Emergence and Electrophysiological Analogies in Jellium Models for Cortical Brain Matter

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The complicated and puzzling neuronal structure of human and animal brains is responsible for mental abilities. Concerning a mechanistic understanding of brain activities, the crucial question refers to the properties of a single neuron versus neurons' spatial arrangement and interconnection as a whole. In this paper we adopt the point of view that a significant share of neurons in a being can be modeled by (in our approach complex-valued) dynamical systems based on a manageable number of phase-space dimensions, thus representing a macroscopic overall description of the totality of highly redundant neuronal processes. This agrees with the general theory of interacting many-particle systems that usually undergo a dramatic reduction of complexity in the spirit of the Kolmogorov entropy, due to collective behavior. Then, emergence is understood as a complexity increase in the dynamics under consideration, where the K-entropy characterizes and summarizes the time evolution of many physiological details. Analogies and their limits with respect to the dynamics of selected physical many-particle systems are investigated.

1. Introduction

It is probably impossible to trace brain research back to some inception, but as far as we can see, two major goals have always been in the focus of interest. First, we want to understand how the brain operates, since it is of utmost importance and ability. Second, much effort goes into medicating diseases and malfunctions. The pertinent methods and approaches frequently are cross-disciplinary and sometimes also involve analogies to computational or engineering questions, for

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example, neuronal networks and cellular automata. Particularly in the mind-brain context, even philosophers like to address related issues.

Concerning analytical tools of investigation, electrophysiology and particularly electroencephalograms (EEGs) are quite successful, since they measure an effective physical output of the brain, in an unconscious or "unbiased" manner. We cannot even prevent our own ongoing EEG pattern, whereas reaction to mental tasks strongly depends on external circumstances, such as the way a task gets posed or introduced. In [1] a far-reaching overview of most aspects in the field of EEG techniques and interpretations is provided. Then there is magnetoencephalography (MEG), a related and nevertheless complementary way to look at the internal electric brain activity mainly caused by ionic currents within the neurons. We mention the work of [2], particularly because of its thorough way of tackling the effect of stimuli. However, we do not enter the vast research domain concerning spatial resolution between functional "modules" within the brain. In addition to EEG and MEG, there are various pertinent tomography techniques, ultrasound imaging, magnetic resonance imaging and more.

Instead, we go back to EEG properties, where we highlight an interesting category of features due to nonlinearity in the underlying dynamics. The question of how the EEG relates to the theory of differential equation sets, particularly ordinary, finite dimensional and autonomous ones, has a long and sometimes doleful tradition (doleful primarily due to the largely unknown amount and origin of noise). In short, we deal with the question of determinism, attractor and phase-space dimensionality, dissipation and their implications for mental brain performance. We think [3–7] have contributed interesting new insights, but particularly toward the end of the past century, many further instructive and seminal investigations have emerged. The field continues to be active and sometimes beyond electrophysiology; meanwhile it extends to qualitatively new viewpoints of neuronal dynamics; for contemporary examples see [8–11].

We also dedicated ourselves to related questions, with particular emphasis on generalized entropies [12]. In the past we mainly investigated semi-invasive and also invasive recordings, and so we were in quite a comfortable situation with respect to a trustworthy distinction between noise and "true" signal. Our present article, however, intends to make a significant additional step. To a great extent we abandon the ambition to explain observable brain functions in a mechanistic way; instead, we ask what the minimum requirements of emergence are. We anticipate that one of our major ingredients will be the use of complex-valued sets of nonlinear differential equations, since their special properties prove extremely beneficial. However, we still maintain the demand for an EEG-like output of our "brain simulations." But, analogous to several amazingly powerful solid-state theories, we deal with a jellium approach that "smears out" the internal microstructure of neurons. At first glance this appears rather unrealistic, but we carefully restrict ourselves to the quite limited catalog of properties and abilities of such models. Conversely, in our view there are also firm justifications within electrophysiology, particularly since an EEG channel represents a summary output produced by many largely synchronized neurons. Thus, we measure voltage fluctuations that in fact originate from neuronal activity, but on the other hand, the EEG is rather unspecific to neuronal interconnection in the near (say, a few millimeters) environment. On top of that, even EEG channels all over the scalp are not too different in their statistical long-term properties, at least in the absence of external stimuli. Further, the EEG certainly cannot settle the question of why we are able to think, to feel or to communicate. Regarding EEGs, apparently the overall situation is not that far from our intention to scrutinize the dynamics of jellium models.

Thus, we are left with the question of what physical quantities comprise the power to notice or to assess emergence. We adopt the point of view that the Kolmogorov entropy (see below) largely suits the pertinent demands, although the realm of living beings definitely exhibits mysterious things far beyond such well-defined quantities. The K-entropy measures the complexity in dynamical systems, and hence, it shows how much such a system is unpredictable or "creative." Moreover, an increase in any kind of entropy is also inherently an information loss, and again we think this points to a potential emergence of the unexpected.

2. Dynamical Equation Sets and Electroencephalograms

2.1 Jellium (or Mean Field) Models

As we understand it, in a physiology context, models within the category under consideration are mainly addressed as mean field models, rather than jellium approaches. The two terms just accentuate different aspects, and so we do not distinguish them anymore (with regard to dynamics, actually both expressions appear somewhat inadequate or misleading). For instance, [13, 14] have attracted our attention, and [15] provides an interesting attempt to bridge the gap between physics and physiology as well as between micro- and macroscopic theories.

