactions.

6.2 Collisions of Two Particles

With the reactions of equation (35), we can already find out simple facts about the particles and their interactions. One fact is hidden in equation (36b): the reaction

$$\overrightarrow{w} \overleftarrow{w} \to e_+ g_o e_- \tag{37}$$

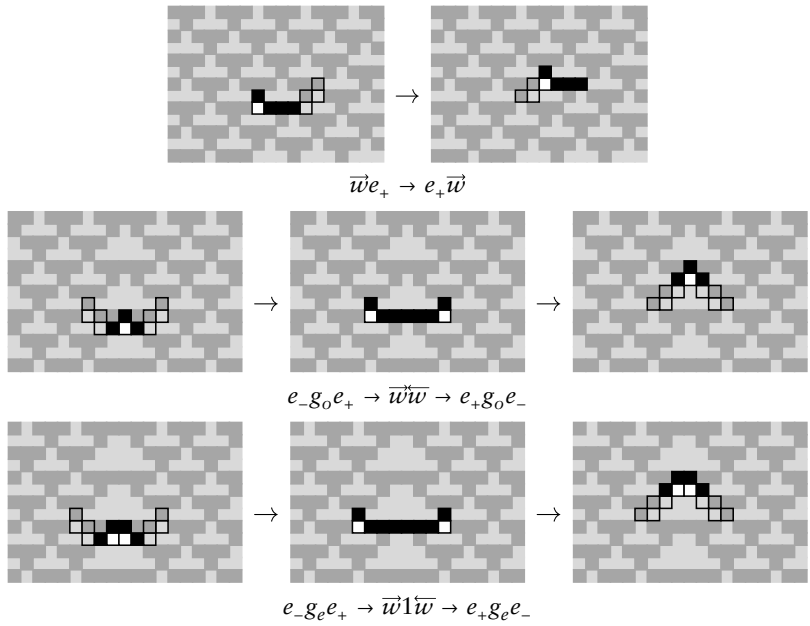

Figure 13. Evolution of the rule 54 particles. The particles are shown in strong colors, and the outlined squares are ether.

can easily be recognized once we remember that $\vec{w}\overleftarrow{w} = \varepsilon_- 1^2\, 1^2 \varepsilon_+$. This is the reaction in which two colliding w particles create a g_o. It is in fact the only reaction that is possible between the two w particles. To see this, we note that if $\vec{w}$ moves toward $\overleftarrow{w}$ with nothing other than ether between them, this must be represented by a situation $\vec{w}E\overleftarrow{w}$, where E is a product of an arbitrary number of e_- and e_+ terms. Then there must be a reaction $E \to e_+^m e_-^n$, where m is the number of e_+ factors in E and n is the number of e_- factors. This leads to a reaction chain

$$\vec{w}E\overleftarrow{w} \to \vec{w}e_+^m e_-^n \overleftarrow{w} \to e_+^m \vec{w}\overleftarrow{w} e_-^n \tag{38}$$

to which we can apply equation (37). We have thus seen that two w gliders always move toward each other unchanged until they react to the position $\vec{w}\overleftarrow{w}$, and that therefore equation (37) is their only possible collision.

The same principle can be applied to any pair of colliding particles. We then have the following theorem:

Theorem 1. (Particle Collisions) Let p and p' be two particles of types (m, n) and (m', n'), with p left of p'. Then p moves toward p' if

$nm' > mn'$ and away from p' if $nm' < mn'$; otherwise, they keep the same distance.

If they collide, then there are nm' possible interactions between them.

Proof. If p and p' collide, the relative speed of p must be greater than that of p'. This means that

$$\frac{n-m}{n+m} > \frac{n'-m'}{n'+m'},$$

or equivalently that $nm' > mn'$. The other two cases are similar.

For the second statement, we represent the relative positions of p and p' by a situation $apbp'c$ with $a, b, c \in \{b_-, b_+\}^*$. Here a and c represent the empty space left and right of the particles. We can make them arbitrarily large without changing the relative positions of p and p'. (A change of a changes the absolute positions of p and p', but that has no influence on their behavior.) Especially, we can assume that $a = b_-^m$ and $c = b_+^{n'}$. The situation b represents the space between p and p', and we can always bring it about by the application of background reactions to the form $b_+^i b_-^j$.

