

# Period-Halving Bifurcation of a Neuronal Recurrence Equation

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The sequences generated by neuronal recurrence equations of the form  $x(n) = 1\left[\sum_{j=1}^h a_j x(n-j) - \theta\right]$  are studied. From a neuronal recurrence equation of memory size  $h$  that describes a cycle of length  $\rho(m) \times \text{lcm}(p_0, p_1, \dots, p_{-1+\rho(m)})$ , a set of  $\rho(m)$  neuronal recurrence equations is constructed with dynamics that describe respectively the transient of length  $O(\rho(m) \times \text{lcm}(p_0, \dots, p_d))$  and the cycle of length  $O(\rho(m) \times \text{lcm}(p_{d+1}, \dots, p_{-1+\rho(m)}))$  if  $0 \leq d \leq -2 + \rho(m)$  and  $1$  if  $d = \rho(m) - 1$ . This result shows the exponential time of the convergence of the neuronal recurrence equation to fixed points and the existence of the period-halving bifurcation.

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## 1. Introduction

Caianiello and De Luca [1] have suggested that the dynamic behavior of a single neuron with a memory that does not interact with other neurons can be modeled by the following recurrence equation:

$$x(n) = 1\left[\sum_{j=1}^k a_j x(n-j) - \theta\right] \quad (1)$$

where:

- $x(n)$  is a variable representing the state of the neuron at  $t = n$ .
- $x(0), x(1), \dots, x(k-2), x(k-1)$  are the initial states.
- $k$  is the memory length, that is, the state of the neuron at time  $t = n$  depends on the states  $x(n-1), \dots, x(n-k)$  assumed by the neuron at the  $k$  previous steps  $t = n-1, \dots, n-k$ .
- $a_j$  ( $j = 1, \dots, k$ ) are real numbers called the weighting coefficients. More precisely,  $a_j$  represents the influence of the state of the neuron at time  $n-j$  on the state assumed by the neuron at time  $n$ .

- $\theta$  is a real number called the threshold.
- $1[u] = 0$  if  $u < 0$ , and  $1[u] = 1$  if  $u \geq 0$ .

The system obtained by interconnecting several neurons is called a neural network. These networks were introduced by McCulloch and Pitts [2] and are quite powerful. Neural networks are able to simulate any sequential machine or Turing machine if an infinite number of cells is provided. Neural networks have been studied extensively as tools for solving various problems such as classification, speech recognition, and image processing [3]. The field of application of threshold functions is large [3–6]. The spin moment of the spin glass system is one of the most cited examples in solid state physics that has been simulated by neural networks.

Neural networks are usually implemented by using electronic components or are simulated in software on a digital computer. One way in which the collective properties of a neural network may be used to implement a computational task is by way of the concept of *energy minimization*. The Hopfield network is a well-known example of such an approach. It has attracted great attention in literature as a *content-addressable memory* [7].

Given a finite neural network, the configuration assumed by the system at time  $t$  is ultimately periodic. As a consequence, there is an integer  $p > 0$  called the period (or a length of a cycle) and another integer  $T \geq 0$  called the transient length such that  $Y(p + T) = Y(T)$ , and  $\nexists T', p' (T', p') \neq (T, p) T \geq T',$  and  $p \geq p'$  such that  $Y(p' + T') = Y(T')$  where  $Y(t) = (x(t), x(t - 1), \dots, x(t - k + 2), x(t - k + 1))$ . The period and the transient length of the sequences generated are good measures of the complexity of the neuron. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden “qualitative” or topological change in its behavior. A period-halving bifurcation in a dynamical system is a bifurcation in which the system switches to a new behavior with half the period of the original system. A great variety of results have been established on recurrence equations modeling neurons with memory [4, 8–15]. However, some mathematical properties are still very intriguing and many problems are being posed. For example, the question remains as to whether there exists one neuronal recurrence equation with transients of exponential lengths [16]. In [17], we give a positive answer to this question by exhibiting a neuronal recurrence equation with memory that generates a sequence of exponential transient length and exponential period length with respect to the memory length. Despite this positive answer, one question remains: does there exist one neuronal recurrence equation with exponential transient length and fixed point?

In this paper, from a neuronal recurrence equation of memory size  $(6m - 1) \times (\rho(m))^2$  whose dynamics contain a cycle of length

$\rho(m) \times \text{lcm}(p_0, p_1, \dots, p_{-1+\rho(m)})$ , we build a set of  $\rho(m)$  neuronal recurrence equations whose dynamics describe respectively:

- the transient of length  $\rho(m) \times \text{lcm}(p_0, p_1, \dots, p_d) + b + d + 1 - (\rho(m) \times (1 + p_d))$ , if  $0 \leq d \leq -1 + \rho(m)$ , and
- the cycle of length  $\rho(m) \times \text{lcm}(p_{d+1}, p_{d+2}, \dots, p_{-1+\rho(m)})$  if  $0 \leq d \leq -2 + \rho(m)$  and 1 if  $d = \rho(m) - 1$ .

Thus, we give a positive answer to the preceding question.

The technique used in this paper to obtain the period-halving bifurcation is to modify some parameters (weighting coefficients and threshold) of the neuronal recurrence equation. This technique relies on control theory. Controllability is related to the possibility of forcing the system into a particular state by using an appropriate control signal.

The paper is organized as follows: in Section 2, some previous results are presented. Section 3 presents some preliminaries. Section 4 is devoted to the construction of neuronal recurrence equation  $z(n, d)$ . Section 5 deals with the behavior of neuronal recurrence equation  $z(n, d)$ . Concluding remarks are stated in Section 6.

## 2. Previous Results

The only study of bifurcation was done by Cosnard and Goles in [10], which studied the bifurcation in two particular cases of neuronal recurrence equations.

*Case 1: Geometric coefficients and bounded memory.* Cosnard and Goles completely described the structure of the bifurcation of the following equation:

$$x_{n+1} = 1 \left[ \theta - \sum_{i=0}^{k-1} b^i x_{n-i} \right]$$

when  $\theta$  varies. They showed that the associated rotation number is an increasing number of the parameter  $\theta$ .

*Case 2: Geometric coefficients and unbounded memory.* Cosnard and Goles completely described the structure of the bifurcation of the following equation:

$$x_{n+1} = 1 \left[ \theta - \sum_{i=0}^n b^i x_{n-i} \right]$$

when  $\theta$  varies. They showed that the associated rotation number is a devil's staircase.

In [9], this conclusion is drawn: "This shows that, if there is a neuronal recurrence equation with memory length  $k$  that generates se-

quences of periods  $p_1, \dots, p_r$ , then there is a neuronal recurrence equation with memory length  $kr$  that generates a sequence of period  $\text{lcm}(p_1, \dots, p_r)r$ , where  $\text{lcm}$  denotes the least common multiple.” This allows us to write the following fundamental lemma of composition of a neuronal recurrence equation.

**Lemma 1.** [9] If there is a neuronal recurrence equation with memory length  $k$  that generates sequences of periods  $p_1, p_2, \dots, p_r$ , then there is a neuronal recurrence equation with memory length  $kr$  that generates a sequence of period  $r \cdot \text{lcm}(p_1, \dots, p_r)$ .

Lemma 1 does not take into account the study of the transient length. Lemma 1 can be amended to obtain the following lemma.