Usually such mean field approaches deal with the summary effect of excitatory and inhibitory neuronal activity, the latter being a suppression of the postsynaptic overall potential caused by a large num-

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ber of neurons. Many of the potential generating mechanisms may be found in the literature we have quoted so far. However, in this paper we pursue another course. We model important aspects of brain dynamics by means of a manageable number of degrees of freedom; that is, we consider dynamical equation sets attended with a limited phase space. Obviously this gateway to a description of many-neuron activity has already been considered ([8, 14] are examples), but these authors largely focused on aspects we do not primarily have in mind. Our equation sets are not supposed to map physically true neuronal processes such as firing patterns or variations in a summary voltage onto a few dynamical variables. We consider systems that are not motivated or justified on a physical construction level, and so their individual variables generally do not correspond to physical quantities that might be subject to a measurement. Instead, our chosen equation sets are carriers for mathematical properties that we purposely select in order to meet certain electrophysiological facts. At first glance this is just a fit, but we have obtained evidence that there is more to it. Such models also reproduce things we did not at all plug in initially.

We may exemplify the preceding statements by a well-known physical theory, conveniently also in a jellium context. We think of Sommerfeld's way of describing the electron gas in solids. This ansatz ignores the atomic microstructure in a lattice, but on the other hand, it makes use of the most important properties of (quasi) free electrons; see [16]. Electrons with energies close to the Fermi edge are relevant to most (but not all) of the macroscopic properties of metals, including electronic, magnetic, thermal and even elastic phenomena. Therefore, the Sommerfeld approach in some sense is much more successful than a cursory guess (based on the theory of atoms and on geometrical lattice structure arguments) might suggest. Likewise, we ignore the abilities of a single neuron as well as the internal spatial structure of brain tissue, and we purposely focus on aspects where we have a chance to get by without these difficult-to-handle things.

On top of that, dynamical equation sets based on "true" physiological quantities and known neuronal mechanisms suffer from a notoriously large parameter space; [13] brings the problem to light. However, there is also an opposing fact. Complex systems (in the spirit of processes attended with nonzero Kolmogorov entropies) described in high-dimensional phase spaces usually are "less complex" than the number of phase-space dimensions might admit. Sometimes this observation is addressed as the presence of sleeping degrees of freedom. Atmospheric physics probably provides the most striking example, since there we face collective modes that summarize the overall dynamics into macroscopic motions and processes. Thus, the number of actual phase-space dimensions gets dramatically reduced. Apparently this applies to all complex systems, and in this particular respect, we assume brain complexity to be no exception. Neurons generally tend to synchronize, and in the case of epileptic seizures this even becomes strongly manifest in the EEG. Based on intracranial multichannel EEGs, we quantified the "remaining" complexity [12], namely what finally is left over in the signal where these (poorly understood) collective long-range effects are operative. Besides certain unavoidable numerical ambiguities, in our view the resulting Kolmogorov entropy as well as phase-space dimensionality then are irreducible; that is, they cannot be further lowered by a new representation of brain activity in another set of dynamical variables.

2.2 A Suitable Example of Jellium Models

At this point we would like to present a numerical example that qualitatively and to a fair extent also quantitatively meets our claims and intentions. We look for a dynamical system with three major characteristics, namely: (*i*) to be irregular in the sense of deterministic chaos; (*ii*) to be almost oscillatory in all the solution components, that is, without spikes, steps, too-strong frequency changes or other extraordinary variations as a function of time; and (*iii*) it should be strongly dissipative, since we deal with biological processes. The archetypal Lorenz model indeed satisfies all these needs, and later we shall also consider alternate possibilities. Recall that we just intend to establish certain mathematical properties, and we largely will get rid of the specific features of this particular system. For instance, concerning physiological interpretation, the form of the nonlinearities in the following equations does not enter our present considerations. We rewrite the Lorenz system in complex variables $z_i(t) = x_i(t) + i y_i(t)$ with $y_i(t)$ being the imaginary parts, namely

$$\begin{aligned} \dot{z}_1 &= \sigma(z_2 - z_1) \\ \dot{z}_2 &= -z_1 z_3 + r z_1 - z_2 \\ \dot{z}_3 &= z_1 z_2 - b z_3. \end{aligned} \tag{1}$$

With regard to the (six altogether) real and imaginary parts and the renaming $y_i = x_{i+3}$, as well as complex multiplication, we get the six-dimensional system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= -x_1 x_3 + x_4 x_6 + r x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - x_4 x_5 - b x_3 \\ \dot{x}_4 &= \sigma(x_5 - x_4) \\ \dot{x}_5 &= -x_1 x_6 - x_3 x_4 + r x_4 - x_5 \\ \dot{x}_6 &= x_1 x_5 + x_2 x_4 - b x_6 \end{aligned} \tag{2}$$

with eight nonlinearities, and we point to the fact that even the first three equations for the real parts differ from the original Lorenz model. We consider one of the standard parameter settings, namely $(\sigma / r / b) = (16 / 40 / 4)$, which in the original system leads to chaotic solutions or trajectories $\mathbf{x}(t)$. However, with these parameters, equation (2) produces just dissipative limit cycles. In other words, if we take the $z_i(t)$ in equation (1) to be real numbers, as is the case in the original Lorenz system, we notice that properties such as chaos or periodicity (and also fixed points) generally do not survive transformations between related systems in the spirit of equations (1) and (2).