So we can assume that the environment of the particles has the form $b_-^m p b_+^i b_-^j p' b_+^{n'}$. Since p and p' collide, none of the reactions $b_-^m p b_+^n \to b_+^n p b_-^m$ and $b_-^{m'} p' b_+^{n'} \to b_+^{n'} p' b_-^{m'}$ can be applied to this situation. This means that $i < n$ and $j < m'$, for which there are nm' possibilities. □

6.3 Interaction between the Static Particles and the *w* Gliders

When we start with a random initial configuration and let it evolve for a short time, we typically see some g_o and g_e particles on a background, with $\overrightarrow{w}$ and $\overleftarrow{w}$ moving between them (Figure 1). The formalism for rule 54 is now developed far enough to describe with it the behavior of these particles in reasonable detail.

Specifically, we can now describe the behavior of isolated g_o and g_e particles, which never interact with each other, only with $\overrightarrow{w}$ and $\overleftarrow{w}$. In Flexible Time we can express this requirement by restricting ourselves to the reactions that start from a situation xgy with $x \in \{e_-, \overrightarrow{w}\}^*$, $g \in \{g_o, g_e\}$ and $y \in \{e_+, \overleftarrow{w}\}^*$.

The g_o case is the simplest, since the collision with a w always destroys this particle. Up to symmetry, we have only the following reactions:

$$\overrightarrow{w} g_o e_+ \to e_+ \overrightarrow{w} e_-, \quad \overrightarrow{w} g_o \overleftarrow{w} \to e_+^2 e_-^2. \tag{39}$$

They could be verified directly, but we will now compute them in a way that is also useful in the more complex case of g_e. For this, we

begin with $\overrightarrow{w}g_o$, a common factor of the two left sides in equation (39), and also the smallest situation that represents a collision of $\overrightarrow{w}$ and g_o. Their reaction is $wg_o = \underline{\varepsilon_- 1^2 \varepsilon_+} \varepsilon_- \to \varepsilon_+^2 \varepsilon_-^3 = e_+ \varepsilon_- e_-$. The end result is here interpreted as an ε_- surrounded by two ether fragments. We can consider it as a short-lived intermediate stage, or a *resonance*, if we use once again the jargon of particle physics. In the next step, we ignore the ether fragments and consider only the development of the ε_-. There are two ways in which it can interact with an ether fragment or with a w particle, namely through the reactions $\varepsilon_- e_+ = \varepsilon_- \varepsilon_+^2 \to 1^2 \varepsilon_+ = \overleftarrow{w}$ and $\varepsilon_- \overrightarrow{w} = \varepsilon_- 1^2 \varepsilon_+ \to \varepsilon_+^2 \varepsilon_-^2 = e_+ e_-$. No further resonances arise from these reactions, so we can stop here.

The result is a scheme of three reactions; they describe the behavior of g_o in the same way as equation (39):

$$\overrightarrow{w}g_o \to e_+ \varepsilon_- e_-, \tag{40a}$$

$$\begin{aligned} \varepsilon_- e_+ &\to \overrightarrow{w}, \\ \varepsilon_- \overleftarrow{w} &\to e_+ e_-. \end{aligned} \tag{40b}$$

We can use these reactions to derive the reactions of equation (39), for example, with the reaction chain

$$\underline{\overrightarrow{w} g_o} e_+ \to e_+ \varepsilon_- \underline{e_- e_+} \to e_+ \underline{\varepsilon_- e_+} e_- \to e_+ \overleftarrow{w} e_-$$

for the first reaction. But for most purposes, equation (40) can be interpreted directly as a two-step scheme that describes how an ε_- is created (equation (40a)) and how it decays to $\overrightarrow{w}$ or ether (equation (40b)). The ether particles at the right side of equation (40a) can be thought of as becoming part of the surrounding space, which is why they do not appear in equation (40b).

