The Time Evolution of a Greenberg–Hastings Cellular Automaton on a Finite Graph

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The Greenberg–Hastings cellular automaton (GHCA) is a probabilistic two-dimensional cellular automaton with a Moore or von Neumann neighborhood to mimic pattern formations of excitable media. It is also defined on a graph, where a vertex corresponds to a cell and its adjacent vertices to the neighborhood of the cell. In this paper, we study a three-valued GHCA on an arbitrary finite connected graph analytically, though it has been mainly investigated numerically. We prove that "maximum cycle density" completely decides asymptotic behavior of its time evolution.

Keywords: Greenberg–Hastings cellular automaton; graph cellular automaton

1. Introduction

A cellular automaton (CA) is a discrete dynamical system in both space and time. Each cell in a CA takes only a finite number of states. The state of a cell changes according to a transition rule, which is defined only by the state of each cell and those of its neighborhoods; that is, the state of each cell at time $t+1$ is decided by its state and those of neighborhoods at time t. A graph cellular automaton (GCA) whose space is expanded to an arbitrary graph has been studied [1, 2]. In a GCA, cells correspond to vertices of the graph. The neighborhood type of a typical two-dimensional CA is either a Moore neighborhood (eight nearest neighbors) or a von Neumann neighborhood (four nearest neighbors). In a GCA, the neighborhood of a vertex is defined as all of its adjacent vertices.

The Greenberg–Hastings cellular automaton (GHCA) is a two-dimensional CA to mimic pattern formations of excitable media [3, 4]. In a GHCA, each cell can be in one of three states: quiescent, excited or refractory. The transition rule is described as follows.

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- An excited cell at time t becomes refractory at time $t+1$.
- A refractory cell at time t becomes quiescent at time $t+1$.
- A quiescent cell at time t becomes excited at time $t+1$ with probability $p(0 \leqq p \leqq 1)$ if at least one of its neighborhoods is excited at time t; otherwise, the cell remains quiescent at time $t+1$.

The space on which a GHCA is defined can be extended to an arbitrary graph from a two-dimensional lattice, owing to the characteristics of the transition rule. Such a GHCA on an arbitrary graph (graph GHCA) has been studied mainly by numerical simulations [5–11]. In this paper, we analytically investigate the time evolution of deterministic graph GHCA ($p = 1$) on a connected, finite, simple graph. Sometimes we say that a cell *gets excited* when it transitions from a quiescent state to an excited state.

In Section 2, a graph GHCA on a connected, finite, simple graph is defined. In Section 3, we prove that the graph GHCA converges to either an equilibrium state where all the cells are quiescent or a periodic state where all the cells get excited with a fixed rhythm. In Section 4, two quantities, *phase difference of a walk* and *change amount of a vertex*, are introduced to perform quantitative analysis of the graph GHCA. In Section 5, we define the notions n^{th} *excitation time of a vertex* and n^{th} *excitation walk of a vertex* and show the necessary and sufficient condition for a vertex to get excited. In Section 6, we prove that if a phase difference of each circuit in a graph GHCA is equal to 0, then the graph GHCA converges to an equilibrium state where all vertices are quiescent, while if there is at least one circuit whose phase difference is not equal to 0, it converges to a periodic state. In Section 7, we define the notion *convergent excitation rate of a vertex* as the ratio N / T for a vertex in a periodic state where it gets excited N times during T time steps. We also introduce the notion *maximum cycle density* d_M and prove that the convergent excitation rate of each vertex is equivalent to d_M.

2. Graph Greenberg–Hastings Cellular Automaton

Let $G = (V, E)$ be a connected, finite, simple, undirected graph. Here V denotes a set of vertices of G, and E denotes a set of edges of G. A vertex will take a value in $\{0, 1, 2\}$, each of which corresponds respectively to a quiescent state, an excited state and a refractory state. We let $v(t)$ be the value of a vertex v at time t. The time evolution rule for vertices is defined as follows.

- In case of $v(t) = 1$, we always have $v(t+1) = 2$.
- In case of $v(t) = 2$, we always have $v(t+1) = 0$.
- In case of $v(t) = 0$, if at least one adjacent vertex is excited, we have $v(t+1) = 1$. Otherwise, we have $v(t+1) = 0$.

Definition 1. We call the cellular automaton defined here a graph Greenberg–Hastings cellular automaton (graph GHCA).

A finite sequence of vertices $w = v_0 v_1 \ldots v_n$ is called a *walk* when there is an edge that connects v_i with v_{i+1} $(0 \leqq \forall\, i < n)$. We denote by $-w$ a walk whose vertices and edges are the same as w but whose direction is opposite to w, that is, $-w = v_n \ldots v_1 v_0$. A walk $w = v_0 v_1 \ldots v_n$ is called a *path* when $v_i \neq v_j$ $(\forall\, i, j : 0 \leqq i < j \leqq n)$ and a *circuit* when $v_0 = v_n$. A circuit $w = v_0 v_1 \ldots v_n$ is called a *cycle* if the walk $v_0 v_1 \ldots v_{n-1}$ in w is a path. We denote by $|w|$ the length of w. For $w = v_0 v_1 \ldots v_n$, $|w| = n$.

3. The Time Evolution of a Graph Greenberg–Hastings Cellular Automaton

We prove from the finiteness of G that a graph GHCA converges to either an equilibrium state where all vertices take 0 or a periodic state.

Lemma 1. If there exists one vertex that converges to a state with value 0, then all vertices converge to states with value 0.

Proof. We suppose that there is a vertex v_0 that satisfies $v_0(t) = 0$ $(t \geqq t_0)$. When v_1 is an adjacent vertex of v_0, $v_1(t) \neq 1$ $(t \geqq t_0)$ from the transition rule of a graph GHCA. If $v_1(t_0) = 0$, $v_1(t) = 0$ $(t \geqq t_0)$, while $v_1(t_0) = 2$, $v_1(t) = 0$ $(t \geqq t_0 + 1)$. Therefore, $v_1(t) = 0$ $(t \geqq t_0 + 1)$ in any case. Hence, for a walk $v_0 v_1 \ldots v_n$, we find that $v_n(t) = 0$ $(t \geqq t_0 + n)$ by repeating the same arguments. We let v be an arbitrary vertex different from v_0. Then there is a walk w satisfying that the initial vertex is v_0 and the terminal vertex is v from the connectivity of G. Therefore, $v_n(t) = 0$ $(t \geqq t_0 + |w|)$, which completes the proof. □

Proposition 1. Each vertex eventually takes on a periodic sequence of states.

Proof. The time evolution rule of a graph GHCA is deterministic, and its orbit is confined to a finite number of states. Therefore, an arbitrary vertex will tend to be either a state with constant value 0 or a periodic state. When there is one vertex that converges to a state with value 0, all vertices converge to the same quiescent states from Lemma 1. Otherwise, all vertices converge to a periodic state. □

4. Phase Difference of a Walk and Change Amount of a Vertex

We introduce two quantities, *phase difference of a walk* and *change amount of a vertex*, in order to perform quantitative analysis of a graph GHCA and investigate the relationship between these two quantities.

Definition 2. For a walk $w = v_0 v_1 \ldots v_n$, we define a value $w(t) \in \mathbb{Z}$ as

$$w(t) := \sum_{i=0}^{n-1} \text{mod}_3(v_{i+1}(t) - v_i(t)). \tag{1}$$

Here, $\text{mod}_3(\cdot) : \mathbb{Z}_3 \to \mathbb{Z}$ is a mapping from an element of $\mathbb{Z}_3$ to a corresponding representative element in $\{-1, 0, 1\}$; that is, if $k \in \mathbb{Z}$ is a representative of an element in $\mathbb{Z}_3$ and $\text{mod}_3(k) = \hat{k} \in \{-1, 0, 1\}$, then $k \equiv \hat{k} \pmod 3$. This means that $\text{mod}_3(-2) = 1$, $\text{mod}_3(1) = 1$, $\text{mod}_3(2) = -1$ and so on. We call $w(t)$ the *phase difference* of w at time t.