Now we may apply a parameter modification to equation (2) that, in our view, entails a remarkable change. The setting $(\sigma / r / b)$ no longer means just three real numbers; instead we consider these parameters to be three projection operators onto the real axis in the Gaussian plane. For the sake of transparency, we present the altered equation (2) in full. This reads

$$\begin{aligned} \dot{x}_1 &= 16(x_2 - x_1) \\ \dot{x}_2 &= -x_1 x_3 + x_4 x_6 + 40 x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - x_4 x_5 - 4 x_3 \\ \dot{x}_4 &= 0(x_5 - x_4) \\ \dot{x}_5 &= -x_1 x_6 - x_3 x_4 + 0 x_4 - x_5 \\ \dot{x}_6 &= x_1 x_5 + x_2 x_4 - 0 x_6, \end{aligned} \tag{3}$$

where the terms in the equations for the imaginary parts, provided they include any of the parameters, just vanish. Note that one of the equations reduces to $\dot{x}_4 = 0$, and so $x_4(t)$ keeps just the constant value of its initial condition (we might condense equation (3) into five irreducible ones). As indicated previously, the solutions of equation (3) are worthy to be inspected in more detail.

Figure 1 presents a three-dimensional phase plot of the real parts in equation (3), namely the components (or coordinates in a scalar perception) x_1 through x_3. Obviously, the major properties of the original Lorenz system are still there. However, we stress that the additional contributions (beyond the Lorenz system) $x_4 x_5$ and $x_4 x_6$ are by no means negligible. By virtue of the chosen initial conditions, there is $x_4(t) \equiv 1.0$ permanently, and $x_5(t)$ as well as $x_6(t)$ exhibits amplitudes comparable to the three real parts plotted in Figure 1. In short, Figure 1 just mimics the Lorenz system; in fact, however, it displays a different one that largely brings along the same properties.

Next we may look at the two nonconstant imaginary parts as a function of time; that is, at last we consider the relevant quantities that are supposed to simulate true EEGs. In Figure 2, we picture $x_6(t)$

or Im $z_3(t)$, and it is one of our more important findings that the characteristics of the Lorenz model have disappeared almost completely. We shall disregard the scaling of time and intensity, since in equation (3) this might be organized arbitrarily. In the untouched Lorenz system, time and state variables are rescaled in a peculiar way that comprises thermal and geometrical quantities of the respective convection problem, and we do not discuss these issues any further.

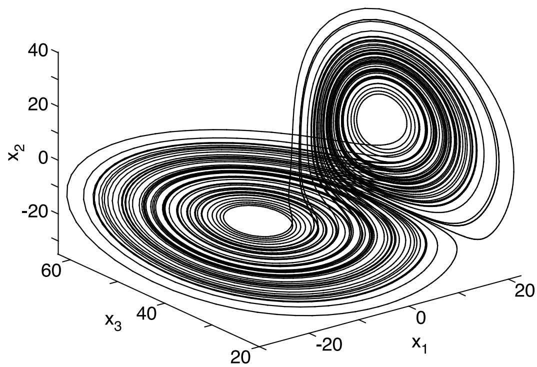

Figure 1. This is *not* the Lorenz system; we plot the three real parts x_1 through x_3 out of the six-dimensional equation (3); see text. With regard to the Lorenz model, the three solution components used already involve two more nonlinearities that numerically are of the same order as the variables x_i in the equations for the $\dot{x}_i$ $(i = 1, 2, 3)$.

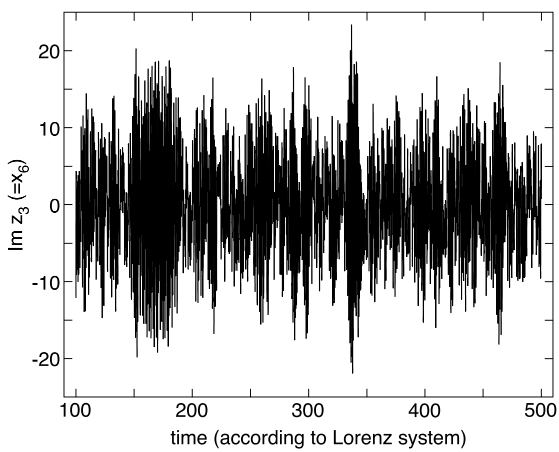

Figure 2. Again we look at equation (3), but this time we plot the time dependence of one of the imaginary parts, namely $x_6(t)$ or Im $z_3(t)$, that picture our "simulated EEG." The peculiarities of the Lorenz model have vanished. Note

also that in equations (2) and (3) (just as in the original Lorenz model) there is constant divergence $\nabla \cdot \dot{\mathbf{x}}(t)$ of the vector field $\dot{\mathbf{x}}(t)$, and so this divergence just equals the sum of all the d Lyapunov exponents $\sum_{i=1}^{d} \lambda_i$ (see text), which is the dissipation.