**Lemma 2.** [13, 17] If there is a neuronal recurrence equation with memory length  $k$  that generates a sequence  $\{x^j(n) : n \geq 0\}$ ,  $1 \leq j \leq g$  of transient length  $T_j$  and of period  $p_j$ , then there is a neuronal recurrence equation with memory length  $kg$  that generates a sequence of transient length  $g \cdot \max(T_1, T_2, \dots, T_g)$  and of period  $g \cdot \text{lcm}(p_1, p_2, \dots, p_g)$ .

In the following example, we will show that Lemmas 1 and 2 are incomplete.

**Example 1.** Let us suppose that the neuronal recurrence equation defined by equation (1) generated six sequences

$$\{x^i(n) : n \geq 0\}, \quad 0 \leq i \leq 5 \quad (2)$$

of periods

$$p_i = 1, \quad 0 \leq i \leq 5. \quad (3)$$

It is clear that each sequence defined by equation (2) is a fixed point. We present two different cases of evolution.

*First case:* We suppose that

$$x^{2i}(n) = 0; \forall n, i \text{ such that } n \geq 0 \text{ and } 0 \leq i \leq 2 \quad (4)$$

$$x^{2i+1}(n) = 1; \forall n, i \text{ such that } n \geq 0 \text{ and } 0 \leq i \leq 2. \quad (5)$$

It is easy to verify that the shuffle of the neuronal recurrence equation defined by equations (4) and (5) is

$$\begin{aligned} x^0(0)x^1(0) \dots x^5(0)x^0(1)x^1(1) \dots x^5(1) \dots x^0(i)x^1(i) \dots \\ x^5(i) \dots = 010\ 101\ 010\ 101\ 010\ 101\ 010\ 101 \\ \dots 010\ 101\ 010\ 101 \dots \end{aligned} \quad (6)$$

The sequence defined by equation (6) describes a period of length 2. By application of Lemma 1, the period of the sequence defined by equation (6) should be 6 (more precisely,  $6 \times \text{lcm}(1, 1, 1, 1, 1, 1)$ ).

*Second case:* We suppose that

$$x^i(n) = 0, \forall n, i \text{ such that } n \geq 0 \text{ and } i \in \{0, 1, 3, 4\} \tag{7}$$

$$x^i(n) = 1, \forall n, i \text{ such that } n \geq 0 \text{ and } i \in \{2, 5\}. \tag{8}$$

It is easy to verify that the shuffle of the neuronal recurrence equation defined by equations (7) and (8) is

$$\begin{aligned} &x^0(0)x^1(0) \dots x^5(0)x^0(1)x^1(1) \\ &\dots x^5(1) \dots x^0(i)x^1(i) \dots x^5(i) \dots = \\ &001\ 001\ 001\ 001\ 001\ 001\ 001\ 001\ 001\ 001\ 001\ 001\ 001 \\ &\dots 001\ 001 \dots \end{aligned} \tag{9}$$

The sequence defined by equation (9) describes a period of length 3. By application of Lemma 1, the period of the sequence defined by equation (9) should be 6 (more precisely,  $6 \times \text{lcm}(1, 1, 1, 1, 1, 1)$ ).

The first and second cases of Example 1 show that Lemmas 1 and 2 do not take into account all of the cases.

Lemma 1 can be amended as follows.

**Lemma 3.** If there is a neuronal recurrence equation with memory length  $k$  that generates  $r$  sequences of periods  $p_1, p_2, \dots, p_r$ , then there is a neuronal recurrence equation with memory length  $kr$  that generates a sequence of period Per. Per is defined as follows.

*First case:*  $\exists j, 1 \leq j \leq r$  such that  $p_j \geq 2$

$$\text{Per} = r \times \text{lcm}(p_1, \dots, p_r).$$

*Second case:*  $p_j = 1; \forall j, 1 \leq j \leq r$ .

Per is a divisor of  $r$ .

The improvement of Lemma 1 does not modify all the main results about periods obtained in the papers [9, 11–14] because all these main results consider only the case where the periods  $p_1, p_2, \dots, p_r$  of the  $r$  sequences are greater than or equal to 2.

We can also amend Lemma 2 as follows.

**Lemma 4.** If there is a neuronal recurrence equation with memory length  $k$  that generates a sequence  $\{x^j(n) : n \geq 0\}$ ,  $1 \leq j \leq g$  of transient length  $T_j$  and of period  $p_j$ , then there is a neuronal recurrence equation with memory length  $kg$  that generates a sequence of transient length  $g \cdot \max(T_1, T_2, \dots, T_g)$  and of period Per. Per is defined as follows.

First case:  $\exists j, 1 \leq j \leq r$  such that  $p_j \geq 2$

$$\text{Per} = r \times \text{lcm}(p_1, \dots, p_g).$$

Second case:  $p_j = 1; \forall j, 1 \leq j \leq r$ .

Per is a divisor of  $g$ .

### 3. Preliminaries

Let  $k$  be a positive integer for a vector  $a \in \mathbb{R}^k$ , a real number  $\theta \in \mathbb{R}$ , and a vector  $\phi \in \{0, 1\}^k$ . We define the sequence  $\{x(n) : n \in \mathbb{N}\}$  by the following recurrence:

$$x(t) = \begin{cases} \phi(t); & t \in \{0, \dots, k-1\} \\ 1 \left( \sum_{i=1}^k a_i x(t-i) - \theta \right); & t \geq k. \end{cases} \quad (10)$$

We denote by  $S(a, \theta, \phi)$  the sequence generated by equation (10), with  $\text{Per}(a, \theta, \phi)$  as its period and  $\text{Tra}(a, \theta, \phi)$  as its transient length.

Let  $m$  be a positive integer. We denote the cardinality of the set  $\mathcal{P} = \{p : p \text{ prime and } 2m < p < 3m\}$  by  $\rho(m)$ . Let us denote by  $p_0, p_1, \dots, p_{-1+\rho(m)}$  the prime numbers belonging to the set  $\{2m+1, 2m+2, \dots, 3m-2, 3m-1\}$ . The sequence  $\{\alpha_i : 0 \leq i \leq -1 + \rho(m)\}$  is defined as  $\alpha_i = 3m - p_i, 0 \leq i \leq -1 + \rho(m)$ .

We also suppose that

$$p_{-1+\rho(m)} < p_{-2+\rho(m)} < \dots < p_{i+1} < p_i < \dots < p_1 < p_0. \quad (11)$$

Subsequently, we consider only the integers  $m$  such that  $\rho(m) \geq 2$ .

It is easy to check that  $\{2m+1, 2m+2, \dots, 3m-2, 3m-1\}$  contains at most  $\left\lceil \frac{m-1}{2} \right\rceil$  odd integers. It follows that

$$\rho(m) \leq \left\lceil \frac{m-1}{2} \right\rceil. \quad (12)$$

We set  $k = (6m-1)\rho(m)$  and  $\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$ . We define

$$\mu(m, \alpha_i) = \left\lfloor \frac{k}{3m - \alpha_i} \right\rfloor$$

$$\beta(m, \alpha_i) = k - ((3m - \alpha_i) \mu(m, \alpha_i)).$$

From the previous definitions, we find  $k = ((3m - \alpha_i) \mu(m, \alpha_i) + \beta(m, \alpha_i))$ .