Note that $w(t)$ is expressed as a sum of the phase difference of walks with two vertices:

$$w(t) = \sum_{i=0}^{n-1} v_i v_{i+1}(t). \tag{2}$$

Lemma 2. For a walk $w = v_0 v_1 \cdots v_n$,

$$v_n(t) \equiv v_0(t) + w(t) \pmod 3. \tag{3}$$

Proof.

$$\begin{aligned} v_0(t) + w(t) &= v_0(t) + \sum_{i=0}^{n-1} \text{mod}_3(v_{i+1}(t) - v_i(t)) \\ &\equiv v_0(t) + \sum_{i=0}^{n-1} \{v_{i+1}(t) - v_i(t)\} \pmod 3 \\ &\equiv v_n(t) \pmod 3. \ \square \end{aligned}$$

Definition 3. For a vertex v, we define a value $v(t, t+t') \in \mathbb{Z}$ as

$$v(t, t+t') := \sum_{i=t}^{t+t'-1} \mathrm{mod}_3\big(v(i+1) - v(i)\big). \tag{4}$$

We call $v(t, t+t')$ the change amount of v from time t to time $t+t'$.

Note that

$$v(t, t+1) = \mathrm{mod}_3\big(v(t+1) - v(t)\big), \tag{5}$$

and

$$v(t, t+t') := \sum_{i=t}^{t+t'-1} v(i, i+1). \tag{6}$$

Lemma 3. For a walk $w = v_0 v_1 \ldots v_n$,

$$w(t+t') - w(t) = v_n(t, t+t') - v_0(t, t+t'). \tag{7}$$

Proof. (a) For a walk with two vertices $w = v_0 v_1$,

$$\begin{aligned} w(t+1) - w(t) &= \mathrm{mod}_3\big(v_1(t+1) - v_0(t+1)\big) - \mathrm{mod}_3(v_1(t) - v_0(t)) \\ &\equiv \{v_1(t+1) - v_1(t)\} - \{v_0(t+1) - v_0(t)\} \pmod 3. \end{aligned}$$

From equation (5), we have

$$w(t+1) - w(t) \equiv v_1(t, t+1) - v_0(t, t+1) \pmod 3. \tag{8}$$

From the definition of $\mathrm{mod}_3(\cdot)$, $-2 \leqq w(t+1) - w(t) \leqq 2$.

Suppose that $w(t+1) - w(t) = -2$. Then it must hold that

$$\mathrm{mod}_3\big(v_1(t+1) - v_0(t+1)\big) = -1, \; \mathrm{mod}_3(v_1(t) - v_0(t)) = 1.$$

The possible values of the pair $(v_1(t), v_0(t))$ are $(1, 0)$, $(2, 1)$ and $(0, 2)$. For $(v_1(t), v_0(t)) = (1, 0)$ and $(2, 1)$, the time evolution rule gives that $(v_1(t+1), v_0(t+1)) = (2, 1)$ and $(0, 2)$, respectively. Hence

$$\mathrm{mod}_3\big(v_1(t+1) - v_0(t+1)\big) = 1.$$

For $(v_1(t), v_0(t)) = (0, 2)$, $(v_1(t+1), v_0(t+1)) = (0, 0)$ or $(1, 0)$, and

$$\mathrm{mod}_3\big(v_1(t+1) - v_0(t+1)\big) = 1 \text{ or } 0.$$

Thus, we have

$$\mathrm{mod}_3\big(v_1(t+1) - v_0(t+1)\big) \neq -1,$$

and we conclude that $w(t+1) - w(t) \neq -2$.

Using similar arguments, we conclude that $w(t+1) - w(t) \neq 2$. Therefore,

$$-1 \leqq w(t+1) - w(t) \leqq 1. \tag{9}$$

On the other hand, it holds that

$$-1 \leqq v_1(t, t+1) - v_0(t, t+1) \leqq 1, \tag{10}$$

because $v(t, t+1) = 0$ or 1 for an arbitrary vertex v. From equations (8) through (10), we obtain

$$w(t+1) - w(t) = v_1(t, t+1) - v_0(t, t+1). \tag{11}$$

The preceding results can be easily extended to $w(t+t') - w(t)$ with an arbitrary positive integer t'. In fact, since

$$w(t+t') - w(t) = \sum_{i=t}^{t+t'-1} \{w(i+1) - w(i)\},$$

we have from equation (11)

$$w(t+t') - w(t) = \sum_{i=t}^{t+t'-1} \{v_1(i, i+1) - v_0(i, i+1)\}.$$

Thus, from equation (6),

$$w(t+t') - w(t) = v_1(t, t+t') - v_0(t, t+t'). \tag{12}$$

(b) For a walk with $n+1$ ($n \geqq 2$) vertices, $w = v_0 v_1 \ldots v_n$; using equation (2), we have

$$w(t+t') - w(t) = \sum_{i=0}^{n-1} \{v_{i+1} v_i(t+t') - v_{i+1} v_i(t)\}.$$

From equation (12),

$$w(t+t') - w(t) = \sum_{i=0}^{n-1} \{v_{i+1}(t, t+t') - v_i(t, t+t')\} = \\ v_n(t, t+t') - v_0(t, t+t'),$$

which completes the proof. □

Lemma 4. For a circuit L, $L(t)$ is invariant in time; that is,

$$^{\forall} t' L(t') = L(t). \tag{13}$$

Proof. Without loss of generality, we can assume that $t' > t$. For a circuit $L = v_0 v_1 \ldots v_0$, from Lemma 3,

$$L(t') - L(t) = v_0(t, t') - v_0(t, t') = 0. \quad \square$$

5. Excitation Time of a Vertex and Excitation Walk of a Vertex

Definition 4. If a vertex v gets excited at time step t_1 for the first time, we define $T(v, 1) := t_1$. Similarly, the time step at which the vertex v gets excited for the n^{th} time is denoted by $T(v, n)$ and is called the n^{th} *excitation time* of v.

Proposition 2. When a vertex v gets excited n times, there is a walk $w = v v_{T(v,n)-1} \ldots v_0$ satisfying the following condition (Figure 1):

$$v_i(i) = 1 \left(0 \leqq i \leqq T(v, n) - 1\right). \tag{14}$$

This walk w is called the n^{th} *excitation walk* of v.

Proof. Since $v(T(v, n)) = 1$, it holds that $v(T(v, n) - 1) = 0$. Therefore, there is an adjacent vertex of v, v^*, satisfying that $v^*(T(v, n) - 1) = 1$ due to the transition rule of a graph GHCA. We let $v_{T(v,n)-1} = v^*$. The existence of the walk w satisfying the preceding condition can be proved by repeating the same arguments. □

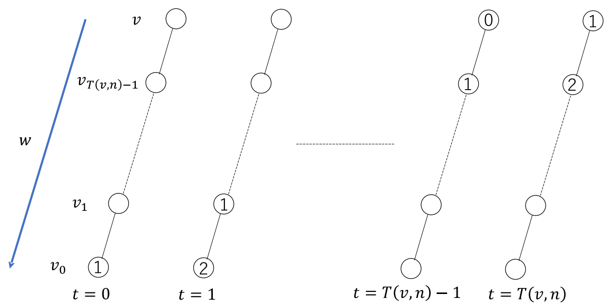

Figure 1. n^{th} excitation walk of a vertex v.

Note that if w is an n^{th} excitation walk of a vertex v,

$$|w| = T(v, n) \tag{15}$$

by definition.