While Figure 2 shows the long-term evolution, Figures 3 and 4 are devoted to the shape of typical EEG features. In a sense, Figure 3 may be taken as an "EEG" that exhibits certain artifacts, whereas Figure 4 appears not too different from a surface EEG of a healthy individual with (preferably) closed eyes. Regarding physiological interpretation, we may hardly be able to proceed further, but this is not really our goal. We intend to simulate the EEG-generating mechanisms in more depth than just optical similarity between the calculation's output and a true measurement. We have the strong impression that agreement in certain mathematical properties of the dynamics indeed means comparable peculiarities or basic characteristics in the signal-generating processes. Apparently some of the complexity indicators (above all Kolmogorov entropy and Lyapunov spectrum, see below) together with a suitable phase-space dimensionality have the power to act in the desired manner. We are aware that an integral quantity such as the EEG or its numerical simulation cannot admit too far-reaching conclusions on a neuronal level; we have already addressed the point. However, in this paper we would like to single out a major issue, namely model simulation of emergence, where our tools and methods presented here actually might turn out to be sufficient.

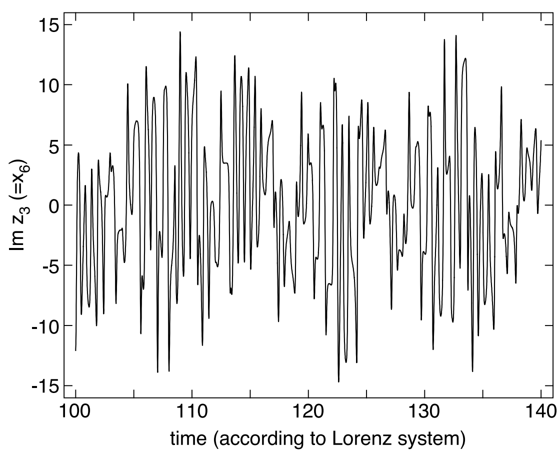

Figure 3. As in Figure 2, but this time we stretch a small segment of the abscissa. We observe substantial similarities to a typical surface EEG that, however, still exhibits certain artifacts that are not likely to be of physiological origin. Again we use the Lorenz default units; see text.

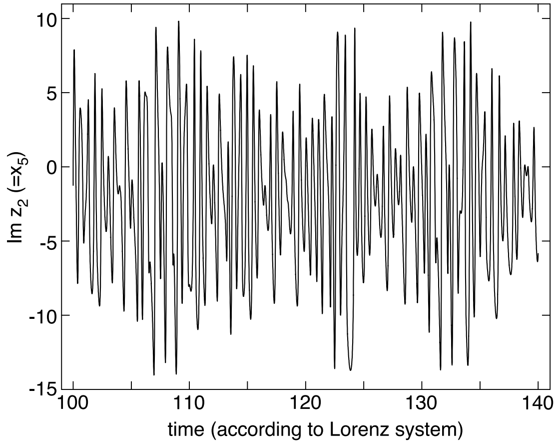

Figure 4. Unlike in Figure 3, we now display Im $z_2(t)$. This resembles a surface EEG of a (healthy) individual with closed eyes. In the text we substantiate that the units may well be adapted to physiologically realistic situations, a procedure that then simply would be part of the numerical model.

2.3 Alternate Possibilities for Neuronal Dynamics

The previous section exemplifies that it is quite possible to separate off the solution specifics of a dynamical model system from the wanted properties in terms of generalized entropies, dissipation and to some extent also frequency spectrum; see our pertinent Figures 2 through 4. Nevertheless, the question arises concerning what alterations appear if we start with another system with entirely different properties. We experienced that our three criteria (*i*)–(*iii*) in Section 2.2, among them the requirement of strong dissipation, in fact are indispensable. This may be visualized by means of the generalized driven Van der Pol oscillator that, for the sake of practical ease, again is represented as a set of three autonomous first-order equations. A sketchy survey over the parameter space and even more useful information is provided by [17]. In analogy to the above, we again apply the complex variables $z_i(t) = x_i(t) + i y_i(t)$. Altogether, the system

$$\begin{aligned} \dot{z_1} &= z_2 \\ \dot{z_2} &= a\left(1 - z_1^2\right) z_2 - z_1^n + k \ \cos z_3 \\ \dot{z_3} &= \Omega \end{aligned} \tag{4}$$

serves now as our new example that replaces equation (1). For parameters attended with chaotic solutions, in the original system (real variables and, for the sake of full comparability, rescaled time according to equally rapid time variations as in the Lorenz system) there is only minor dissipation. If we take the discussed Lorenz model as a benchmark, things get worse if we gradually approach the Hamiltonian limit case with no dissipation. Clearly, from a technical viewpoint, a related procedure as in the previous section then is possible again. But the outcome of the enlarged system that also accommodates the imaginary parts no longer exhibits the outward appearance of an EEG.