It is clear that  $\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$

$$2m + 1 \leq m - \alpha_i \leq 3m - 1.$$

This implies that

$$\frac{(6m - 1)\rho(m)}{3m - 1} \leq \frac{k}{3m - \alpha_i} \leq \frac{(6m - 1)\rho(m)}{2m + 1}.$$

Therefore,

$$2\rho(m) \leq \mu(m, \alpha_i) \leq 3\rho(m), \tag{13}$$

$\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$ . We want to construct a neuronal recurrence equation  $\{x^{\alpha_i}(n) : n \geq 0\}$  with memory of length  $k$  that evolves as follows:

$$\underbrace{00 \dots 0}_{\beta(m, \alpha_i)} \underbrace{100 \dots 0}_{3m - \alpha_i} \underbrace{100 \dots 0}_{3m - \alpha_i} \dots \underbrace{100 \dots 0}_{3m - \alpha_i} \dots \underbrace{100 \dots 0}_{3m - \alpha_i} \dots \tag{14}$$

and that describes a cycle of length  $3m - \alpha_i = p_i$ .

$\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$ ; let  $\phi^{\alpha_i} \in \{0, 1\}^k$  be the vector defined by

$$\phi^{\alpha_i}(0) \dots \phi^{\alpha_i}(k - 1) = \underbrace{0 \dots 0}_{\beta(m, \alpha_i)} \underbrace{10 \dots 0}_{p_i} \dots \underbrace{10 \dots 0}_{p_i}. \tag{15}$$

$\mu(m, \alpha_i) p_i$

In other words,  $\phi^{\alpha_i}$  is defined by

$$\phi^{\alpha_i}(j) = \begin{cases} 1 & \text{if } \exists \ell, 0 \leq \ell \leq \mu(m, \alpha_i) - 1 \text{ such that } j = \beta(m, \alpha_i) + \ell p_i \\ 0 & \text{otherwise.} \end{cases}$$

We define the neuronal recurrence equation  $\{x^{\alpha_i}(n) : n \geq 0\}$  by the following recurrence:

$$x^{\alpha_i}(t) = \begin{cases} \phi^{\alpha_i}(t); & t \in \{0, \dots, k - 1\} \\ \mathbf{1} \left( \sum_{j=1}^k \bar{a}_j x^{\alpha_i}(t - j) - \bar{\theta} \right); & t \geq k \end{cases} \tag{16}$$

where  $\bar{a}_j$  is defined as follows.

*First case:*  $\rho(m)$  is even and  $\forall i_2 \in \mathbb{N}, 0 \leq i_2 \leq -1 + \rho(m)$

$$\bar{a}_j = \begin{cases} 2 & \text{if } j \in \text{Pos}(\alpha_{i_2}) \text{ and } j \leq \frac{3 \times \rho(m) \times p_{i_2}}{2}, \\ -2 & \text{if } j \in \text{Pos}(\alpha_{i_2}) \text{ and } j > \frac{3 \times \rho(m) \times p_{i_2}}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Second case:  $\rho(m)$  is odd,  $\rho(m) \geq 3$  and  $\forall i_2 \in \mathbb{N}$ ,  $0 \leq i_2 \leq -1 + \rho(m)$

$$\bar{a}_j = \begin{cases} 2 & \text{if } j \in \text{Pos}(\alpha_{i_2}) \text{ and } j \leq \frac{(3\rho(m)-1)}{2} \times p_{i_2}, \\ -2 & \text{if } j \in \text{Pos}(\alpha_{i_2}) \text{ and } \frac{(3\rho(m)+1)}{2} \times p_{i_2} \leq j \leq (2\rho(m)-2) \times p_{i_2}, \\ -1 & \text{if } j \in \{(2\rho(m)-1) \times p_{i_2}, 2\rho(m) \times p_{i_2}\}, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

We also define

$$\text{Pos}(\alpha_i) = \{j p_i : j = 1, \dots, 2\rho(m)\} \quad (19)$$

$$= \{p_i, 2p_i, \dots, (-1 + 2\rho(m))p_i, 2\rho(m)p_i\}, \quad (20)$$

$$0 \leq i \leq -1 + \rho(m)$$

$$D = \{i : i = 1, \dots, k\} = \{1, 2, \dots, k-1, k\} \quad (21)$$

$$F = \bigcup_{i=0}^{1+\rho(m)} \text{Pos}(\alpha_i) \quad (22)$$

$$G = D \setminus F \quad (23)$$

$$\bar{\theta} = 2 \times \rho(m) \quad (24)$$

$$k = (6m-1) \times \rho(m). \quad (25)$$

By definition,  $\text{Pos}(\alpha_i)$  represents the set of indices  $j$ ,  $1 \leq j \leq k$  such that  $x^{\alpha_i}(k-j) = 1$ .

From the definition of  $\text{Pos}(\alpha_i)$  and from equation (15), it can easily be verified that

$$j \in \text{Pos}(\alpha_i) \implies x^{\alpha_i}(k-j) = 1 \quad (26)$$

$$j \in D \setminus \text{Pos}(\alpha_i) \implies x^{\alpha_i}(k-j) = 0. \quad (27)$$

$\forall d \in \mathbb{N}$ ,  $0 < d < p_i$ ; we also denote  $P\text{Pos}(\alpha_i, d)$  (the set of indices  $j$ ) such that  $x^{\alpha_i}(k+d-j) = 1$ . In other words:

$$P\text{Pos}(\alpha_i, d) = \{j : x^{\alpha_i}(k+d-j) = 1 \text{ and } 1 \leq j \leq k\}.$$



$\forall i, d \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$ , and  $0 < d < p_i$ . We denote

$$\begin{aligned} Q(\alpha_i, d) &= \{d + j p_i : j = 0, 1, \dots, \mu(m, \alpha_i)\}, \quad 0 < d \leq \beta(m, \alpha_i) \\ Q(\alpha_i, d) &= \{d + j p_i : j = 0, 1, \dots, -1 + \mu(m, \alpha_i)\}, \\ &\quad \beta(m, \alpha_i) < d < p_i \\ E(\alpha_i, d) &= Q(\alpha_i, d) \cap F. \end{aligned}$$

The neuronal recurrence equation  $\{x^{\alpha_i}(n) : n \geq 0\}$  with memory of length  $k$  is defined by equations (15) and (16).

We will show that the neuronal recurrence equation  $\{x^{\alpha_i}(n) : n \geq 0\}$  evolves as specified in equation (14).

In the following proposition, we present an important property.

**Proposition 1.** [13]  $\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$  and  $\forall d \in \mathbb{N}, 1 \leq d < p_i$ :

$$\text{card } E(\alpha_i, d) \leq \rho(m) - 1.$$

The following proposition characterizes the sum of the interaction coefficients  $\bar{a}_j$  when  $j \in \text{Pos}(\alpha_i)$ .

**Proposition 2.**  $\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$ , so

$$\sum_{j \in \text{Pos}(\alpha_i)} \bar{a}_j = 2 \times \rho(m).$$

The following lemma characterizes the evolution of the sequence  $\{x^{\alpha_i}(n) : n \geq 0\}$  at time  $t = k$ .

**Lemma 5.**  $x^{\alpha_i}(k) = 1$ .

From Lemma 5 and equation (15), it is easy to verify that

$$P \text{Pos}(\alpha_i, 1) = Q(\alpha_i, 1). \tag{28}$$

From the definition of  $E(\alpha_i, 1)$ , equation (15), equation (28), and Lemma 5, we easily check that

$$\ell \in D \setminus E(\alpha_i, 1) \implies x^{\alpha_i}(k + 1 - \ell) = 0 \text{ or } \bar{a}_\ell = 0. \tag{29}$$

The values of the sequence  $\{x^{\alpha_i}(n) : n \geq 0\}$  at time  $t = k + 1, \dots, k - 1 + p_i$  are given by the following lemma.

**Lemma 6.**  $\forall t \in \mathbb{N}$  such that  $1 \leq t \leq 3m - 1 - \alpha_i$ ; we find  $x^{\alpha_i}(k + t) = 0$ .