Lemma 5. If a vertex v gets excited n times, then

$$v\left(0, T(v, n)\right) = 3n + 1 - \text{mod}_3^*\left(v(0)\right). \tag{16}$$

Here,

$$\text{mod}_3^*(\cdot) : \mathbb{Z}_3 \to \mathbb{Z}$$

is a mapping from an element of $\mathbb{Z}_3$ to the corresponding value in $\{1, 2, 3\}$; that is, if $k \in \mathbb{Z}$ is a representative of an element in $\mathbb{Z}_3$ and

$\mathrm{mod}_3^*(k) = \hat{k} \in \{1, 2, 3\}$, then $k \equiv \hat{k} \pmod 3$. This means that $\mathrm{mod}_3^*(-2) = 1$, $\mathrm{mod}_3^*(0) = 3$, $\mathrm{mod}_3^*(2) = 2$ and so on.

Proof. (a) In case of $v(0) = 0$.

By definition, $v(T(v, n)) = 1$ and $v(T(v, n) - 1) = 0$. Furthermore, $v(t)$ takes $n - 1$ times the root: $0 \to 1 \to 2 \to 0$. Because of the time evolution rule, $\mathrm{mod}_3(v(i+1) - v(i)) \in \{0, 1\}$ and $\mathrm{mod}_3(v(i+1) - v(i)) = 0$ if and only if $v(i) = v(i+1) = 0$. Hence we have

$$\begin{aligned} v(0, T(v, n)) &:= 3 \sum_{i=0}^{T(v,n)} \mathrm{mod}_3(v(i+1) - v(i)) \\ &= 3(n-1) + 1 = 3n - 2 = 3n + 1 - \mathrm{mod}_3^*(v(0)). \end{aligned}$$

(b) In case of $v(0) = 1$.

$v(1) = 2$, $v(2) = 0$. Hence,

$$\mathrm{mod}_3(v(1) - v(0)) = \mathrm{mod}_3(v(2) - v(1)) = 1.$$

In the time interval $3 \leqq t \leqq T(v, n-1)$, $v(t)$ takes $n - 1$ times the root: $0 \to 1 \to 2 \to 0$. By definition, $\mathrm{mod}_3(v(T(v, n)) - v(T(v, n) - 1)) = 1$. Therefore,

$$v(0, T(v, n)) = 2 + 3(n-1) + 1 = 3n = 3n + 1 - \mathrm{mod}_3^*(v(0)).$$

(c) In case of $v(0) = 2$.

$v(1) = 0$. Hence,

$$\mathrm{mod}_3(v(1) - v(0)) = 1.$$

In the time interval $2 \leqq t \leqq T(v, n-1)$, $v(t)$ takes $n - 1$ times the root: $0 \to 1 \to 2 \to 0$. By definition, $\mathrm{mod}_3((T(v, n)) - v(T(v, n) - 1)) = 1$. Therefore,

$$v(0, T(v, n)) = 1 + 3(n-1) + 1 = 3n - 1 = 3n + 1 - \mathrm{mod}_3^*(v(0)).$$

Thus we have proved Lemma 5. □

Lemma 6. When w is an n^{th} excitation walk of a vertex v,

$$w(0) = v(0, T(v, n))(= 3n + 1 - \mathrm{mod}_3^*(v(0))). \tag{17}$$

Proof. Let an n^{th} excitation walk of a vertex v be $w = v v_{T(v,n)-1} \ldots v_0$. We also put $v_{T(v,n)} := v$. Note that the subscripts of the vertices in w are arranged in reverse order compared to that of a walk in Definition 2.

By definition of an n^{th} excitation walk, each vertex v_i satisfies equation (14); that is, $v_i(i) = 1$. From Lemma 3, we have

$$w(0) = w(1) - \{v_0(0, 1) - v(0, 1)\}.$$

Since

$$v_0(0, 1) = \text{mod}_3(v_0(1) - v_0(0)) = \text{mod}_3(2 - 1) = 1,$$

and

$$\begin{aligned} w(1) &= \sum_{i=0}^{T(v,n)-1} \text{mod}_3(v_i(1) - v_{i+1}(1)) \\ &= \sum_{i=1}^{T(v,n)-1} \text{mod}_3(v_i(1) - v_{i+1}(1)) + \text{mod}_3(v_0(1) - v_1(1)) \\ &= v_{T(v,n)} v_{T(v,n)-1} \ldots v_1(1) + \text{mod}_3(2 - 1) \\ &= v_{T(v,n)} v_{T(v,n)-1} \cdots v_1(1) + 1, \end{aligned}$$

we have

$$w(0) = v v_{T(v,n)-1} \ldots v_1(1) + v(0, 1).$$

Similarly, we have

$$v v_{T(v,n)-1} \ldots v_1(1) = v v_{T(v,n)-1} \ldots v_2(2) + v(1, 2).$$

Hence, using equation (6), we have

$$\begin{aligned} &w(0) = v v_{T(v,n)-1} \ldots v_2(2) + v(1, 2) + \\ &\quad v(0, 1) = v v_{T(v,n)-1} \ldots v_2(2) + v(0, 2). \end{aligned}$$

Repeating similar arguments, finally we have

$$\begin{aligned} &w(0) = v v_{T(v,n)-1}(T(v, n) - 1) + v(0, T(v, n) - 1) = \\ &\quad \text{mod}_3(v_{T(v,n)-1}(T(v, n) - 1) - v(T(v, n) - 1)) + v(0, T(v, n) - 1) = \\ &\quad \text{mod}_3(1 - v(T(v, n) - 1)) + v(0, T(v, n) - 1) = \\ &\quad \text{mod}_3(v(T(v, n)) - v(T(v, n) - 1)) + v(0, T(v, n) - 1) = \\ &\quad v(T(v, n) - 1, T(v, n)) + v(0, T(v, n) - 1) = v(0, T(v, n)). \;\square \end{aligned}$$

Proposition 3. When a vertex v gets excited n times, there is a walk w whose initial vertex is v and that satisfies equation (18). Similarly, when there is a walk w whose initial vertex is v and that satisfies equation (18), v gets excited n times:

$$w(0) = 3n + 1 - \text{mod}_3^*(v(0)). \tag{18}$$

Proof. (I) From Proposition 2, if a vertex v gets excited n times, there exists a walk w that is an n^{th} excitation walk of a vertex v. Then, from Lemmas 5 and 6,

$$w(0) = v(0, T(v, n)) = 3n + 1 - \text{mod}_3^*(v(0)).$$

Hence, the former statement is proved.

(II) Let a walk $w^* = v_N v_{N-1} \ldots v_0$ $(v = v_N)$ be the shortest walk that satisfies equation (18). It is sufficient to prove the latter statement to show that $N = T(v, n)$. For this purpose, we first show that

$$v_i(i) = 1(0 \leqq i \leqq N). \tag{19}$$

If equation (19) holds, from Lemma 6,

$$v(0, N) = w^*(0) = 3n + 1 - \text{mod}_3^*(v(0)).$$

Then, from Lemma 5, we have $N = T(v, n)$. Hence, it is enough to prove equation (19). We prove it by induction.

(1) From Lemma 2,

$$\begin{aligned} v_0(0) &\equiv v_N(0) + w^*(0) && (\text{mod } 3) \\ &\equiv v(0) + 3n + 1 - \text{mod}_3^*(v(0)) \\ &\equiv 1 && (\text{mod } 3). \end{aligned}$$

Since $0 \leqq v_0(0) \leqq 1$, $v_0(0) = 1$.

(2) We shall show $v_1(1) = 1$ by proof of contradiction.

(a) Suppose that $v_1(1) = 2$. Since $v_0(0) = 1$ and $v_1(0) = 1$,

$$w^*(0) = v_N v_{N-1} \ldots v_1 v_0(0) = v_N v_{N-1} \cdots v_1(0) + \text{mod}_3(v_0(0) - v_1(0)) = v_N v_{N-1} \ldots v_1(0).$$

Hence, the walk $v_N v_{N-1} \ldots v_1$ satisfies equation (18) and its length is $N - 1 < |w^*|$. This contradicts the definition of w^* (Figure 2).