We think it is instructive to look at various well-known toy models, since they cover much of the phenomenology within the realm of dynamical systems theory. This, however, cannot be the subject matter of our present paper. There are equation sets with a spiky solution component as well as further off-standard features; for example, the four-dimensional Rössler system [18] is of this type. Conversely, there exist systems whose solutions in almost every respect resemble the Lorenz model: the Rikitake two-disk generator is the example we have in mind; see [19] for an interesting application thereof. It is quite elaborate to explore the situation for higher phase-space dimensionalities, and also for lengthy equations with complicated nonlinearities. Therefore, our model systems, generalizations and parameter settings have been chosen with care, although they are just a selection from all the possibilities.

It may be noteworthy that there is also a qualitatively different way to tackle this category of questions, namely complex-valued dynamics that mimic EEGs in all (or most) of the crucial respects. We still postpone the phenomenon of emergence, and so for the moment we stick to simulated time series. The alternate possibility refers to path integrals in the Gaussian plane, namely

$$\int_\Gamma f(z)dz = \int_a^b f(z(t))\dot{z}(t)\mathrm{dt} = F(z(b)) - F(z(a)), \tag{5}$$

where the function $f(z(t))$ and the time derivative $\dot{z}(t)$ are multiplied in the spirit of complex numbers. Provided $f(z)$ is analytic within some domain under consideration, the Cauchy theorem states that the integral does not depend on the path Γ between the given points (or actually moments) a and b that correspond to the beginning and end of Γ. This way we recognize many interesting results in complex analysis, such as $\oint_\Gamma f(z)dz = 0$ for closed loops, but this is not really what we are after. Rather, we aim at the infinity of possibilities to parametrize the curve Γ. Since equation (5) does not refer to some specific parametrization, for any choice of $f(z)$ as well as $z(a)$ and $z(b)$ we face a wealth of different time-dependent signals. Things get significantly more complicated if the antiderivative $F(z)$ again is a variable that is part of a coupled equations set, but this is not really a matter of principle. Clearly, it is close to impossible to gain an overview about the numerical outcome of these many options, but we appreciate that we definitely are not limited to the few model systems we have shown or at least addressed.

3. Neuronal Firing Patterns, Electroencephalograms and Emergence

3.1 Modeling of Firing Patterns

In view of the hurdles to overcome, we renounce the already introduced parametrized Cauchy integrals. Instead, we return to our complex-valued straightforward generalization of dynamical systems with well-chosen properties. But this time we focus on the peculiarities of firing neurons within the framework of jellium models. These two seemingly contradictory viewpoints for neuronal dynamics become reconciled if we, analogous to the EEG case, concentrate on the relevant mathematical properties of firing in general, rather than on the physically observable action potential of a single neuron.

Again we consider equation (3) with some zero terms in the imaginary parts due to projection onto the real axis. Its last (sixth) equation

$$\dot{x_6} = x_1 x_5 + x_2 x_4 - 0 x_6 \tag{6}$$

gets modified now in such a way that the last term constitutes a drive, rather than a dissipative term (i.e., a damping) with negative prefactor such as in equation (2). This reads

$$\dot{x_6} = x_1 x_5 + x_2 x_4 + 4 x_6, \tag{7}$$

and in Figure 5 we consider just the component $x_6(t)$ out of the complete set of equation (3) with the only alteration according to equation (7). The nearby symmetry of peaks and troughs in fact differs from what we know about neuronal firing patterns (or spike trains). We also may shape such patterns by means of quite simple measures; for example, the minor parameter modulation

$$\dot{x_6} = x_1 x_5 + x_2 x_4 + \left(4 + 0.01 x_6\right) x_6 \tag{8}$$

with 1% of the x_6 amplitude renders things somewhat more irregular; see Figure 6. However, we shall not try to optimize such patterns, since they are just model simulations. Instead, we aim at those mathematical properties that carry the potential for emergent phenomena, and we ask whether they relate to signals that model different types of neuronal dynamics, primarily EEGs and firing patterns. In this respect, we hope for certain complexity measures, above all the Kolmogorov entropy, as we have stated already. We intend to substantiate these anticipations.

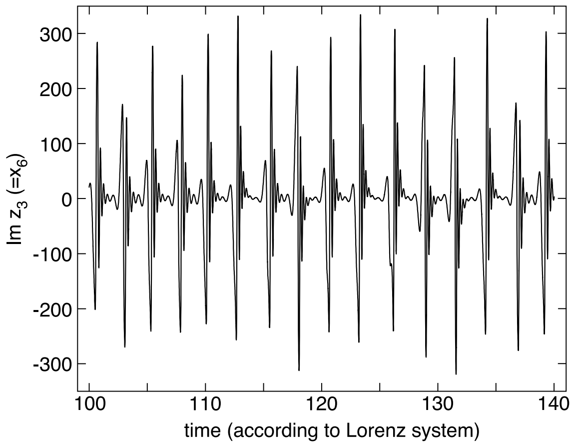

Figure 5. In an oversimplified graphic, we describe the properties of firing neurons, although we never leave the scope of jellium models. On this account, we replace a dissipative term in equation (2) by a "drive"; see text. The outward properties of the signal mimic a stronger chaos than in Figures 3 and 4, but as a numerical fact, the actual K-entropy here is lower than in the EEG case. This is one of the possibilities to achieve emergence caused by interconnection of neurons to a totality that is assembled such that the outcome is EEG-like.