It is easy to verify that  $\forall i \in \mathbb{N}, 0 \leq i \leq -1 + \rho(m)$ :

$$P \text{Pos}(\alpha_i, j) = Q(\alpha_i, j) \quad \forall j, \quad 1 \leq j \leq 3m - 1 - \alpha_i.$$

**Lemma 7.** There exists  $\bar{a}, \phi^{\alpha_i} \in \mathbb{R}^k$ , and  $\bar{\theta} \in \mathbb{R}$  such that

$$\text{Per}(\bar{a}, \bar{\theta}, \phi^{\alpha_i}) = p_i.$$

**Lemma 8.**  $\forall t, i \in \mathbb{N}, t \geq k$ , and  $0 \leq i \leq -1 + \rho(m)$ :

$$\mu(m, \alpha_i) \leq \sum_{j=1}^k x^{\alpha_i(t-j)} \leq 1 + \mu(m, \alpha_i).$$

In order to present some properties of the sequence  $\{x^{\alpha_i}(n) : n \geq 0\}$ , we introduce the following notation.

**Notation 1.** Let us define  $S1(\alpha_i, n)$  as

$$S1(\alpha_i, n) = \sum_{j=1}^k \bar{a}_j x^{\alpha_i(n-j)}$$

and let  $\lambda$  be a strictly negative real number such that  $\forall i, 0 \leq i \leq \rho(m) - 1$ :

$$\max\{S1(\alpha_i, n) - \bar{\theta} : S1(\alpha_i, n) < \bar{\theta} \text{ and } n \geq k\} \leq \lambda.$$

**Lemma 9.**  $\forall i, n \in \mathbb{N}$  such that  $0 \leq i \leq -1 + \rho(m)$  and  $n \geq k$ ,

$$S1(\alpha_i, n) \in [-2(1 + \mu(m, \alpha_i)), \bar{\theta} - 1] \cup \{\bar{\theta}\},$$

$$\lambda \in [-1, 0[.$$

Let  $\{v^{\alpha_i}(n) : n \geq 0\}$  be the sequence where the first  $k$  terms are defined as follows:

$$v^{\alpha_i}(0) v^{\alpha_i}(1) \dots v^{\alpha_i}(k-1) = x^{\alpha_i}(1) \dots x^{\alpha_i}(k-1) \overline{x^{\alpha_i}(k)}, \quad (30)$$

and the other terms are generated by the following neuronal recurrence equation:

$$v^{\alpha_i}(n) = 1 \left[ \sum_{j=1}^k \bar{a}_j v^{\alpha_i}(n-j) - \bar{\theta} \right], \quad n \geq k. \quad (31)$$

**Remark 1.** The term  $x^{\alpha_i}(k)$  is equal to 1; this implies that  $v^{\alpha_i}(k-1)$  is equal to 0.

The parameters  $\bar{a}_j$ ,  $1 \leq j \leq k$ , and  $\bar{\theta}$  used in neuronal recurrence equation (31) are those defined in equations (17), (18), and (24).

The following lemma, which is easy to prove, characterizes the evolution of the sequence  $\{v^{\alpha_i}(n) : n \geq 0\}$ .

**Lemma 10.** In the evolution of the sequence  $\{v^{\alpha_i}(n) : n \geq 0\}$ ,  $\forall t \in \mathbb{N}, t \geq k$ , we find:

- (a)  $v^{\alpha_i}(t) = 0$
- (b)  $\sum_{j=1}^k \bar{a}_j v^{\alpha_i}(t-j) \leq \bar{\theta} - 2$

(c) The sequence  $\{v^{\alpha_i}(n) : n \geq 0\}$  describes a transient of length  $k - p_i$  and a fixed point.

The instability of the sequence  $\{x^{\alpha_i}(n) : n \geq 0\}$  occurs as a result of the convergence of the sequence  $\{v^{\alpha_i}(n) : n \geq 0\}$  to  $00 \dots 00$ .

**Notation 2.**  $h = \rho(m) \times k = (6m - 1) \times (\rho(m))^2$  is the length of the memory of some neuronal recurrence equations.

Let us also note:

$$L_0(d) = \begin{cases} \rho(m) \times \text{lcm}(p_{d+1}, p_{d+2}, \dots, p_{-2+\rho(m)}, p_{-1+\rho(m)}), & \text{if } 0 \leq d \leq -2 + \rho(m) \\ 1, & \text{if } d = -1 + \rho(m) \end{cases} \tag{32}$$

$$L_1(d) = \rho(m) \times \text{lcm}(p_0, p_1, \dots, p_d), \quad 0 \leq d \leq \rho(m) - 1 \tag{33}$$

$$L_2 = \rho(m) \times \text{lcm}(p_0, p_1, \dots, p_{-1+\rho(m)}). \tag{34}$$

$L_0(d)$ ,  $L_1(d)$ , and  $L_2$  represent the periods of some neuronal recurrence equations.

Let  $\{y(n) : n \geq 0\}$  be the sequence whose first  $h$  terms are defined as follows:

$$\forall j \in \mathbb{N}, \quad 0 \leq j \leq k - 1 \tag{35}$$

$$y((\rho(m) \times j) + i) = x^{\alpha_i}(1 + j), \quad 0 \leq i \leq -1 + \rho(m)$$

and the other terms are generated by the following neuronal recurrence equation:

$$y(n) = 1 \left[ \sum_{f=1}^h b_f y(n - f) - \theta_1 \right]; \quad n \geq h \tag{36}$$

where

$$b_f = \begin{cases} \bar{a}_j, & \text{if } f = \rho(m) \times j, \quad 1 \leq j \leq k \\ 0, & \text{otherwise} \end{cases} \tag{37}$$

$$\theta_1 = \bar{\theta}. \tag{38}$$

The parameters  $\bar{a}_j$  are those defined in equations (17) and (18). The parameters  $\bar{\theta}$  and  $k$  are defined in equations (24) and (25).

**Remark 2.** (a) The first  $h$  terms of the sequence  $\{y(n) : n \geq 0\}$  are obtained by shuffling the  $k$  terms of each subsequence  $\{x^{\alpha_i}(n) : 1 \leq n \leq k\}$  where  $0 \leq i \leq -1 + \rho(m)$ .

(b) The neuronal recurrence equation (36) is obtained by applying the construction of Lemma 1 to the neuronal recurrence equation (16) whose parameters are given in equations (17), (18), (24), and (25).

Because the sequence  $\{y(n) : n \geq 0\}$  is the shuffle of the  $\rho(m)$  subsequences and due to its construction, we can write Lemma 11.

**Lemma 11.**  $\forall t \in \mathbb{N}$  such that  $t = q \times \rho(m) + i$  with  $q \in \mathbb{N}$  and  $0 \leq i \leq -1 + \rho(m)$ , so

$$y(t) = x^{\alpha_i}(1 + q).$$

The next lemma gives the period of the sequences  $\{y(n) : n \geq 0\}$ .