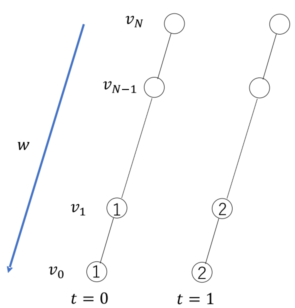

Figure 2. In case of $v_1(1) = 2$.

(b) Suppose that $v_1(1) = 0$. By the time evolution rule of a graph GHCA, $v_1(0) = 0$ or $v_1(0) = 2$. If $v_1(0) = 0$, then $v_1(1) = 1$, because the adjacent vertex v_0 takes $v_0(0) = 1$. Hence, $v_1(0) = 2$ and $v_1 v_0(0) = \mathrm{mod}_3(v_0(0) - v_1(0)) = -1$. We have

$$w^*(0) = v_N v_{N-1} \ldots v_1 v_0(0) = v_N v_{N-1} \ldots v_1(0) + v_1 v_0(0) = v_N v_{N-1} \ldots v_1(0) - 1.$$

Therefore

$$v_N v_{N-1)} \cdots v_1(0) > w^*(0).$$

Then there exists an integer k $(2 \leqq k \leqq N-1)$ satisfying $v_N \ldots v_k(0) = w^*(0)$, because

$$v_i v_{i+1}(0) = \mathrm{mod}_3(v_{i+1}(0) - v_i(0)) \in \{-1, 0, 1\}.$$

Since

$$|v_N \ldots v_k(0)| = N - k < |w^*(0)|,$$

w^* is not the shortest walk satisfying equation (18), which is a contradiction (Figure 3).

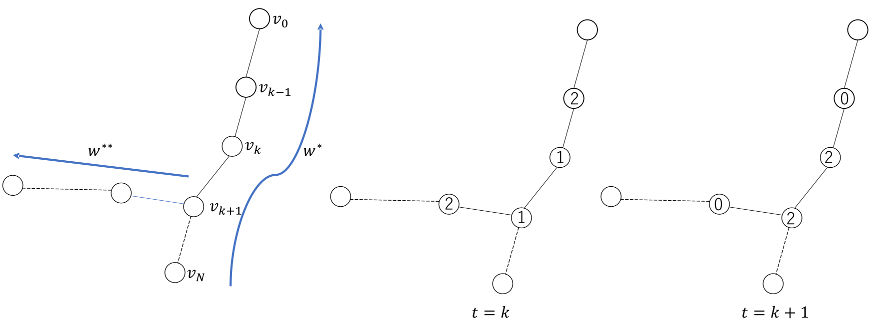

Figure 3. In case of $v_1(1) = 0$.

Thus we find that $v_1(1) = 1$.

(3) For a positive integer k, let us assume that $v_i(i) = 1$ $(0 \leqq i \leqq k)$. We show $v_{k+1}(k+1) = 1$ by proof of contradiction.

(a) Suppose that $v_{k+1}(k+1) = 2$. Then, there exists a positive integer m satisfying that $k = T(v_{k+1}, m)$, because $v_{k+1}(k) = 1$. We let w^{**} be an m^{th} excitation walk of v_{k+1}. From Lemmas 3 and 6,

$$v_k v_{k+1}(k) - v_k v_{k+1}(0) = v_{k+1}(0, k) - v_k(0, k) = w^{**}(0) - v_k \ldots v_0(0).$$

Since

$$v_k v_{k+1}(k) = \mathrm{mod}_3(v_{k+1}(k) - v_k(k)) = 0$$

and

$$v_k v_{k+1}(0) = -v_{k+1} v_k(0),$$

we have

$$w^{**}(0) = v_{k+1} v_k(0) + v_k \ldots v_0(0).$$

Therefore we have

$$w(0) = v_N \ldots v_{k+1}(0) + v_{k+1} v_k (0) + v_k \ldots v_0(0) = v_N \ldots v_{k+1}(0) + w^{**}(0).$$

Let us consider the walk $v_N \ldots v_{k+1} \vee w^{**}$, which is constructed by combining $v_N \ldots v_{k+1}$ with w^{**}. The walk $v_N \ldots v_{k+1} \vee w^{**}$ satisfies equation (18) and the length is $N - 1$ since, from Proposition 2, $|w^{**}| = T(v_{k+1}, m) = k$. This contradicts the definition of w^* (Figure 4).

(b) Suppose that $v_{k+1}(k+1) = 0$. According to the time evolution rule of a graph GHCA, $v_{k+1}(k) = 0$ or $v_{k+1}(k) = 2$. If $v_{k+1}(k) = 0$, then $v_{k+1}(k+1) = 1$, because $v_k(k) = 1$ and v_k is an adjacent vertex of v_{k+1}. Hence $v_{k+1}(k) = 2$, which implies $v_{k+1}(k-1) = 1$. Then there exists a positive integer m satisfying that $k - 1 = T(v_{k+1}, m)$. The argument following is almost the same as that in (a). We let w^{**} be an m^{th} excitation walk of v_{k+1}. From Lemmas 3 and 6,

$$v_{k-1} v_k v_{k+1}(k-1) - v_{k-1} v_k v_{k+1}(0) = v_{k+1} (0, k-1) - v_{k-1}(0, k-1) = w^{**}(0) - v_{k-1} \ldots v_0(0).$$

Noticing the fact that

$$v_{k-1} v_k v_{k+1}(k-1) = \mathrm{mod}_3(v_{k+1}(k-1) - v_k(k-1)) + \mathrm{mod}_3 (v_k(k-1) - v_{k-1}(k-1)) = \mathrm{mod}_3(1 - 0) + \mathrm{mod}_3(0 - 1) = 0$$

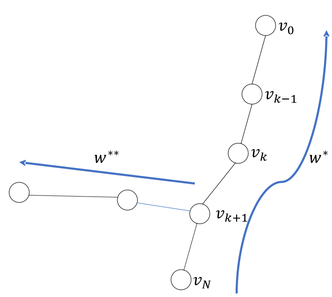

Figure 4. In case of $v_{k+1}(1) = 2$.

and

$$v_{k-1}v_kv_{k+1}(0) = -v_{k+1}v_kv_{k-1}(0),$$

we have

$$w^{**}(0) = v_{k+1}v_kv_{k-1}(0) + v_{k-1}\ldots v_0(0).$$

Thus

$$w(0) = v_N\ldots v_{k+1}(0) + w^{**}(0) = v_N \ldots v_{k+1}(0) + v_{k+1}v_kv_{k-1}(0) + v_{k-1}\ldots v_0(0).$$

By connecting w^{**} to $v_N\ldots v_{k+1}$, we have the walk $v_N\ldots v_{k+1} \vee w^{**}$, which satisfies equation (18). From Proposition 2,

$$|w^{**}| = T(v_{k+1}, m) = k-1$$

and

$$|v_N\ldots v_{k+1} \vee w^{**}| = N-2.$$

This contradicts the definition of w^* (Figure 5).

From (a) and (b), we conclude that $v_{k+1}(k+1) = 1$.