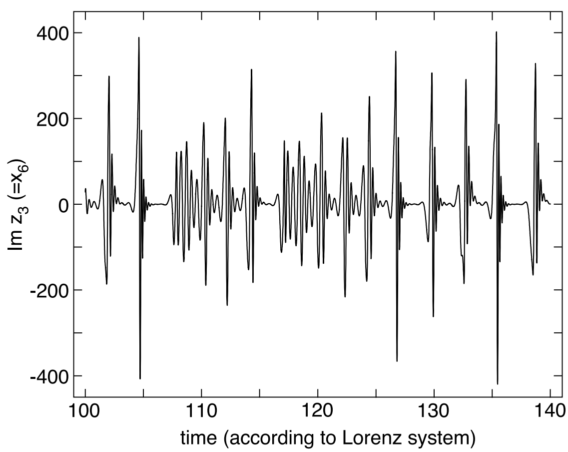

Figure 6. With respect to Figure 5, we now apply a minor change according to equation (8), in order to illustrate a possibility to shape such signals. The full Lyapunov spectrum reads in this example (0.69 / 0 / 0 / −0.79 / −4.17 / −13.61) reciprocal natural time units. The two degenerate zeros arise from the special situation in the fourth line of equation (3) (see

text), and in an irreducible representation we get the same spectrum, as expected, except that there is only a single zero exponent left over.

3.2 Complexity Indicators

The relationship between complexity and emergence is not always evident. Part of the reason is the fact that an increase as well as a decrease in complexity may cause pattern formation or self-organization. To put it simply, an interacting many-particle system on the way to (usually thermodynamic) equilibrium frequently undergoes an "optimum situation" close to equilibrium where spatial and/or temporal structures preferably occur. Then, if we proceed further toward equilibrium from this quasi-equilibrium, we lose the structures and thus approach statistical disorder.

In this paper we cover just one of these aspects or possibilities, with regard to temporal structures only. We purposely focus on situations where an increase in complexity offers the chance for emergent phenomena, in accordance with common sense. Under these assumptions, the Kolmogorov entropy as well as the entire Lyapunov spectrum are our major indicators of complexity, and we recall our brief description thereof in [20].

Summarized even more briefly, we retrace the method of Benettin et al. to calculate Lyapunov exponents (LEs) given in [21], supplemented by an orthogonalization procedure in order to also harvest the local expansion rates in directions other than the principal one. This method is based on a direct evaluation of distances; namely at every mesh point it measures to what extent initially nearby trajectories are separated after a time step. On this score, let $\boldsymbol{\xi}_i(t = 0)(i = 1, \ldots, d)$ be a set of d orthonormal d-dimensional vectors, d being the number of phase-space dimensions. This set may have any orientation, and the length of all the d vectors is normalized to some small value ξ. The points $\mathbf{x}(0) + \boldsymbol{\xi}_i(0)$ serve as initial conditions for the neighboring trajectories of the fiducial one $\mathbf{x}(t)$, and by virtue of a dynamical law such as, for example, the one in equation (2), we inspect their time evolution after a mesh interval τ. Then, the difference vectors between the neighboring trajectories and the fiducial or primary one provide a new set $\boldsymbol{\xi}_i^*(\tau)$ that is no longer orthogonal. These vectors have changed their directions as well as their magnitudes, and after a few further integration steps, all of them would collapse into the direction of maximum average separation. Therefore, at every time step we reorthogonalize them by means of the well-known Gram–Schmidt procedure, and we rescale them to their original magnitude ξ. This way, in the next step we arrive at some new orthonormal set $\xi_i(\tau)$ that, by further integration of the dynamical law under consideration, yields $\xi_i^*(2\tau)$, and so on. The d LEs λ_i then may be written as

$$\lambda_i = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \gamma_i(n\tau), \tag{9}$$

where the $\gamma_i(n\tau)$ are the local expansion exponents

$$\gamma_i(n\tau) = \frac{1}{\tau} \ln \left| \frac{\xi_{i\shortparallel}^*(n\tau)}{\xi} \right|, \tag{10}$$

and the index $\shortparallel$ denotes the projection onto the directions $\xi_i(n\tau)$ of the respective orthonormal set. For $i = 1$, that is, the direction of maximum average growth, this projection has no effect, since at every time step the directions of $\xi_{i=1}^*(n\tau)$ and $\xi_{i=1}(n\tau)$ coincide: the orthogonalization gets started this way. The d LEs λ_i are now evaluated according to equations (9) and (10), and the $\xi_i(n\tau)$ always arrange themselves according to the size of their respective LEs λ_i, namely

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d. \tag{11}$$

After all, the small reorthonormalization steps $n\tau$ are not explicitly stated anymore, and the time evolution of local (thus also momentaneous) quantities such as the $\gamma_i(t)$ is considered to be continuous.