**Lemma 12.** The sequence  $\{y(n) : n \geq 0\}$  describes a cycle of length  $L_2$ .

$\forall d \in \mathbb{N}$  such that  $0 \leq d \leq -1 + \rho(m)$ . We denote by  $\{w(n, d) : n \geq 0\}$  the sequence whose first  $h$  terms are defined as

$$\forall i, 0 \leq i \leq d$$

$$w(\rho(m)j + i, d) = \begin{cases} x^{\alpha_i}(1 + j), & 0 \leq j \leq k - 2 \\ \overline{x^{\alpha_i}(k)}, & j = k - 1 \end{cases} \tag{39}$$

and

$$\forall i, d + 1 \leq i \leq -1 + \rho(m)$$

$$w(\rho(m)j + i, d) = y(\rho(m)j + i + L_1(d)); \quad 0 \leq j \leq k - 1. \tag{40}$$

The first  $h$  terms of the sequence  $\{w(n, d) : n \geq 0\}$  are obtained by shuffling the  $k$  terms of each of the sequences:

$$v^{\alpha_i}(0) v^{\alpha_i}(1) v^{\alpha_i}(2) \dots v^{\alpha_i}(k - 1); \quad 0 \leq i \leq d \tag{41}$$

and

$$x^{\alpha_i}(1 + \gamma_i(d)) x^{\alpha_i}(2 + \gamma_i(d)) x^{\alpha_i}(3 + \gamma_i(d))$$

$$\dots x^{\alpha_i}(k + \gamma_i(d)); \quad d + 1 \leq i \leq -1 + \rho(m) \tag{42}$$

where

$$\frac{L_1(d)}{\rho(m)} \equiv \gamma_i(d) \pmod{p_i}; \quad d + 1 \leq i \leq -1 + \rho(m). \tag{43}$$

The other terms of the sequence  $\{w(n, d) : n \geq 0\}$  are generated by the following neuronal recurrence equation:

$$w(n, d) = 1 \left[ \sum_{f=1}^h b_f w(n - f, d) - \theta_1 \right]; \quad n \geq h. \tag{44}$$

The next lemma gives the period of the sequence  $\{w(n, d) : n \geq 0\}$ .

**Lemma 13.** The sequence  $\{w(n, d) : n \geq 0\}$  generates a transient of length  $(\rho(m) \times (k - p_d - 1)) + d + 1$  and a cycle of length  $L_0(d)$ .

**Notation 3.** Let us define  $S2(n), S3(n, d)$  as

$$S2(n) = \sum_{f=1}^b b_f y(n - f),$$

$$S3(n, d) = \sum_{f=1}^b b_f w(n - f, d).$$

**Remark 3.** On the basis of the composition of automata [9] and the definition of  $\lambda$ , we can conclude that:

- $\max \{S2(n) - \theta_1 : S2(n) < \theta_1 \text{ and } n \geq b\} = \max \{S1(r, n) - \bar{\theta} : S1(r, n) < \bar{\theta} \text{ and } n \geq k\} \leq \lambda$ , and
- $\max \{S3(n, d) - \theta_1 : S3(n, d) < \theta_1 \text{ and } n \geq b\} = \max \{S1(r, n) - \bar{\theta} : S1(r, n) < \bar{\theta} \text{ and } n \geq k\} \leq \lambda$ .

### 3.1 Results on the Dynamics of Sequences $y$ and $w$

In this subsection, we recall and give some interesting results on the dynamics of the sequences  $\{y(n) : n \geq 0\}$  and  $\{w(n, d) : n \geq 0\}$ .

The following lemma characterizes the sequence  $\{y(n) : n \geq 0\}$  and  $\{w(n, d) : n \geq 0\}$  in terms of the sum of  $b$  consecutive terms.

**Lemma 14.**  $\forall n, d \in \mathbb{N}$ , such that  $n \geq b$  and  $0 \leq d \leq \rho(m) - 1$ :

- $\sum_{i=0}^{-1+\rho(m)} \mu(m, \alpha_i) \leq \sum_{f=1}^b y(n - f) \leq \rho(m) + \sum_{i=0}^{-1+\rho(m)} \mu(m, \alpha_i)$ ,
- $\sum_{i=d+1}^{-1+\rho(m)} \mu(m, \alpha_i) \leq \sum_{f=1}^b w(n - f, d) \leq -d - 1 + \rho(m) + \sum_{i=0}^{-1+\rho(m)} \mu(m, \alpha_i)$ .

**Definition 1.** Let us define the terms of the sequence  $\{\text{tim}(i, l, q) : l, q \in \mathbb{N} \text{ and } 0 \leq i \leq -1 + \rho(m)\}$  as

$$\text{tim}(i, l, q) = (l \times \rho(m)) + i + q : l, q \in \mathbb{N} \text{ and } 0 \leq i \leq -1 + \rho(m).$$

From Lemma 11, it can easily be deduced that

$$y(\text{tim}(i, l, 0)) y(\text{tim}(i, l + 1, 0)) \dots y(\text{tim}(i, l + k - 1, 0)) = x^{\alpha_i}(1 + l) \dots x^{\alpha_i}(l + k); \quad 0 \leq i \leq -1 + \rho(m). \tag{45}$$

From Lemma 11 or equation (45), we can also easily deduce that the terms of the sequence  $\{y(n) : n \geq 0\}$  verify the following relation:

$$\begin{aligned}
 & y(\text{tim}(i, 0, -\rho(m) + L_1(d))) y(\text{tim}(i, 1, -\rho(m) + L_1(d))) \\
 & \quad \dots y(\text{tim}(i, k - 1, -\rho(m) + L_1(d))) = \\
 & \frac{00 \dots 0 \ 10 \dots 0 \ 0 \dots 10 \dots 00; \ 0 \leq i \leq d}{\beta(m, \alpha_i) \underbrace{p_i \dots p_i}_{\mu(m, \alpha_i) \times p_i}} \tag{46(a)}
 \end{aligned}$$

$$\begin{aligned}
 & y(\text{tim}(i, 0, -\rho(m) + L_1(d))) y(\text{tim}(i, 1, -\rho(m) + L_1(d))) \\
 & \quad \dots y(\text{tim}(i, k - 1, -\rho(m) + L_1(d))) = \\
 & x^{\alpha_i}(\gamma_i(d)) x^{\alpha_i}(1 + \gamma_i(d)) \dots x^{\alpha_i}(k - 2 + \gamma_i(d)) \\
 & \quad x^{\alpha_i}(k - 1 + \gamma_i(d)); \quad d + 1 \leq i \leq -1 + \rho(m). \tag{46(b)}
 \end{aligned}$$

For all  $d \in \mathbb{N}$  such that  $0 \leq d \leq -1 + \rho(m)$ ,  $B_0(d)$  is the set of integers  $f$  that verify equations (47) and (48):

$$1 \leq f \leq b - d \tag{47}$$

$$y(-f - \rho(m) + b + L_1(d)) = 1. \tag{48}$$

*Comment:* By considering the following terms, it is possible by easy computation to build the set  $B_0(d)$ :  $y(-\rho(m) + L_1(d)) y(1 - \rho(m) + L_1(d)) \dots y(-\rho(m) + L_1(d) + b - 1)$ . In a bid to give the algebraic expression of the set  $B_0(d)$ , let us define the set  $C0(i, p_i)$  as follows:

$$\begin{aligned}
 C0(i, p_i) = \\
 \{ -i + \rho(m) + (\rho(m) \times p_i \times j) : j = 0, 1, \dots, \mu(m, \alpha_i) \}, \\
 0 \leq i \leq \rho(m) - 1. \tag{49}
 \end{aligned}$$

From the definition of the terms  $y(0) y(1) \dots y(b - 1)$  (see equation (35)), it is easy to see that  $\bigcup_{i=0}^{\rho(m)-1} C0(i, p_i)$  represents the set of indices  $j$  such that  $y(b - j) = 1$ .