From (1), (2), (3), we proved that $v_i(i) = 1(0 \leqq i \leqq N)$, which completes the proof. □

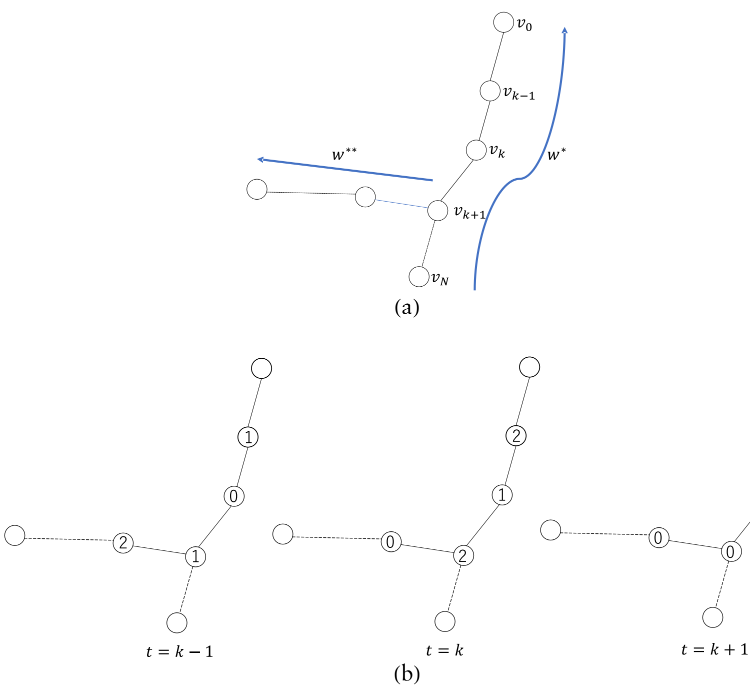

Figure 5. In case of $v_{k+1}(1) = 0$.

From the proof of Proposition 3, we see that Proposition 4 holds.

Proposition 4. The statement that w is an n^{th} excitation walk of a vertex v is equivalent to saying that w is the shortest walk whose initial vertex is v and that satisfies equation (18). It also holds that

$$T(v, n) = \min \left\{|w| \mid w \text{ is a walk with initial vertex } v \text{ and } w(0) = 3\,n + 1 - \text{mod}_3^*\,(v\,(0))\right\}. \tag{20}$$

6. The Convergence of a Graph Greenberg–Hastings Cellular Automaton

In this section, the asymptotic state of a graph GHCA, which is either an equilibrium state where all the vertices take 0 or a periodic state, is determined by the existence or nonexistence of a circuit whose phase difference is not equal to 0.

Theorem 1. (1) If a phase difference of each circuit in a graph GHCA is equal to 0, a value of each vertex asymptotically approaches 0.

(2) If there is at least one circuit whose phase difference is not equal to 0, all vertices converge to a periodic excited state.

Proof of (1). Let us consider a walk w in the graph GHCA with initial vertex v. When w is not a circuit, w is decomposed into one path P and finite cycles $C_1, \ldots, C_m$. Since a phase difference of each cycle is supposed to be equal to 0,

$$w(0) = P(0) + C_1(0) + \cdots + C_m(0) = P(0) \leqq |P| < +\infty.$$

If w is a circuit, $w(0) = 0$ from the assumption of Theorem 2 (1). Hence, in both cases, $w(0)$ has an upper limit. Then, from Proposition 3, $v(t)$ can get excited only finite times, which implies that $v(t)$ converges to 0. □

Proof of (2). Let v be an arbitrary vertex, C be a circuit whose phase difference is not equal to 0 $(C(0) > 0)$, and v^* be a vertex in C. From the connectivity of G, there is such a walk w that the initial vertex is v and the terminal vertex is v^*. We consider the walk that is constituted of w and n cycles of C, and denote it by $w \vee nC$ (Figure 6). Then,

$$\lim_{n\to\infty} \left|w \vee nC(0)\right| = \lim_{n\to\infty} \left|\{w(0) + nC(0)\}\right| = \infty.$$

Therefore, $w \vee nC(0)$ has no upper limit. Then, from Proposition 3, there is no finite time step t_f that satisfies $v(t) = 0$ $({}^\forall t,\ t \geq t_f)$. Thus, from Proposition 1, all vertices converge to a periodic state. □

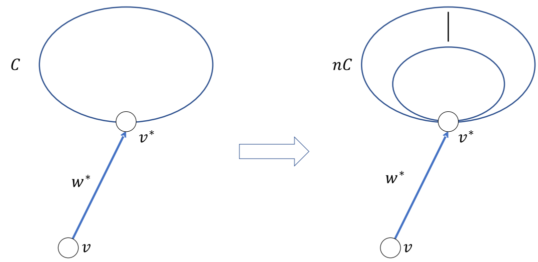

Figure 6. $w \vee nC$.

7. Convergent Excitation Rate of Vertex and Maximum Cycle Density d_M

In the previous section, we found that when there is at least one circuit whose phase difference is not equal to 0, all vertices converge to a periodic state (Theorem 2). In this section, the notion *convergent excitation rate of a vertex* is defined as N/T when the vertex converges to a periodic state in which it gets excited N times during T steps. We also introduce the notion *maximum cycle density* d_M and prove that the convergent excitation rate of each vertex is equivalent to d_M.

Definition 5. When a vertex v converges to a periodic state where it gets excited N times during T time steps, we define

$$e(v) := \frac{N}{T} \tag{21}$$

and call it the convergent excitation rate of v. When a value of a vertex v converges to 0, we put $e(v) = 0$ for convenience.

Lemma 7. When a vertex v converges to a periodic state,

$$\lim_{n \to \infty} \frac{n}{T(v, n)} = e(v). \tag{22}$$

Proof. Let us suppose that after time t_0, a vertex v enters a periodic state where it gets excited N times during T time steps. For a sufficiently large number n_0, we can assume that $t_0 \leqq T(v, n_0)$. An arbitrary integer n ($n_0 \leqq n$) can be expressed as

$$n = n_0 + aN + b, \left(0 \leqq a,\ 0 \leqq b < N\right).$$

Clearly,

$$T(v, n_0 + aN) \leqq T(v, n) \leqq T\left(v, n_0 + \left(a + 1\right)N\right).$$

Since

$$T(v, n_0 + aN) = T(v, n_0) + aT$$

and

$$T\big(v, n_0 + (a+1)N = T(v, n_0) + (a+1)T,$$

we have

$$T(v, n_0) + aT \leqq T(v, n) \leqq T(v, n_0) + (a+1)T.$$

Noticing that $n_0 + aN \leqq n \leqq n_0 + (a+1)N$,

$$\frac{n_0 + aN}{T(v, n_0) + (a+1)T} \leqq \frac{n}{T(v, n)} \leqq \frac{n_0 + (a+1)N}{T(v, n_0) + aT}.$$

Therefore

$$\frac{N}{T} \leqq \lim_{n \to \infty} \frac{n}{T(v, n)} \leqq \frac{N}{T}.$$

Because $e(v) = N / T$, the lemma is proved. □

Definition 6. For a walk w in a graph GHCA G, we define

$$d(w) := \frac{w(0)}{3|w|} \tag{23}$$

and call it the density of w. We also define

$$d_M := \max\{d(w) \mid w \subseteq G \text{ is a cycle}\} \tag{24}$$

and call it the maximum cycle density. When G has no cycle, we define $d_M = 0$ for convenience.

Lemma 8. When a vertex v converges to a periodic state and w_n is an n^{th} excitation walk of a vertex v,

$$\lim_{n \to \infty} d(w_n) = e(v). \tag{25}$$

Proof. From Lemmas 6 and 7,

$$d(w_n) := \frac{w_n(0)}{3|w_n|} = \frac{3n + 1 - \text{mod}_3^*(v(0))}{3T(v, n)} \longrightarrow e(v)(n \to +\infty). \ \square$$

Lemma 9. If the phase difference of a circuit C at t is positive, that is, $C(t) > 0$, then there exists a vertex $v \in C$ that satisfies $v(t) = 1$.

Proof. Let $w = v_0 v_1 \ldots v_{n-1} v_0$. Suppose that $v_i(t)$ is either 0 or 2 $(0 \leqq {}^\forall i \leqq n-1)$. We consider the case of $G = C$. Clearly, $v_i(t+1) = 0$ if $v_i(t) = 2$. And $v_i(t+1) = 0$ if $v_i(t) = 0$, because there is no vertex $v \in G$ that satisfies $v(t) = 1$. That is, $v_i(t+1) = 0$ $(0 \leqq {}^\forall i \leqq n-1)$, which implies $C(t+1) = 0$. From Lemma 4, $C(t) = 0$. $C(t)$ in case of an arbitrary G including C is equivalent to one in case of $G = C$ from Definition 2. This contradicts $C(t) > 0$. Hence, there exists a vertex $v \in w$ that satisfies $v(t) = 1$. □

Theorem 2. For an arbitrary vertex v,

$$e(v) = d_M. \tag{26}$$

Proof. (1) If $d_M = 0$, a phase difference of each cycle is equal to 0, and a phase difference of each circuit is equal to 0. Then, from Theorem 2 (1), we have $e(v) = 0$.