Further, in a commonly used notation a Lyapunov spectrum is said to be the set of all the d LEs. At sufficiently large d, these exponents then may be represented as a spectral density. A system's solution or trajectory $\mathbf{x}(t)$ is regarded as chaotic if at least one of the d LEs λ_i is positive. In the most important special case of a single so-called basin of attraction, we may now also quantify the Kolmogorov entropy: it represents the sum of the positive LEs, and frequently this is said to be the "complexity" of a system. Actually it is the sum of the non-negative LEs, since the periodic (or quasi-periodic) case attended with zero complexity is also admitted. Further, the sum of all the d LEs is the dissipation, that is, the (global or average) shrinking rate of the phase volume. A zero dissipation again is a special situation of utmost importance. In this case, a system is meant to be Hamiltonian, but in a biology context that is less common.

3.3 Emergence in Our Category of Models

With regard to the above preliminaries, it is now quite straightforward to recognize (and to quantify) the phenomenon of emergence. However, first we should once again supplement a technical issue. We tacitly have insinuated that to rewrite model systems with desired dynamical properties in a complex-valued manner also answers our most urgent questions concerning emergence. Moreover, we suggested that alternate procedures such as indicated in equation (5) in all probability would prove troublesome. In various respects, the special properties of complex numbers indeed match our needs, and so they do much more than just enlarge a given dynamical system. This may be further illustrated in a "minimum example." Consider the system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 - i\overline{z_1}^2 \end{aligned} \tag{12}$$

that in second-order notation reads $\ddot{z} = -z - i\overline{z}^2$, which does not even have an index anymore, and the bar means complex conjugate. Note that there is just a single nonlinearity, namely the square that actually is the simplest imaginable one. With $z_1 = x_1 + ix_2$ and $z_2 = p_1 + ip_2$, we recover the quite well-known four-dimensional chaotic and conservative Hénon–Heiles (HH) model [22], where the $x_i(t)$ and $p_i(t)$ are the position and momentum coordinates, respectively.

From a physical viewpoint, it appears convenient and instructive to look at the HH system by means of its Hamiltonian function

$$H(x_1, \ldots, x_N, p_1, \ldots, p_N) = \frac{1}{2}\sum_{i=1}^{N} \frac{p_i^2}{m_i} + V(x_1, \ldots, x_N), \tag{13}$$

which is now specified as

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}\left(\frac{p_1^2}{m} + \frac{p_2^2}{m} + x_1^2 + x_2^2\right) + x_1^2 x_2 - \frac{1}{3}x_2^3, \tag{14}$$

and by virtue of the canonical equations, again renders the original HH model (not shown here). It may be helpful to interpret the resulting dynamics as the motion of a point mass in a suitably shaped bowl, the $x_i(t)$ being spatial coordinates in the plane as shown in Figure 7, and the momentaneous height refers to potential energy. Then, the total energy is determined by the initial conditions and serves as a control parameter. As long as we stick to a representation in terms of scalar variables, equation (12) probably constitutes one of the simplest ways to generate chaos, or rather to "write it down." The actual HH model expressed by four real-valued variables looks significantly more complicated, in particular with regard to its three nonlinearities. Clearly, the HH system is Hamiltonian, and hence, it may hardly serve as a representative for dynamical models in biology-related mechanisms. However, complex variables are able to condense remarkable properties into formally simple systems, as we already experienced in the dissipative cases related to equation (1). And the wealth of possibilities that accompany equation (5), we have not even touched yet. Altogether, we are firmly convinced that complex-valued models carry a tremendous potential with respect to nontrivial (and yet unknown) dynamics, particularly within the domain of collective neuronal processes.

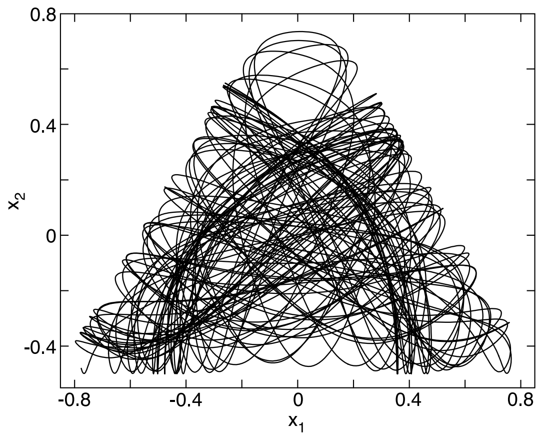

Figure 7. Two-dimensional phase plot (spatial coordinates) of the HH model that is our example of Hamiltonian systems. These are not very suitable for our modeling purposes, since they inherently already comprise the structure achieved by the complexification procedure, and so it is hard to get rid of their special features. This agrees well with the obvious circumstance that biological systems generally are strongly dissipative.

Now we keep our promise to present emergent phenomena, to the extent where they show up in the so-far-presented systems. In the preceding section on complexity indicators, we announced that we would restrict ourselves to cases associated with a (dynamical) complexity increase, and we substantiated that there are also qualitatively different situations. We evaluate the Kolmogorov entropy of equation (3) that simulates our model EEG; see Figures 2 through 4. This is equal to the principal LE $\lambda_{i=1}$, since the system exhibits only one positive LE. In the default units of the Lorenz system (see above), we obtain $K = 1.27$ reciprocal time units, a value that serves now as a benchmark for further numerical K-entropies. Figures 5 and 6 based on equations (7) and (8) yield significantly lower entropies, namely $K = 0.67$ and $K = 0.69$, respectively.