Let us define the set  $C1(n, y)$  as follows:

$$C1(n, y) = \{ j : y(b + n - j) = 1 \}. \tag{50}$$

It is easy to see that

$$\begin{aligned}
 \ell \in \bigcup_{i=0}^{\rho(m)-1} C0(i, p_i) & \iff y(b - \ell) = 1 \text{ and } 1 \leq \ell \leq b, \\
 & \iff y(b + n - (\ell + n)) = 1 \text{ and } 1 \leq \ell \leq b, \\
 & \iff \ell + n \in C1(n, y) \text{ and } 1 \leq \ell \leq b.
 \end{aligned}$$

Consequently,

$$\begin{aligned} \ell \in \bigcup_{i=0}^{\rho(m)-1} C0(i, p_i) &\iff \ell + n \in \\ C1(n, y) \text{ and } 1 \leq \ell \leq h. & \end{aligned} \tag{51}$$

Based on the period of the subsequence  $\{x^{\alpha_i(n)} : n \geq 0\}$  and on the fact that  $y$  is a shuffle of the subsequences  $x^{\alpha_i}(0 \leq i \leq \rho(m) - 1)$ , we deduce that

$$\left. \begin{aligned} \ell + n \in C1(n, y), \\ \ell \in C0(i, p_i), \end{aligned} \right\} \implies \ell + n - (\rho(m) \times p_i) \in C1(n, y).$$

Let us denote  $C2(i, d)$  the following set:

$$\begin{aligned} C2(i, d) = \\ \{j : -i + \rho(m) + L_1(d) - \rho(m) \equiv j \pmod{\rho(m) \times p_i}\}. \end{aligned} \tag{52}$$

The set  $C2(i, d)$  contains the only element  $j$  ( $0 \leq j < \rho(m) \times p_i$ ) such that  $-i + \rho(m) + L_1(d) - \rho(m) \equiv j \pmod{\rho(m) \times p_i}$ . By using equation (51) and the fact that  $-i + \rho(m) \in C0(i, p_i)$ , we deduce the following implications:

$$\left. \begin{aligned} -i + \rho(m) \in C0(i, p_i), \\ j \in C2(i, d), \\ j \neq 0, \end{aligned} \right\} \implies j \in C1(n, y)$$

$$\left. \begin{aligned} -i + \rho(m) \in C0(i, p_i), \\ j \in C2(i, d), \\ j = 0, \end{aligned} \right\} \implies \rho(m) \times p_i \in C1(n, y).$$

We build the set  $C3(i, d)$  as follows:

$$\begin{aligned} C3(i, d) = \\ \{\ell : \ell \equiv j \pmod{\rho(m) \times p_i}, 1 \leq \ell \leq h, \ell \in C2(i, d)\}. \end{aligned} \tag{53}$$

It is easy to see that

$$B_0(d) = \bigcup_{i=0}^{\rho(m)-1} C3(i, d).$$

Let us denote  $A(d)$  in the following set:

$$A(d) = \bigcup_{i=0}^d B_i(d) \tag{54}$$

where

$$B_{i+1}(d) = \{1 + j : j \in B_i(d)\}.$$

Based on the evolution of the neuronal recurrence equation  $y$  and the definition of the set  $B_i(d)$ , it is easy to verify that

$$\begin{aligned} \forall \ell \in \mathbb{N}, 0 \leq \ell \leq d \text{ and } \forall f \in B_\ell(d), \\ \text{so } y(-f - \rho(m) + h + \ell + L_1(d)) = 1. \end{aligned} \tag{55}$$

#### 4. Construction of the Neuronal Recurrence Equation $z$

The basic idea is to construct a sequence  $\{z(n, d) : n \geq 0\}$  whose terms are generated by the neuronal recurrence equation

$$z(n, d) = 1 \left[ \sum_{f=1}^b c(f, d) z(n - f, d) - \theta_2(d) \right] \tag{56}$$

and whose first  $b$  terms are initialized as follows:

$$z(f, d) = y(f); \quad 0 \leq f \leq b - 1, \tag{57}$$

which exploits the instability of the sequences  $\{x^{\alpha_i(n)} : n \geq 0\}$  to converge to the sequence  $\{w(n, d) : n \geq 0\}$ .

We define the parameters  $c(f, d)$  and  $\theta_2(d)$  as follows:

$$c(f, d) = \begin{cases} b_f & 1 \leq f \leq b \text{ and } f \notin A(d), \\ b_f + \beta(d) & 1 \leq f \leq b \text{ and } f \in A(d). \end{cases} \tag{58}$$

$$\theta_2(d) = \theta_1 + \xi(d), \tag{59}$$

where

$$\beta(d) = \frac{\lambda}{\text{Tot}(d)}, \tag{60}$$

$$\xi(d) = \lambda - \frac{\beta(d)}{8}, \tag{61}$$

$$\text{Tot}(d) = \text{card } B_\ell(d), \quad 0 \leq \ell \leq d. \tag{62}$$

We have defined the parameters  $c(f, d)$  so that the sequence  $\{z(n, d) : n \geq 0\}$  converges to the sequence  $\{w(n, d) : n \geq 0\}$ .



**Remark 4.** The terms of the sequence  $\{z(n, d) : n \geq 0\}$  verify the following relation:

$$z(\text{tim}(i, 0, 0), d) z(\text{tim}(i, 1, 0), d) \dots z(\text{tim}(i, k - 1, 0), d) = x^{\alpha_i}(1) x^{\alpha_i}(2) \dots x^{\alpha_i}(k - 1) x^{\alpha_i}(k); \quad 0 \leq i \leq -1 + \rho(m).$$

**Notation 4.** Let  $Q2(n, d)$  be defined as follows:

$$Q2(n, d) = \beta(d) \sum_{f \in A(d)} z(n - f, d) - \xi(d).$$

We now establish a proposition that states a relation between  $z(n, d)$ ,  $S2(n)$ , and  $Q2(n, d)$ .

**Proposition 3.** If

$$z(n - i, d) = y(n - i) \text{ for all } i \text{ such that } 1 \leq i \leq h \text{ and } n - i \equiv 0 \pmod{\rho(m)},$$

then

$$z(n, d) = 1[S2(n) + Q2(n, d) - \theta_1].$$

The next two propositions provide the link between the value of  $Q2(n, d)$  and the sequence  $z(n, d)$ .

**Proposition 4.** If

$$\sum_{f \in A(d)} z(n - f, d) = \text{Tot}(d),$$

then

$$Q2(n, d) = \frac{\beta(d)}{8}.$$

**Proposition 5.** If

$$\sum_{f \in A(d)} z(n - f, d) \leq \text{Tot}(d) - 1,$$

then

$$\frac{-7\beta(d)}{8} \leq Q2(n, d) \leq -\xi(d).$$

In the next two lemmas, we show the relations between  $B_0(d)$ ,  $A(d)$ , and  $\text{Tot}(d)$ .

**Lemma 15.**  $\forall d, n \in \mathbb{N}$  such that  $0 \leq d \leq \rho(m) - 1$ .

If

$$n \not\equiv h - \rho(m) + L_1(d) \pmod{L_2},$$

then

$$\sum_{f \in B_0(d)} y(n-f) < \text{Tot}(d).$$

**Lemma 16.**  $\forall n \in \mathbb{N}$  such that  $n \equiv \ell \pmod{L_2}$ .

If

$$\begin{aligned} \ell \notin \{ & -\rho(m) + h + L_1(d), \\ & 1 - \rho(m) + h + L_1(d), \dots, d - \rho(m) + h + L_1(d) \}, \end{aligned}$$

then

$$\sum_{f \in A(d)} y(n-f) < \text{Tot}(d).$$

We want to exploit the following facts:

- $\sum_{f \in A(d)} z(n-f, d) = \text{Tot}(d)$  implies that  $Q2(n, d) = \frac{\beta(d)}{8} < 0$ .
- $\sum_{f \in A(d)} z(n-f, d) \leq \text{Tot}(d) - 1$  implies that  $\frac{-7\beta(d)}{8} \leq Q2(n, d) \leq -\xi(d)$ .
- $\forall t, 0 \leq t \leq \rho(m) - 1$ , so  $\sum_{f=1}^h b_f y(h - \rho(m) + t - f + L_1(-1 + \rho(m))) = \bar{\theta}$ .
- The sequence  $\{x^{\alpha_i}(n) : n \geq 0\}$  is unstable.