(2) If $d_M \neq 0$, from Theorem 2 (2), all vertices converge into a periodic state.

(a) We show that $d_M \leqq e(v)$.

Let C be a cycle whose density is d_M and hence $C(0) > 0)$. Then, from Lemma 9, there is a vertex v^* in C that satisfies $v^*(0) = 1$. Due to the connectivity of G, there exists a walk w^* whose initial vertex is v and whose terminal vertex is v^*. We connect m cycles of C to w^* and denote the resulting walk by $w^* \vee mC$. From Lemma 2,

$$C(0) \equiv v^*(0) - v^*(0) \equiv 0 \pmod 3$$

and

$$w^*(0) \equiv v^*(0) - v(0) \equiv 1 - v(0) \pmod 3.$$

Putting

$$C(0) = 3\alpha,\ w^*(0) = 3\beta + 1 - \text{mod}_3^*(v(0)) \quad (\alpha, \beta \in \mathbb{Z}),$$

we have

$$w^* \vee mC(0) = w^*(0) + mC(0) = 3(m\alpha + \beta) + 1 - \text{mod}_3^*(v(0)).$$

Let $w_{m\alpha+\beta}$ be an $m\alpha + \beta^{\text{th}}$ excitation walk of the vertex v. From equation (15) and Lemma 6,

$$d(w_{m\alpha+\beta}) := \frac{w_{m\alpha+\beta}(0)}{3|w_{m\alpha+\beta}|} = \frac{3(m\alpha + \beta) + 1 - \text{mod}_3^*(v(0))}{3T(v, m\alpha + \beta)}.$$

From Proposition 4, we have

$$T(v, m\alpha + \beta) \leqq |w^* \vee mC|,$$

and

$$\frac{3(m\alpha + \beta) + 1 - \text{mod}_3^*(v(0))}{3|w^* \vee mC|} = \frac{3(m\alpha + \beta) + 1 - \text{mod}_3^*(v(0))}{3(|w^*| + m|C|)} \leqq d(w_{m\alpha+\beta}).$$

Therefore

$$\lim_{m\to\infty} \frac{3(m\alpha+\beta)+1-\mathrm{mod}_3^*(v(0))}{3(|w^*|+m|C|)} = \frac{\alpha}{|C|} \leqq \lim_{m\to\infty} d(w_{m\alpha+\beta}).$$

On the other hand,

$$d_M = \frac{C(0)}{3|C|} = \frac{\alpha}{|C|}.$$

Thus we obtain

$$d_M \leqq e(v). \tag{27}$$

(b) We show that $e(v) \leqq d_M$.

We let w_n be the walk of a vertex v excited n times. If w_n is a circuit, w_n is decomposed into finite cycles $C_{n1}, \ldots, C_{nf}$. If w_n is not a circuit, w_n is constituted of one path P_n and finite cycles $C_{n1}, \ldots, C_{nf}$. We denote by $|V|$ the number of vertices in G. Clearly, $P_n \leqq |V|$ and for an arbitrary cycle C, $C(0) \leqq 3d_M|C|$ by equation (24). Thus, we have

$$w_n(0) \leqq |V| + C_{n1}(0) + \cdots + C_{nf}(0) \leqq |V| + 3d_M(|C_{n1}| + \cdots + |C_{nf}|),$$

and

$$|w_n| \geqq |C_{n1}| + \cdots + |C_{nf}|.$$

Therefore,

$$d(w_n) = \frac{w_n(0)}{|w_n|} \leqq \frac{|V| + 3d_M(|C_{n1}| + \cdots + |C_{nf}|)}{3(|C_{n1}| + \cdots + |C_{nf}|)}.$$

By $n \to \infty$, from Lemma 8, we obtain

$$e(v) \leqq d_M. \tag{28}$$

From equations (27) and (28), we have $d_M = e(v)$, which completes the proof. □

8. Concluding Remarks

In this paper, we have investigated analytically a deterministic graph Greenberg–Hastings cellular automaton (graph GHCA). We proved that all vertices converge to either an equilibrium state where the value of each vertex is equal to 0 or a periodic excited state, and that nonexistence or existence of a circuit whose phase difference is not equal to 0 determines if the graph GHCA converges to either an equilibrium state or a periodic state. When a vertex v converges to a periodic state where it gets excited N times during T time steps, we define $e(v) = N / T$ and call it a convergent excitation rate of v. We have showed that all the vertices converge to a periodic state, and the convergent excitation rate of each vertex is equivalent to d_M. (Example: Figures 7 through 13.)

In a general graph GHCA, a quiescent vertex at time t becomes excited at time $t+1$ with probability p $(0 \leqq p \leqq 1)$. As is the case in this paper, d_M is time invariant for $p = 1$ because the phase difference of a circuit is time invariant. But in a general graph GHCA with $p \neq 1$, d_M is not time invariant. We consider it important to investigate the time evolution of d_M by a stochastic approach for $p \neq 1$, which is a project we wish to address in the future.

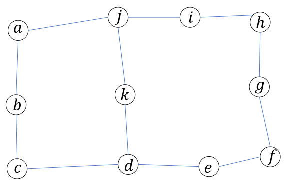

Figure 7. An example (1) of G. G has six cycles, $\pm C_1$, $\pm C_2$, $\pm C_3$. $C_1 = ajkdcba$, $C_2 = dkjihgfed$, $C_3 = ajihgfedcba$.

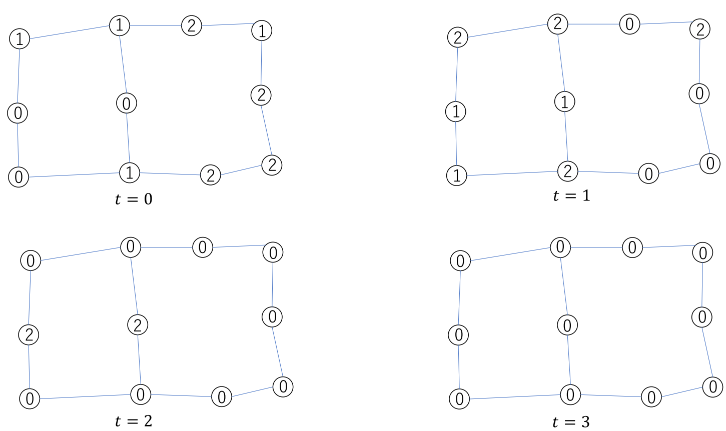

Figure 8. An example (1) of $d_M = 0$. $d(\pm C_1) = d(\pm C_2) = d(\pm C_3) = 0$, $d_M = 0$. All vertices converge to 0 in value after time 3.