At first glance these numerical findings are not overly surprising, but we point to certain facts that in our view are remarkable. The spiky solutions in Figures 5 and 6 that simulate the neuronal firing properties actually should manifest a stronger chaos, particularly since the "basic frequency" apart from the spikes compares to the one in Figure 2. In most cases (i.e., mathematical models) that can be encountered, this is so. We think the EEG-like situation in Figure 2 has undergone a kind of emergence, relative to the neuron-like case.

We are aware of contributions that provide quite a straightforward manner to tackle emergence, and some of the proposed concepts in fact overlap with our own work. This is expected, since the pertinent ideas certainly cannot be completely disjoint. A recent analysis on a general level is presented in [23], while [24, 25] are much closer to neuronal physiology. The latter approaches introduce, among other issues, new aspects concerning the interesting viewpoint of phase transitions in the brain, and this is also the crucial point in the seminal work of [26]. Clearly, our present approach ignores or at least underestimates all the spatial aspects in the greater context of emergence. However, the spatial neuronal complexity is really immense, and so we are still far from a macroscopically sound description. Hence, to "split off" EEG-related dynamical aspects in our view is justified.

Another issue concerns the general phenomenon of complexity reduction due to collective behavior of many constituents; we have already addressed this point. The dynamics of an EEG by no means can map the actual brain complexity in the sense of physiological mechanisms and mental abilities. However, this circumstance cannot fully compensate for the complexity increase caused by the assembly of many neurons, and we stick to our plausible (but oversimplified) picture of emergence detected in the K-entropy.

As an obvious improvement, we might repeatedly apply our suggested procedure and further complexify, say, the set of equation (2), and so the newly achieved number of phase-space dimensions would be as many as 12 now. This then already reaches the real-life value we presented in [12] based on particularly low-noise intracranial EEGs. This way, in all probability we (realistically) would attain more than one positive LE where, however, the parameter handling would be less trivial. On the other hand, pragmatically speaking, we do not expect too-prominent new dynamical features as we proceed further along this path, since the major novelties are "used up" after a one-time complexification.

In this respect, it may be instructive to consider the HH system of this section again. This system is conservative, and so the preceding complex expansion is already present. Conversely, the reduction to a system such as equation (12) thus is always possible (although it may not everywhere look that simple), which is easiest seen in conjugate variables. Hamiltonian systems do not (or to a much lower extent) carry the potential for emergence in the bearing of dynamical complexity increase by virtue of simple measures, since the expansion under consideration inherently is done already. In simple terms, dissipation is an indispensable prerequisite to emergence in the spirit of this article, namely a type of emergence that likely occurs without circumstances that too rarely are present.

4. Conclusion and Outlook

In sum, we have established a relation between electrophysiological output of the brain and a signal-generating mechanism that models various important dynamical aspects, among them the outward appearance of an electroencephalogram (EEG) amplitude's time dependence. We adopt the point of view that the more properties (and details) match, the more it becomes unlikely that this is all accidental. The agreement between theory and experiment is not yet a proof of the theory under consideration, but in an overall view, it is still one of the strongest and most meaningful tools in science. We substantiated that the Kolmogorov entropy together with phase-space and attractor dimensionality as well as dissipation are the most instructive quantities for our ansatz to tackle the question of emergence. Then, in a neuronal context it is quite obvious that we scrutinized the dynamical difference between single neuron behavior and a large coupled assembly of them. Our pertinent finding there is the complexity increase in the EEG case, although this is not evident in a visual inspection of its dynamical properties. Moreover, the gain in dynamical complexity is only moderate, in agreement with general theory of many-particle systems that exhibit self-organization.

Admittedly, these issues taken by themselves may hardly provide new insights into mental processes in the implication of the mind-brain problem. But in our view, to follow this line of inquiry someday will answer long-standing questions. Apparently it is not really necessary to know and to interpret all the details of neuronal interconnections on a "hardware" level. There is an amazing degree of redundancy in the way these neuronal assemblies operate, which can readily be seen in the case of accident victims who have endured massive brain surgery. In addition, not all of these neuronal links are of equal importance. How much and what is important await further investigations, and here we advance just a few of the pertinent aspects. We think the role of redundancy will prove significant, since this offers a chance to uncover the difference between the essentials and other things that are also present, maybe because they are relics from former times in evolution.

Altogether, we face various reasons to favor the dynamical aspects in the spatio-temporal phenomenon of neuronal emergence, that is, the "sudden" improvement of mental abilities that undoubtedly and frequently happens in the brains of humans and animals. We clearly admit that practical feasibility is one of these reasons. Further, we feel encouraged to explore the consequences of complex variables in various types of differential equation sets in more depth. In equations (2) and (3) we exemplify that some useful properties arise from the special situation that these systems can be formally condensed into complex-valued form, namely into equation (1) in our instance. Maybe there are other higher-structured (scalar) variables that would do a similar job, but in all probability the outstanding properties of complex numbers are important. A practical use of equation (5) might turn into a tremendous effort, but again we see the potential for refined modeling, that is, for a way to tackle desired properties more directly. With regard to equation (5), we might even get closer to pertinent spatio-temporal approaches, but this is clearly beyond our present possibilities. In this paper we just try to open the gateway to a largely new and in our view promising type of neuronal dynamics modeling.

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