This proves that the sequence  $\{z(n, d) : n \geq 0\}$  converges to the sequence  $\{w(n, d) : n \geq 0\}$ . We intend to divide the dynamic of the sequence  $\{z(n, d) : n \geq 0\}$  into five phases.

## 5. Dynamical Behavior of the Neuronal Recurrence Equation $z$

This study is done in two steps: first, we analyze the transient phase and next the cyclic phase. Subsequently, we suppose that  $d$  verifies the following equation:

$$d \in \mathbb{N} \text{ and } 0 \leq d \leq \rho(m) - 1.$$

### 5.1 Transient Phase

The transient phase of the sequence  $\{z(n, d) : n \geq 0\}$  unfolds during four phases.

*Phase 1.* We want the dynamics of the sequence  $\{z(n, d) : n \geq 0\}$  to verify the relation  $\forall t \in \mathbb{N}$  such that  $0 \leq t \leq L_1(d) + h - 1 - \rho(m)$ . We find

$$z(t, d) = y(t).$$

Phase 1 starts at time 0 and finishes at time  $L_1(d) + h - 1 - \rho(m)$ .

In Lemma 17, we prove that the sequence  $\{z(n, d) : n \geq 0\}$  verifies the properties of phase 1.

**Lemma 17.** In the evolution of the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ ,  $\forall t \in \mathbb{N}$  such that  $0 \leq t \leq L_1(d) + h - 1 - \rho(m)$ . We find

$$z(t, d) = y(t).$$

*Phase 2.* We want  $\forall t \in \mathbb{N}$  such that  $L_1(d) + h - \rho(m) \leq t \leq L_1(d) + h - \rho(m) + d$ . We find

$$z(t, d) = 0 \text{ even when } y(t) = 1.$$

Phase 2 occurs from time  $h - \rho(m) + L_1(d)$  to time  $L_1(d) + h - \rho(m) + d$ . In Lemma 18, we prove that the sequence  $\{z(n, d) : n \geq 0\}$  verifies the properties of phase 2.

**Lemma 18.**  $\forall t \in \mathbb{N}$  such that  $L_1(d) + h - \rho(m) \leq t \leq L_1(d) + h - \rho(m) + d$ . We find

$$z(t, d) = 0 \text{ even when } y(t) = 1.$$

After phase 2, the behavior of the sequence  $\{z(n, d) : n \geq 0\}$  begins to be different from the behavior of the sequence  $\{y(n) : n \geq 0\}$ . After phase 2, the sequence  $\{z(n, d) : n \geq 0\}$  begins its convergence to the sequence  $\{w(n, d) : n \geq 0\}$ .

*Phase 3.* This phase starts at time  $L_1(d) + h - \rho(m) + d + 1$  and finishes at time  $L_1(d) + h - 1$ .

**Lemma 19.** In the evolution of the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ ,  $\forall t \in \mathbb{N}$  such that  $L_1(d) + h - \rho(m) + d + 1 \leq t \leq L_1(d) + h - 1$ . We find

$$z(t, d) = y(t).$$

**Remark 5.** Based on Lemmas 11, 17, 18, and 19, we easily deduce that the terms of the sequence  $\{z(n, d) : n \geq 0\}$  verify the following relations:

$$\begin{aligned} & z(\text{tim}(i, 0, L_1(d)), d) z(\text{tim}(i, 1, L_1(d)), d) \\ & \dots z(\text{tim}(i, k-1, L_1(d)), d) = \\ & v^{\alpha_i}(0) v^{\alpha_i}(1) \dots v^{\alpha_i}(k-1), \quad 0 \leq i \leq d \end{aligned}$$

$$\begin{aligned}
 & z(\text{tim}(i, 0, L_1(d)), d) z(\text{tim}(i, 1, L_1(d)), d) \\
 & \dots z(\text{tim}(i, k - 1, L_1(d)), d) = \\
 & x^{\alpha_i}(1 + \gamma_i(d)) x^{\alpha_i}(2 + \gamma_i(d)) \dots x^{\alpha_i}(k + \gamma_i(d)), \\
 & d + 1 \leq i \leq -1 + \rho(m)
 \end{aligned}$$

where  $\gamma_i(d)$  is defined in equation (43).

**Remark 6.** If

$$d = \rho(m) - 1,$$

then it is clear that phase 3 does not exist.

*Phase 4.* This phase starts at time  $L_1(d) + h$  and finishes at time  $L_1(d) + h + d + (\rho(m) \times (k - 1 - p_d))$ . This phase corresponds to the transient phase of the neuronal recurrence equation  $\{w(n, d) : n \geq 0\}$ .

**Lemma 20.** In the evolution of the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ ,  $\forall t \in \mathbb{N}$  such that  $0 \leq t \leq (\rho(m) \times (k - 1 - p_d)) + d$ ; we find

$$z(L_1(d) + h + t, d) = w(h + t, d).$$

**Notation 5.** We set  $L_3(d) = L_1(d) + 2h + d - (\rho(m) \times (1 + p_d))$ , which represents the end of the fourth phase, and we set  $L_4(d) = L_3(d) - h + 1$ , which represents the beginning of the cyclic phase.

**Remark 7.** Based on Lemmas 17, 18, 19, and 20, we easily deduce that the terms of the sequence  $\{z(n, d) : n \geq 0\}$  verify the following relations:

$$\begin{aligned}
 & z(\text{tim}(i, 0, L_4(d)), d) z(\text{tim}(i, 1, L_4(d)), d) \\
 & \dots z(\text{tim}(i, k - 1, L_4(d)), d) = \\
 & x^{\alpha_{i+d+1}}(\gamma_{i+d+1}(d) + k - p_d) x^{\alpha_{i+d+1}}(1 + \gamma_{i+d+1}(d) + k - p_d) \dots \\
 & x^{\alpha_{i+d+1}}(2k - 1 + \gamma_{i+d+1}(d) - p_d); 0 \leq i \leq \rho(m) - d - 2 \\
 & z(\text{tim}(i, 0, L_4(d)), d) z(\text{tim}(i, 1, L_4(d)), d) \\
 & \dots z(\text{tim}(i, k - 1, L_4(d)), d) = \underbrace{000 \dots 000}_k; \tag{63}
 \end{aligned}$$

$$\rho(m) - d - 1 \leq i \leq \rho(m) - 1.$$

The sequence  $\{z(n, d) : n \geq 0\}$  describes a cycle during its fifth step. Section 5.2 is devoted to this study.

**5.2 Cyclic Phase**

*Phase 5.* This phase starts at time  $L_4(d)$  and describes a cycle of length  $L_0(d)$ .

**Lemma 21.**

$$z(t + L_4(d), d) = w(t + h + d + 1 + (\rho(m) \times (-1 - p_d)), d); \forall t \in \mathbb{N}.$$

We have shown the following.