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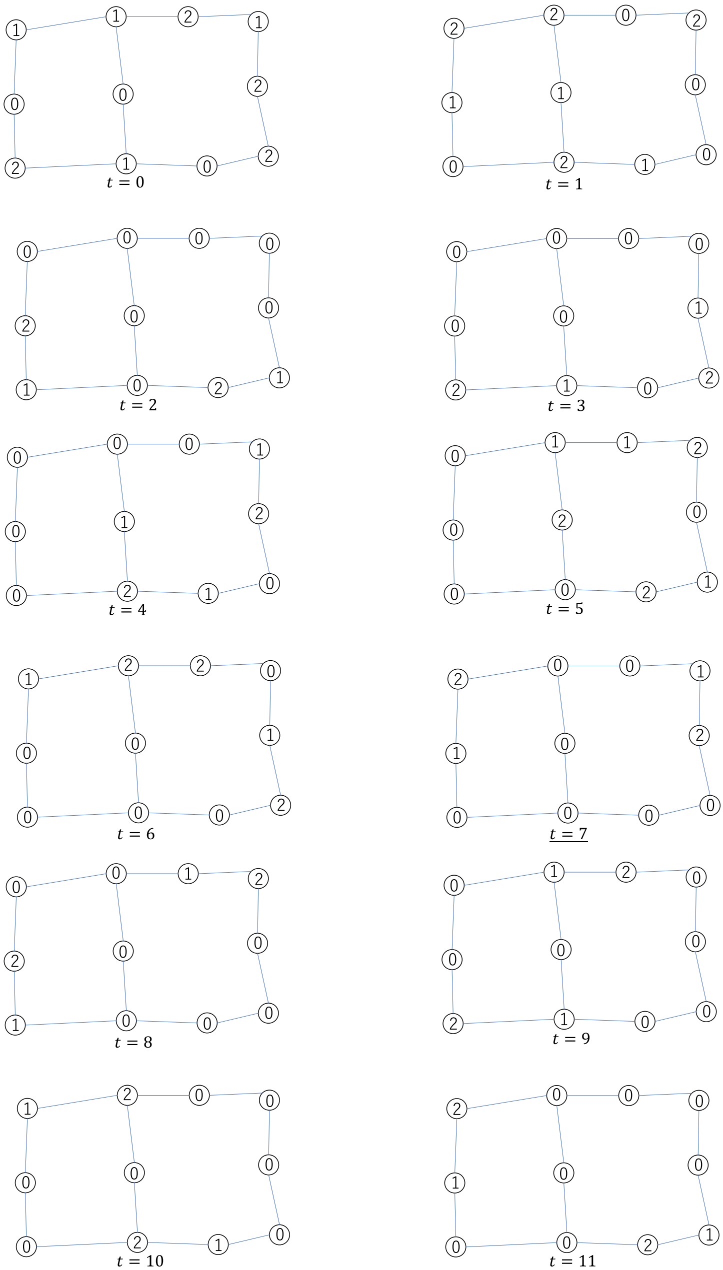

Figure 9. (*continues*)

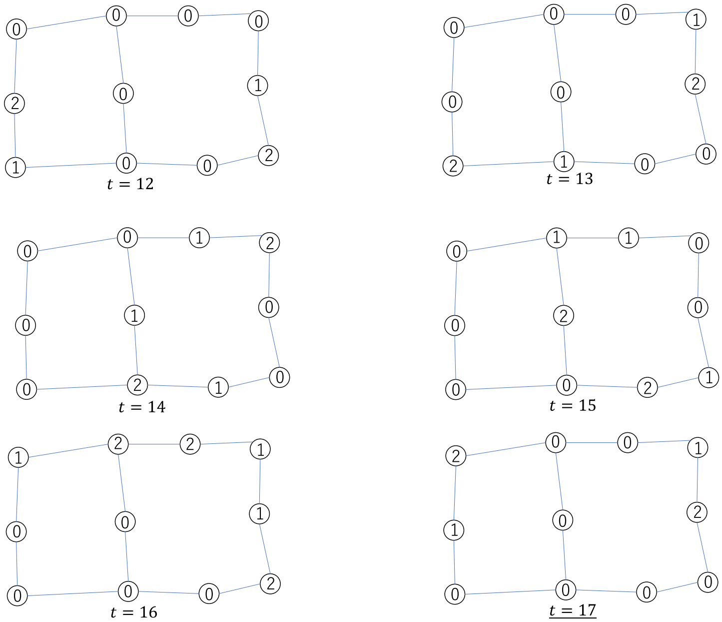

Figure 9. An example (1) of $d_M \neq 0$. $d(\pm C_1) = \pm 1/6$, $d(C_2) = \pm 1/8$, $d(C_3) = \pm 1/5$, $d_M = 1/5$. The values of each vertex at time 7 are equal to the values of each vertex at time 17. That is, each vertex converges to a periodic state 10 steps after time 7. In addition, each vertex gets excited two times in such a cycle, so the excitation rate of each vertex converges to $2/10 = 1/5 = d_M$.

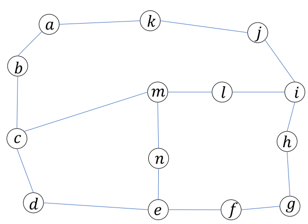

Figure 10. An example (2) of G. G has 14 cycles, $\pm C_1$, $\pm C_2$, $\pm C_3$, $\pm C_4$, $\pm C_5$, $\pm C_6$, $\pm C_7$. $C_1 = abcmlijka$, $C_2 = cdenmc$, $C_3 = mnefghilm$, $C_4 = abcdenmlijka$, $C_5 = cdefghilmc$, $C_6 = abcmnefghijka$, $C_7 = abcdefghijka$.

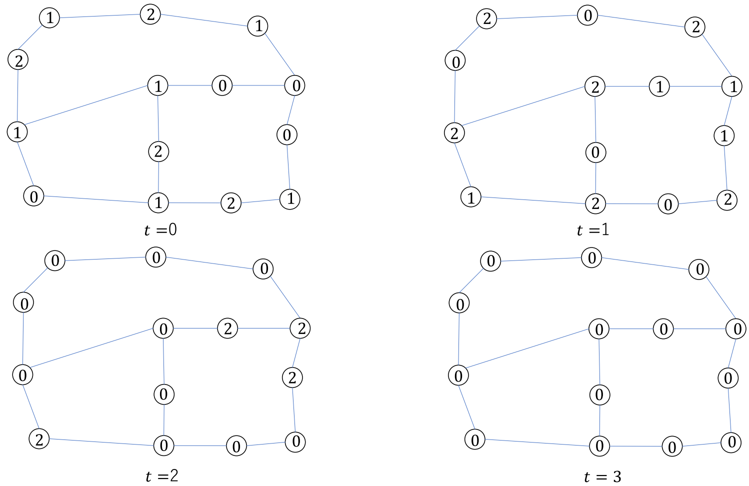

Figure 11. An example (2) of $d_M = 0$.
$d(\pm C_1) = d(\pm C_2) = d(\pm C_3) = d(\pm C_4) = d(\pm C_5) = d(\pm C_6) = d(\pm C_7) = 0$,
$d_M = 0$. All vertices converge to 0 in value after time 3.

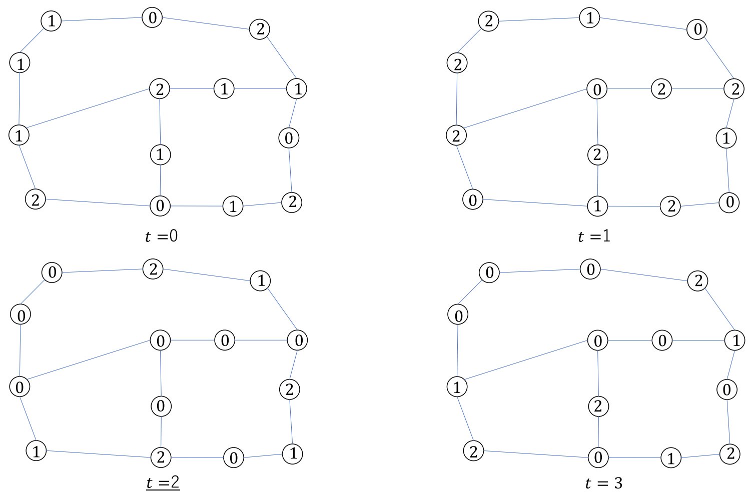

Figure 12. (*continues*)