A similar but more complex scheme describes the collision of g_e with one or more w particles. Up to symmetry it has the intermediate states 1, $1\varepsilon_-$ and $1^5\langle 1 \rangle$ and can be written as follows:

$$\overrightarrow{w}g_e \to e_+ e_- 1 \varepsilon_-, \tag{41a}$$

$$\begin{aligned} 1\varepsilon_- e_+ &\to 1\overleftarrow{w} \\ 1\varepsilon_- \overleftarrow{w} &\to 1 e_+ e_-, \end{aligned} \tag{41b}$$

$$\begin{aligned} e_- 1 e_+ &\to 1^5\langle 1 \rangle \\ e_- 1 \overleftarrow{w} &\to \overleftarrow{w} 1 e_- \\ \overrightarrow{w} 1 \overleftarrow{w} &\to e_+ g_e e_-, \end{aligned} \tag{41c}$$

$$\begin{aligned} e_- 1^5 e_+ &\to \overleftarrow{w} g_e \overrightarrow{w} \\ e_- 1^5 \overleftarrow{w} &\to \overleftarrow{w} e_+^2 1 e_-^2 \\ \overrightarrow{w} 1^5 \overleftarrow{w} &\to e_+^2 g_e e_-^2. \end{aligned} \tag{41d}$$

All these reactions are short and can be verified directly. They show that an isolated g_e can neither be destroyed nor does it explode to a larger structure. (See [14] for the deeper reasons behind this.) The intermediate states can, however, persist for an indefinite time if the right pattern of incoming w gliders is given. We can see this, for example, from the reaction $e\_1\overline{w} \to \overline{w}1e\_$ in equation (41c). It can be iterated to $\left(e\_^k 1\overline{w}^k \to \overline{w}^k 1e\_^k\right)_k$, which shows how the intermediate state 1 can be kept alive indefinitely by a sequence of incoming $\overline{w}$ gliders.

In summary, we get a description of the behavior not just of a single g_o and g_e, but also of a whole system of particles, provided that the g particles and their intermediate states all keep a distance from each other. The distance must be so large that next to each g particle or intermediate state there is always a w particle or an ether fragment. As long as this is true, the g_o particles are created (equation (37)) and destroyed (equation (40)) by w gliders, while the g_e persist but go through intermediate states (equation (41)).

7. Summary

This text consists of two interleaving tracks, one with the goal of understanding rule 54 better, the other to find concepts that are valid for all cellular automata.

After a recapitulation of the results derived in [3], we began with constructing a shorter representation of the local reaction system for rule 54 (Table 2). We then described how the transition rule φ influences the local reaction system Φ and at the end introduced two slogans to summarize the generator reactions of the local system.

With equations (16) and (17), we learned how to iterate reactions. This helped to derive expressions for the triangles under rule 54 and to find a subsystem in equation (35) of Φ that consists only of modified triangle reactions. It also introduced the situations $\varepsilon\_$ and ε_+, which, together with the situation 1, were the building blocks of the following construction.

We introduced definitions for the background and for particles and explored particle collisions. A formula for the number of particle interactions was already found in [13] under a different framework, but the proof here seems more direct.

Expressions for the ether and the main particles of rule 54 were found and the collisions of the particles computed. We could see that an isolated g_e is stable under all collisions with incoming w gliders. This extends in a way a result in [14], which already showed that a

single g_e could not be destroyed, but the current, more detailed investigation also shows that it could not "explode" either and become a steadily growing perturbation in the ether.

On the way to this result, we saw an efficient method to display all possible interactions of an isolated particle with all other particles and the background (equation (41)).

The track about rule 54 led therefore to results about the interaction of its particles, while the general track led to generic definitions of triangles, background and particles and a theorem about glider collisions. Both show how Flexible Time helps to understand an automaton like rule 54 as a system of interacting particles.

7.1 Changes in the Formalism

One of the aims of this work was to extend the capabilities of Flexible Time by applying it to the understanding of a "naturally occurring" cellular automaton, that is, one that was not constructed for a specific purpose. This resulted in the following changes with respect to the version in [3]:

1. The interpretations of $\ominus$ and $\oplus$ were changed silently in equation (10). In [3], they were abbreviations for $\ominus_r$ and $\oplus_r$, where r was the radius of the cellular automaton. Now the horizontal offsets associated to $\ominus$ and $\oplus$ depend on the context in which the symbols occur.
2. Reaction families, which were already present in [3], got a shorter notation.
3. A short notation for overlapping situations was introduced in Definition 6. There was already an overlap notation in [3], but it was more clumsy. Now overlapping situations are part of the normal formalism.

The new interpretations of $\ominus$ and $\oplus$ allowed us to write the formulas of the local reaction system completely without indices and to make the similarities between the basic reactions more visible.

With overlaps, definitions like those of a background pair (equation (29)) could be written in a concise way.

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