**Lemma 22.** The sequence  $\{z(n, d) : n \geq 0\}$  describes a transient of length  $L_4(d)$  and a cycle of length  $L_0(d)$ .

It is easy to see that by perturbation, we can build the neuronal recurrence equations  $\{z(n, 0) : n \geq 0\}$  from  $\{y(n) : n \geq 0\}$  and  $\{z(n, d + 1) : n \geq 0\}$  from  $\{z(n, d) : n \geq 0\}$ .

This second item is obtained by the following transformations:

$$c(f, 1 + d) = \begin{cases} c(f, d), & \text{if } f \notin A(d) \cup A(1 + d) \\ c(f, d) - \beta(d) + \beta(1 + d), & \text{if } f \in A(d) \cap A(1 + d) \\ c(f, d) - \beta(d), & \text{if } f \in A(d) \cap \overline{A(1 + d)} \\ c(f, d) + \beta(1 + d), & \text{if } f \in \overline{A(d)} \cap A(1 + d) \end{cases} \quad (64)$$

$$\theta_2(d + 1) = \theta_2(d) - \xi(d) + \xi(d + 1). \quad (65)$$

The main result of the paper is Theorem 1.

**Theorem 1.**  $\forall m, d \in \mathbb{N}$  such that  $\rho(m) \geq 2$  and  $0 \leq d \leq -1 + \rho(m)$ . We construct a set of  $\rho(m) + 1$  neuronal recurrence equations that verify the following:

- The neuronal recurrence equation  $\{y(n) : n \geq 0\}$  describes a cycle of length  $L_1(-1 + \rho(m))$ .
- By perturbation, we can build the neuronal recurrence equation  $\{z(n, 0) : n \geq 0\}$  from the neuronal recurrence equation  $\{y(n) : n \geq 0\}$ . The period of the neuronal recurrence equation  $\{z(n, 0) : n \geq 0\}$  is a divisor of the period of the neuronal recurrence equation  $\{y(n) : n \geq 0\}$ .
- By perturbation, we can build the neuronal recurrence equation  $\{z(n, d + 1) : n \geq 0\}$  from the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ . The period of the neuronal recurrence equation  $\{z(n, d + 1) : n \geq 0\}$  is a divisor of the period of the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ .
- The period of the neuronal recurrence equation  $\{z(n, -1 + \rho(m)) : n \geq 0\}$  is 1 (i.e., a fixed point).

**Remark 8.** The new contributions in this paper with respect to the previous work [17] are as follows:

- In [17], the sequence  $\{z(n) : n \geq 0\}$  is a composition of the  $s + 1$  subsequences of periods  $p_0, p_1, \dots, p_s$  and  $3m - 1$ . In the evolution of the sequence  $\{z(n) : n \geq 0\}$ , the subsequence of period  $3m - 1$  vanishes and converges to the null sequence  $000 \dots 000 \dots$ . This fact appears in the formula of transient length and in the formula of the cycle length of the

sequence  $\{z(n) : n \geq 0\}$ , which are respectively  $(s + 1)(3m + 1 + \text{lcm}(p_0, p_1, \dots, p_{s-1}, 3m - 1))$  and  $(s + 1)\text{lcm}(p_0, p_1, \dots, p_{s-1}, 3m - 1)$ . It is clear that in this case, the periods  $p_0, p_1, \dots, p_{s-1}$  of the subsequences intervene in the transient formula and in the period formula.

- In this paper, the sequence  $\{z(n, d) : n \geq 0\}$  is a composition of the  $\rho(m)$  subsequences of periods  $p_0, p_1, \dots, p_{-1+\rho(m)}$  and in the evolution of the sequence  $\{z(n, d) : n \geq 0\}$ , the subsequences of period  $p_0, p_1, \dots, p_d$  vanish and converge to the null sequence  $000 \dots 000 \dots$ . This fact appears in the formula of transient length and on the cycle length of the sequence  $\{z(n, d) : n \geq 0\}$ , which are respectively  $\rho(m) \times \text{lcm}(p_0, p_1, \dots, p_d) + h + d + 1 - (\rho(m) \times (1 + p_d))$ ,  $0 \leq d \leq -1 + \rho(m)$  and  $\rho(m) \times \text{lcm}(p_{d+1}, p_{d+2}, \dots, p_{-1+\rho(m)})$  if  $0 \leq d \leq -2 + \rho(m)$  and  $1$  if  $d = \rho(m) - 1$ . It is clear that in this case, there does not exist a period  $p_i$  of a subsequence that intervenes in the transient formula and in the period formula.
- The difference mentioned in the expressions of the transient formula and in the period formula in this paper and in [17] imply that the concept used in the construction of the set  $A$  (see page 520 of [17]) is fundamentally different from the concept used in the construction of the set  $A(d)$  (see equation (54)).
- In [17], we build only one neuronal recurrence equation  $\{z(n) : n \geq 0\}$ , whereas in this paper we build  $\rho(m)$  neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ ,  $0 \leq d \leq \rho(m) - 1$ .

Let us note  $e$  an integer such that

$$\forall i \in \mathbb{N}, 0 \leq i \leq \rho(m) - 1; \quad \text{we find } \beta(m, \alpha_e) \leq \beta(m, \alpha_i).$$

Subsequently, we suppose that

$$d < \beta(m, \alpha_e).$$

Let us note  $\tilde{z}(0, d) \tilde{z}(1, d) \dots \tilde{z}(h - 1, d)$  for the following  $h$  terms:

$$\tilde{z}(i, d) = y(i) \text{ for } i \text{ such that } \beta(m, \alpha_e) - d \leq i \leq h - 1 \tag{66}$$

$$\tilde{z}(i, d) \in \{0, 1\} \text{ for } i \text{ such that } 0 \leq i \leq \beta(m, \alpha_e) - d - 1. \tag{67}$$

The following lemma characterizes the basin of attraction of the sequence  $\{z(n, d) : n \geq 0\}$ .

**Lemma 23.** If  $d < \beta(m, \alpha_e)$ , then from the following initial configurations

$$y(0) y(1) \dots y(h - 1),$$

$$\tilde{z}(0, d) \tilde{z}(1, d) \dots \tilde{z}(h - 1, d);$$

the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$  converges to the same basin of attraction.

## 6. Conclusion

We improve the fundamental lemma of composition of neuronal recurrence equations. From a neuronal recurrence equation that describes a cycle of length  $L_2$ , we construct a set of  $\rho(m)$  neuronal recurrence equations  $\{z(n, d) : n \geq 0\}$ , the dynamics of which describe respectively the cycle of length  $L_0(d)$  where  $L_0(d)$  are the divisors of  $L_2$ . The neuronal recurrence equation  $\{z(n, -1 + \rho(m)) : n \geq 0\}$  describes an exponential transient and a fixed point. By perturbation, we have built the neuronal recurrence equation  $\{z(n, d + 1) : n \geq 0\}$  from the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$  such that the period of the neuronal recurrence equation  $\{z(n, d + 1) : n \geq 0\}$  is a divisor of the period of the neuronal recurrence equation  $\{z(n, d) : n \geq 0\}$ . Thus, we have built a period-halving bifurcation of a neuronal recurrence equation. This result is inscribed in the framework of results on the convergence time of neural networks [18–20]. The exponential convergence time of neuronal recurrence equations can be useful when we want to use it in a cryptographic toolbox (e.g., remote authentication, generation of pseudo-random numbers, etc.).

*Note:* The extended version of the paper with proofs can be found at arXiv:1110.3586.

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