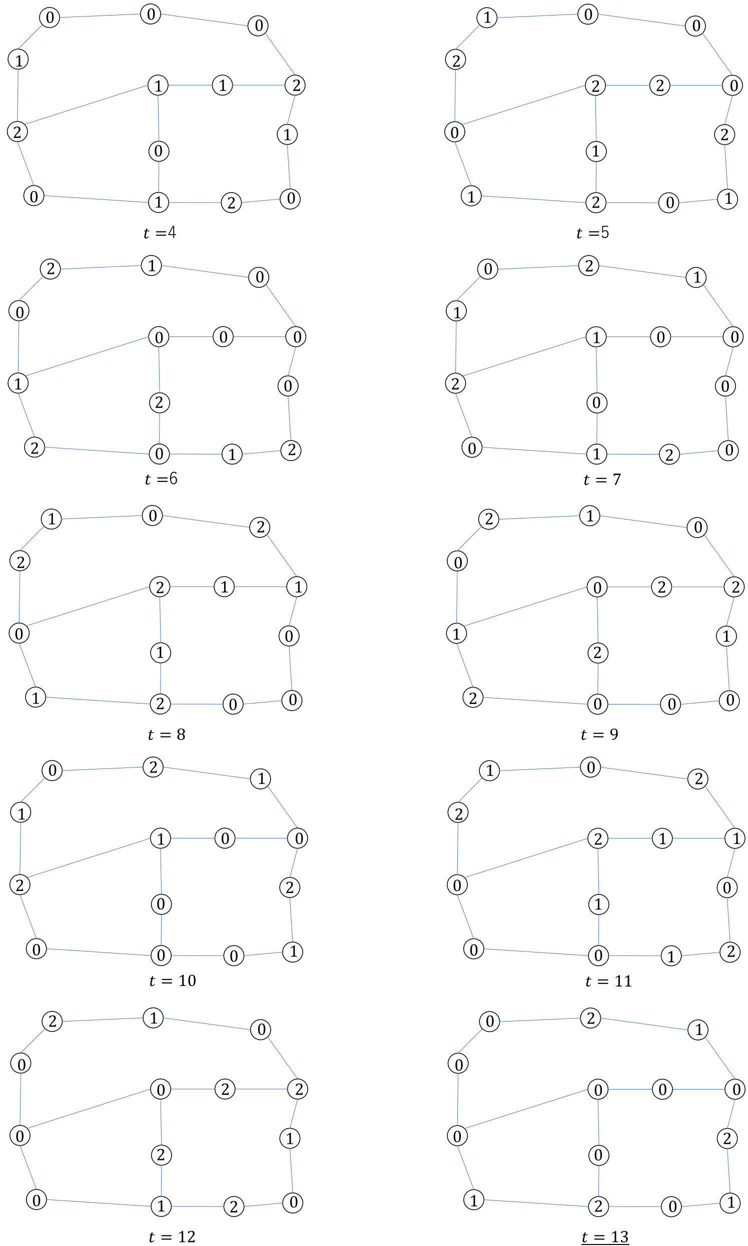

Figure 12. An example (2) of $d_M \neq 0$. $d(\pm C_1) = \pm 1/8$, $d(\pm C_2) = \pm 1/5$, $d(\pm C_3) = \pm 1/8$, $d(\pm C_4) = \pm 2/11$, $d(\pm C_5) = \pm 2/9$, $d(\pm C_6) = \pm 1/4$, $d(\pm C_7) = \pm 3/11$, $d_M = 3/11$. The values of each vertex at time 2 are equal to those at time 13, thus converging into a cycle of period 11 after time 2.

In addition, each vertex gets excited three times in the 11-cycle; thus, the convergent excitation rate of each vertex is equal to $3 / 11 = d_M$.

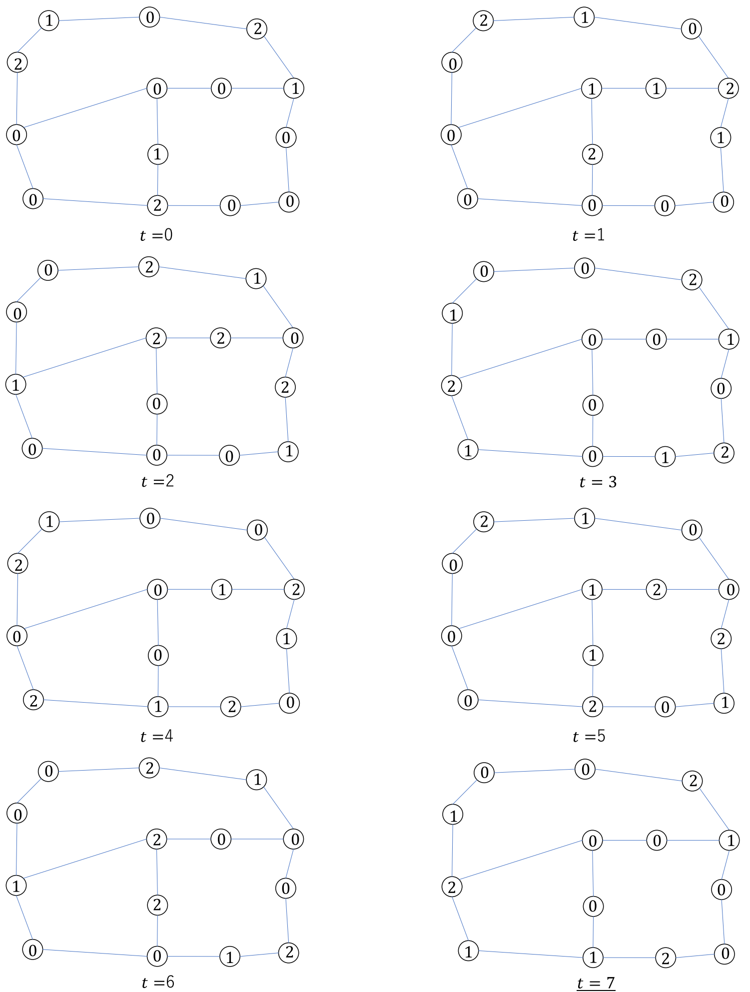

Figure 13. (*continues*)

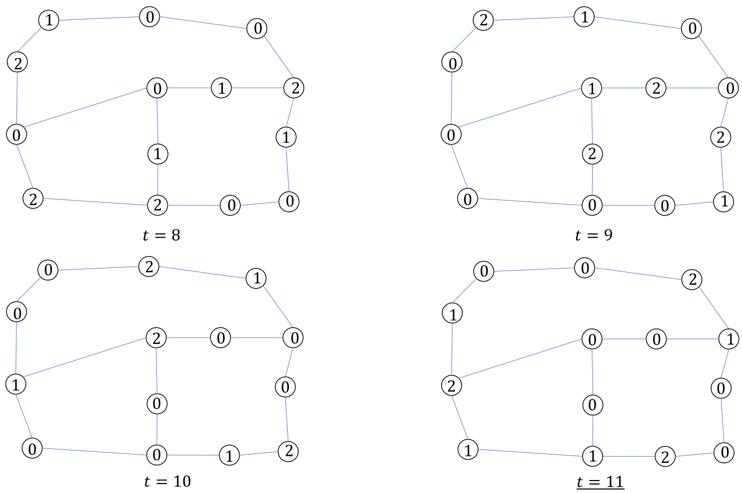

Figure 13. An example (3) of $d_M \neq 0$. $d(\pm C_1) = \pm 1/4$, $d(\pm C_2) = \mp 1/5$, $d(\pm C_3) = \pm 1/8$, $d(\pm C_4) = \pm 1/11$, $d(\pm C_5) = 0$, $d(\pm C_6) = \pm 1/4$, $d(\pm C_7) = \pm 2/11$, $d_M = 1/4$. The values of each vertex at time 7 are equal to those at time 11, thus getting into a cycle of period 4. Each vertex gets excited only one time in that period, and thus the convergent excitation rate of each vertex is equal to $1/4 = d_M$.

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